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A NOTE ON FIXED POINT THEOREMS FOR WEAKLY T-KANNAN AND WEAKLY T-CHATTERJEA CONTRACTIONS IN b-METRIC SPACES

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ABSTRACT. In this paper, we consider fixed point results in the setup of b-metric spaces for weakly T-Kannan and weakly T-Chatterjea contractive mappings. Thus, we improve and complement recent results established by Z. Mustafa et al. [J. Inequal. Appl. 2014, 2014:46] and A. Razani, V. Parvaneh [Russ. Math. (Izv. VUZ), 57 (3), 38–45 (2013)], with much shorter proofs. Moreover, we show that the results of A.H. Ansari et al. [J. Inequal. Appl. 2014, 2014:429] are not generalizations of the known ones. An example is given illustrating our results.

1. Introduction and preliminaries

Fixed point theory is one of the most important topics in development of nonlinear and mathematical analysis in general. Also, fixed point theory has been used effectively in many other branches of science, such as computer science, engineering, chemistry, biology, economics ... It is well known that Banach's contraction principle theorem is one of the pivotal results of nonlinear analysis. A mapping $f : X \to X$, where (X, d) is a metric space, is said to be a contraction if there exists $k \in [0, 1)$ such that for all $x, y \in X$

$$d(fx, fy) \leq kd(x, y). \tag{1.1}$$

If the space (X, d) is complete then the mapping f satisfying (1.1) has a unique fixed point.

There are many generalizations of metric spaces: Menger spaces, fuzzy metric spaces, generalized metric spaces, abstract metric spaces, b-metric spaces or, by some authors, metric type spaces ... Czerwik introduced in [6] the concept of a b-metric space (metric type spaces in [8, 9, 11]). Since then, several papers have dealt with fixed point theory for single-valued and multi-valued operators in b-metric spaces (see, e.g., [8, 9, 11, 16, 18, 20]).

Let X be a (nonempty) set and $s \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{R}^+$ is a b-metric if the following conditions are satisfied:

(b1) $d(x, y) = 0$ if and only if $x = y$,

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(b2) $d(x,y) = d(y,x)$,
(b3) $d(x,z) \le s[d(x,y) + d(y,z)]$,

for all $x, y, z \in X$. In this case, the pair (X, d) is called a b-metric space.

It should be noted that the class of b-metric spaces is effectively larger than that of metric spaces since a b-metric is a metric when $s = 1$, and there are b-metric spaces which are not metric spaces. For some useful examples of b-metric spaces see, e.g., [1, 8, 9, 16, 18, 20]. Also, the notions such as b-convergent sequence, b-Cauchy sequence and b-complete space are defined in an obvious way.

Using different forms of contractive conditions in various generalized metric spaces, a large number of extensions of Banach contraction principle have been obtained (see, e.g., [4, 5, 10, 19]). Some of these generalizations have been obtained via altering distances or some other types of functions (see, e.g., [7, 12, 13, 14, 17]).

In this paper, we consider fixed point results in the setup of b-metric spaces for weakly T-Kannan and weakly T-Chatterjea contractive mappings. Thus, we improve and complement recent results established in the papers [15] and [17], with much shorter proofs. Also, a T-Hardy-Rogers-type fixed point result is presented. Moreover, we show that the recent results of A.H. Ansari et al. [2] are not generalizations of the known ones. An example is given illustrating our results.

2. Main results

Let (X, d) be a metric space. Recall that a mapping $T : X \to X$ is said to be sequentially convergent (resp. subsequentially convergent) if, for a sequence $\{x_n\}$ in X for which $\{Tx_n\}$ is convergent, $\{x_n\}$ is also convergent (resp. $\{x_n\}$ has a convergent subsequence).

Recently, Z. Mustafa et al. proved the following two theorems:

Theorem 2.1. [15, Theorem 4] *Let (X, d) be a complete b-metric space with parameter $s \ge 1$, $T, f : X \to X$ be such that, for some $\psi \in \Psi$, $\varphi \in \Phi$, and all $x, y \in X$,*

$$\begin{aligned}&\psi(sd(Tfx, Tfy))\\ &\quad \le \psi\left(\frac{d(Tx, Tfy) + d(Ty, Tfx)}{s+1}\right) - \varphi(d(Tx, Tfy), d(Ty, Tfx)),\end{aligned} \tag{2.1}$$

and let T be one-to-one and continuous. Then the following hold:

(1) *For every $x \in X$ the sequence $\{Tf^n x\}$ is b-convergent.*
(2) *If T is subsequentially convergent then f has a unique fixed point.*
(3) *If T is sequentially convergent then for each $x \in X$ the sequence $\{f^n x\}$ b-converges to the fixed point of f.*

Theorem 2.2. [15, Theorem 5] *The conclusions of Theorem 2.1 remain valid if the condition (2.1) is replaced by*

$$\begin{aligned}&\psi(d(Tfx, Tfy))\\ &\quad \le \psi\left(\frac{d(Tx, Tfx) + d(Ty, Tfy)}{s+1}\right) - \varphi(d(Tx, Tfx), d(Ty, Tfy)),\end{aligned} \tag{2.2}$$

The conditions of previous theorems include:

(**I**) The conditions (2.1), resp. (2.2), involving functions $\psi \in \Psi$ and $\varphi \in \Phi$, where

$$\Psi = \{\psi : [0,\infty) \to [0,\infty) \mid \ \psi \text{ is an altering distance function}\}$$

and

$$\Phi = \{\varphi : [0,\infty)^2 \to [0,\infty) \mid \ \varphi(x,y) = 0 \iff x = y = 0 \text{ and}$$
$$\varphi(\liminf_{n\to\infty} a_n, \liminf_{n\to\infty} b_n) \le \liminf_{n\to\infty} \varphi(a_n, b_n)\}.$$

(**II**) The assumption that mapping T is continuous and (for certain parts) (sub)sequentially convergent.

Also, the following auxiliary result was used in the proof.

Lemma 2.3. [15, Lemma 1] *Let (X,d) be a b-metric space with $s \ge 1$, and suppose that sequences $\{x_n\}$ and $\{y_n\}$ in X are b-convergent to x, y, respectively. Then we have*

$$\frac{1}{s^2} d(x,y) \le \liminf_{n\to\infty} d(x_n, y_n) \le \limsup_{n\to\infty} d(x_n, y_n) \le s^2 d(x,y).$$

In particular, if $x = y$, then we have $\lim_{n\to\infty} d(x_n, y_n) = 0$. Moreover, for each $z \in X$, we have

$$\frac{1}{s} d(x,z) \le \liminf_{n\to\infty} d(x_n, z) \le \limsup_{n\to\infty} d(x_n, z) \le s d(x,z).$$

In what follows, we will prove that the conclusions about fixed points in previous results are valid without assumption (**II**) and under much more general assumptions than (**I**). Also, we will not need Lemma 2.3 in our proofs.

Theorem 2.4. *Let (X,d) be a complete b-metric space with parameter $s > 1$, $T, f : X \to X$ be such that for $x, y \in X$,*

$$d(Tfx, Tfy) \le \frac{d(Tx, Tfx) + d(Ty, Tfy)}{s+1} \tag{2.3}$$

and let T be one-to-one. Also assume that either

(a) T is a surjective selfmap or

(b) at least one of $Tf(X)$ or $T(X)$ is closed.

Then f has a unique fixed point.

Proof. It is clear that f has a fixed point (say z) if and only if $Tfz = Tz$ (because T is one-to-one). Further, it is obvious that if f has a fixed point then it is unique. Now, since $Tf(X) \subseteq T(X)$, for a given point x_0 in X, we can construct a standard Jungck sequence

$$y_n = Tfx_n = Tx_{n+1} \ (n = 0, 1, 2, \dots). \tag{2.4}$$

If $y_{n_0} = y_{n_0+1}$ for some $n_0 \in \mathbb{N}$ then x_{n_0+1} is a unique fixed point of f. Assume now that $y_n \ne y_{n+1}$ for all $n \in \mathbb{N}$. In this case (putting in (2.3) $x = x_{n+1}, y = x_n$)

we obtain

$$\begin{aligned} d(y_{n+1}, y_n) &= d(Tfx_{n+1}, Tfx_n) \\ &\le \frac{d(Tx_{n+1}, Tfx_{n+1}) + d(Tx_n, Tfx_n)}{s+1} \\ &= \frac{d(y_n, y_{n+1}) + d(y_{n-1}, y_n)}{s+1}, \end{aligned}$$

that is,

$$d(y_{n+1}, y_n) \le kd(y_n, y_{n-1}) \le \cdots \le k^n d(y_1, y_0),$$

where $k = \frac{1}{s} < 1$. Hence, $d(y_{n+1}, y_n) \downarrow 0$. Now we will show that $\{y_n\}$ is a b-Cauchy sequence in each of $Tf(X), T(X)$ and X.

Suppose that this is not true. Then there exists $\varepsilon > 0$ for which we can find subsequences $\{y_{m(k)}\}$ and $\{y_{n(k)}\}$ of $\{y_n\}$ such that $n(k)$ is the smallest index for which $n(k) > m(k) > k$ and $d(y_{m(k)}, y_{n(k)}) \ge \varepsilon$. This means that

$$d(y_{m(k)}, y_{n(k)-1}) < \varepsilon.$$

Further, by (2.3), we get

$$\begin{aligned} \varepsilon \le d(y_{m(k)}, y_{n(k)}) &= d(Tfx_{m(k)}, Tfx_{n(k)}) \\ &\le \frac{d(Tx_{m(k)}, Tfx_{m(k)}) + d(Tx_{n(k)}, Tfx_{n(k)})}{s+1} \\ &= \frac{d(y_{m(k)-1}, y_{m(k)}) + d(y_{n(k)-1}, y_{n(k)})}{s+1} \to 0, \text{ as } k \to \infty. \end{aligned}$$

A contradiction. Therefore, in both cases (a) or (b), there is $u \in X$ such that $Tfx_n = Tx_{n+1} \to Tu$ when $n \to \infty$. We will prove that $Tfu = Tu$.

Using (2.3) we conclude that

$$\begin{aligned} d(Tfu, Tu) &\le s[d(Tfu, Tfx_n) + d(Tfx_n, Tu)] \\ &\le s\left[\frac{d(Tu, Tfu) + d(Tx_n, Tfx_n)}{s+1} + d(Tfx_n, Tu)\right] \\ &= \frac{sd(Tu, Tfu)}{s+1} + \frac{sd(y_{n-1}, y_n)}{s+1} + sd(Tfx_n, Tu). \end{aligned} \tag{2.5}$$

Passing to the limit in (2.5) as $n \to \infty$, we obtain that

$$d(Tfu, Tu) \le \frac{sd(Tu, Tfu)}{s+1},$$

that is $Tfu = Tu$. Hence, f has a unique fixed point and the proof is completed. □

Theorem 2.5. *Let (X, d) be a complete b-metric space with parameter $s > 1$, $T, f : X \to X$ be such that for $x, y \in X$,*

$$d(Tfx, Tfy) \le \frac{d(Tx, Tfy) + d(Ty, Tfx)}{s(s+1)} \tag{2.6}$$

and let T be one-to-one. Also assume that either

(a) T is a surjective selfmap or

(b) at least one of $Tf(X)$ or $T(X)$ is closed.
Then f has a unique fixed point.

Proof. The proof of this theorem is similar to the previous one, and so we will prove just the Cauchyness of the sequence given by (2.4).

Suppose that it is not a Cauchy sequence and choose sequences $n(k)$ and $m(k)$ as in the proof of Theorem 2.4. Then, using (2.6), we get that

$$\begin{aligned}\varepsilon \le d(y_{m(k)}y_{n(k)}) &= d(Tfx_{m(k)}, Tfx_{n(k)})\\ &\le \frac{d(Tx_{m(k)}, Tfx_{n(k)}) + d(Tx_{n(k)}, Tfx_{m(k)})}{s(s+1)}\\ &\le \frac{d(Tx_{m(k)}, Tfx_{m(k)}) + d(Tfx_{m(k)}, Tfx_{n(k)})}{s+1}\\ &\quad + \frac{d(Tx_{n(k)}, Tfx_{n(k)}) + d(Tx_{n(k)}, Tfx_{m(k)})}{s+1}\\ &= \frac{d(Tx_{m(k)}, Tfx_{m(k)}) + 2d(Tfx_{m(k)}, Tfx_{n(k)}) + d(Tx_{n(k)}, Tfx_{n(k)})}{s+1}\\ &= \frac{d(y_{m(k)-1}, y_{m(k)}) + 2d(y_{m(k)}, y_{n(k)}) + d(y_{n(k)-1}, y_{n(k)})}{s+1}.\end{aligned}$$

Hence,

$$\left(1 - \frac{2}{s+1}\right) d(y_{m(k)}, y_{n(k)}) \le \frac{d(y_{m(k)-1}, y_{m(k)}) + d(y_{n(k)-1}, y_{n(k)})}{s+1},$$

and so

$$0 \le (s-1)d(y_{m(k)}, y_{n(k)}) \le d(y_{m(k)-1}, y_{m(k)}) + d(y_{n(k)-1}, y_{n(k)}) \to 0,$$

as $k \to \infty$. Since $s > 1$ and $d(y_{m(k)}, y_{n(k)}) \ge \varepsilon$, we have obtained a contradiction. □

Corollary 2.6. *In addition to the hypothesis of Theorem 2.4 (resp. 2.5), suppose that T is continuous and sequentially convergent. Then for each $x \in X$ the sequence $\{f^n x\}$ converges to the fixed point of f.*

Proof. In the same way as in the proof of Theorem 2.4 we obtain that

$$Tu = \lim_{n\to} Tff^n x = \lim_{n\to\infty} Tf^n x, \tag{2.7}$$

where u is a unique fixed point of f. Further, since T is sequentially convergent we have that $f^n x \to v$ for some $v \in X$. From the continuity of T it follows that $Tf^n x \to Tv$. Now, by (2.7) we conclude that $Tu = Tv$, or equivalently $u = v$. Hence, the sequence $\{f^n x\}$ converges to the fixed point of f. □

Remark 2.7. Very recently, A.N. Ansari et al. proved in [2] the same conclusions as in our Theorems 2.4 and 2.5, under the following contractive conditions:

$$\psi(d(Tfx, Tfy)) \le \frac{\psi\left(\frac{d(Tx,Tfx)+d(Ty,Tfy)}{s+1}\right)}{1 + \varphi(d(Tx, Tfx), d(Ty, Tfy))} \tag{2.8}$$

[2, Theorem 3.3], resp.

$$\psi(sd(Tfx,Tfy)) \leq \frac{\psi\left(\frac{d(Tx,Tfy)+d(Ty,Tfx)}{s+1}\right)}{1+\varphi(d(Tx,Tfy),d(Ty,Tfx))} \tag{2.9}$$

[2, Theorem 3.1]. Here, $\psi : [0,+\infty) \to [0,+\infty)$ is a continuous and strictly increasing function and $\varphi : [0,+\infty)^2 \to [0,+\infty)$ satisfies certain conditions. They claimed that these results were more general than those obtained under conditions (2.3), resp. (2.6). However, it is trivial to see that they are in fact equivalent, since (2.8) implies that

$$\psi(d(Tfx,Tfy)) \leq \psi\left(\frac{d(Tx,Tfx)+d(Ty,Tfy)}{s+1}\right),$$

and this implies the condition (2.3) since the function ψ is strictly increasing. Similarly, the condition (2.9) implies the condition (2.6).

Moreover, [2, Theorem 3.2] is also not a new result, since the condition

$$\psi(sd(Tfx,Tfy)) \leq \left(\psi\left(\frac{d(Tx,Tfy)+d(Ty,Tfx)}{s+1}\right)+l\right)^{\frac{1}{1+\varphi(d(Tx,Tfy),(Ty,Tfx))}}-l$$

implies

$$\psi(sd(Tfx,Tfy)) \leq \psi\left(\frac{d(Tx,Tfy)+d(Ty,Tfx)}{s+1}\right)+l-l,$$

i.e.

$$d(Tfx,Tfy) \leq \frac{d(Tx,Tfy)+d(Ty,Tfx)}{s(s+1)}.$$

(It was used that

$$(A+l)^{\frac{1}{1+B}} < A+l$$

for all $A>0$, $B>0$, since the function $x \mapsto a^x$ is increasing when $a>1$.)

Remark 2.8. It is obvious that Theorem 2.4 (resp. Theorem 2.5) improves Corollary 2 (resp. Corollary 1) from [15] if $s>1$, in several directions, in particular because $k \in [0, \frac{1}{s+1}]$ is allowed.

If $s=1$ then (X,d) is a complete metric space. Thus the conditions (2.1) and (2.2) become

$$\psi(d(Tfx,Tfy)) \leq \psi\left(\frac{d(Tx,Tfy)+d(Ty,Tfx)}{2}\right)-\varphi(d(Tx,Tfy),d(Ty,Tfx)), \tag{2.10}$$

and

$$\psi(d(Tfx,Tfy)) \leq \psi\left(\frac{d(Tx,Tfx)+d(Ty,Tfy)}{2}\right)-\varphi(d(Tx,Tfx),d(Ty,Tfy)), \tag{2.11}$$

respectively. If (2.10) (resp. (2.11)) holds then f is called a generalized weak T-Chatterjea (resp. T-Kannan) contraction (see [17]). For both types of mappings we have the following improvements.

Theorem 2.9. *Let (X, d) be a complete metric space, $f : X \to X$ be a generalized weak T-Chatterjea (resp. T-Kannan) contraction and $T : X \to X$ be one-to-one. Also assume that either*
(a) T is a surjective selfmap or
(b) at least one of $Tf(X)$ or $T(X)$ is closed.
Then f has a unique fixed point.

Proof. We will prove only the case when f is a generalized weak T-Chatterjea contraction. Therefore, first of all, suppose that u, v are different fixed points of f. Then, it follows from (2.10) for $x = u$, $y = v$ that

$$\begin{aligned}\psi(d(Tu, Tv)) &= \psi(d(Tfu, Tfv)) \\ &\leq \psi\left(\frac{d(Tu, Tfv) + d(Tv, Tfu)}{2}\right) - \varphi(d(Tu, Tfv), d(Tv, Tfu)) \\ &= \psi\left(\frac{d(Tu, Tv) + d(Tv, Tu)}{2}\right) - \varphi(d(Tu, Tv), d(Tv, Tu)) \\ &= \psi(d(Tu, Tv)) - \varphi(d(Tu, Tv), d(Tv, Tu)).\end{aligned}$$

It follows that $\varphi(d(Tu, Tv), d(Tv, Tu)) = 0$, what is equivalent with $Tu = Tv$. Hence, $u = v$, because T is one-to-one. Contradiction.

Let now $x_0 \in X$ be a given point. From the obvious relation $Tf(X) \subseteq T(X)$ we obtain a standard Jungck sequence defined by

$$y_n = Tfx_n = Tx_{n+1} \ (n = 0, 1, 2, \dots).$$

If $y_{n_0} = y_{n_0+1}$ for some $n_0 \in \mathbb{N}$ then x_{n_0+1} is a unique fixed point of f. Therefore, suppose that $y_n \neq y_{n+1}$ for all $n \in \mathbb{N}$. As in [17], we have that $\{y_n\}$ is a Cauchy sequence. In either of cases (a) or (b), there is $u \in X$ such that $Tfx_n = Tx_{n+1} \to Tu$ when $n \to \infty$. We will prove that $Tfu = Tu$. Indeed, we have

$$d(Tfu, Tu) \leq d(Tfu, Tfx_n) + d(Tfx_n, Tu).$$

Further, we get

$$\begin{aligned}&\psi(d(Tfu, Tfx_n)) \\ &\quad \leq \psi\left(\frac{d(Tu, Tfx_n) + d(Tx_n, Tfu)}{2}\right) - \varphi(d(Tu, Tfx_n), d(Tx_n, Tfu)). \quad (2.12)\end{aligned}$$

Passing to the limit in (2.12) as $n \to \infty$, we get that

$$\begin{aligned}\psi(d(Tfu, Tu)) &\leq \psi\left(\frac{d(Tu, Tu) + d(Tu, Tfu)}{2}\right) - \varphi(d(Tu, Tu), d(Tu, Tfu)) \\ &= \psi(d(Tfu, Tu)) - \varphi(0, d(Tu, Tfu)),\end{aligned}$$

that is, $\varphi(0, d(Tu, Tfu)) = 0$. Hence $Tfu = Tu$, i.e., u is a fixed point of f. The proof is finished. $\square$

Finally, we consider T-Hardy-Rogers type mappings in the setup of b-metric spaces.

Definition 2.10. Let (X, d) be a b-metric space with coefficient $s \geq 1$ and $T, f : X \to X$ be two mappings. The mapping f is said to be a T-Hardy-Rogers contraction, if there exist $a_i \geq 0$, $i = \overline{1, 5}$ with

$$2sa_1 + (s+1)(a_2 + a_3) + (s^2 + s)(a_4 + a_5) < 2 \tag{2.13}$$

such that for all $x, y \in X$

$$\begin{aligned} d(Tfx, Tfy) \leq a_1 d(Tx, Ty) + a_2 d(Tx, Tfx) + a_3 d(Ty, Tfy) \\ + a_4 d(Tx, Tfy) + a_5 d(Ty, Tfx). \end{aligned} \tag{2.14}$$

Note that when $s = 1$ the conditions (2.13) and (2.14) reduce to the T-Hardy-Rogers condition in metric spaces and for $T = i_X$ we get the Hardy-Rogers condition in b-metric spaces [9, Corollary 3.9].

Theorem 2.11. *Let (X, d) be a complete b-metric space, $f : X \to X$ be a T-Hardy-Rogers contraction and $T : X \to X$ be one-to-one. Also assume that either*
(a) T is a surjective selfmap or
(b) at least one of $Tf(X)$ or $T(X)$ is closed.
Then f has a unique fixed point.

Proof. Take an arbitrary $x_0 \in X$ and using that $Tf(X) \subseteq T(X)$, construct a Jungck sequence $\{y_n\}$ defined by $y_n = Tfx_n = Ty_{n+1}$, $n = 0, 1, 2, \dots$. Let us prove that this is a Cauchy sequence. Indeed, using (2.14), we get that

$$\begin{aligned} d(y_{n+1}, y_n) &= d(Tfx_{n+1}, Tfx_n) \\ &\leq a_1 d(Tx_{n+1}, Tx_n) + a_2 d(Tx_{n+1}, Tfx_{n+1}) + a_3 d(Tx_n, Tfy_n) \\ &\quad + a_4 d(Tx_{n+1}, Tfx_n) + a_5 d(Tx_n, Tfx_{n+1}) \\ &= a_1 d(y_n, y_{n-1}) + a_2 d(y_n, y_{n+1}) + a_3 d(y_{n-1}, y_n) \\ &\quad + a_4 d(y_n, y_n) + a_5 d(y_{n-1}, y_{n+1}) \\ &\leq (a_3 + a_3 + sa_5) d(y_{n-1}, y_n) + (a_2 + sa_5) d(y_n, y_{n+1}). \end{aligned}$$

Similarly, we obtain that

$$\begin{aligned} d(y_n, y_{n+1}) &= d(Tfx_n, Tfx_{n+1}) \\ &\leq a_1 d(Tx_n, Tx_{n+1}) + a_2 d(Tx_n, Tfx_n) + a_3 d(Tx_{n+1}, Tfx_{n+1}) \\ &\quad + a_4 d(Tx_n, Tfx_{n+1}) + a_5 d(Tx_{n+1}, Tfx_n) \\ &= a_1 d(y_{n-1}, y_n) + a_2 d(y_{n-1}, y_n) + a_3 d(y_n, y_{n+1}) \\ &\quad + a_4 d(y_{n-1}, y_{n+1}) + a_5 d(y_n, y_n) \\ &\leq (a_1 + a_2 + sa_4) d(y_{n-1}, y_n) + (a_3 + sa_4) d(y_n, y_{n+1}). \end{aligned}$$

Adding up the last two inequalities, we get that

$$d(y_{n+1}, y_n) \leq \lambda d(y_{n-1}, y_n),$$

where

$$\lambda = \frac{2a_1 + a_2 + a_3 + s(a_4 + a_5)}{2 - a_2 - a_3 - s(a_4 + a_5)}.$$

The assumption (2.13) implies that $\lambda s < 1$. According to [9, Lemma 3.1], it follows that $\{y_n\}$ is a Cauchy sequence in complete b-metric space (X, d). In either of cases (a) or (b), there is $u \in X$ such that $Tfx_n = Tx_{n+1} \to Tu$ as $n \to \infty$. We will prove that $Tfu = Tu$.

Using (2.14) we obtain that

$$\begin{aligned}
\frac{1}{s}d(Tfu, Tu) &\leq d(Tfu, Tfx_n) + d(Tfx_n, Tu) \\
&\leq a_1 d(Tu, Tx_n) + a_2 d(Tu, Tfu) + a_3 d(Tx_n, Tfx_n) \\
&\qquad + a_4 d(Tu, Tfx_n) + a_5 d(Tx_n, Tfu) + d(Tfx_n, Tu) \\
&\leq a_1 d(Tu, Tx_n) + a_2 d(Tu, Tfu) + a_3 d(Tx_n, Tfx_n) \\
&\qquad + (a_4 + 1) d(Tu, Tfx_n) + s a_5 d(Tx_n, Tu) + s a_5 d(Tu, Tfu) \\
&= a_1 d(Tu, Tx_n) + (a_2 + s a_5) d(Tu, Tfu) + a_3 d(Tx_n, Tfx_n) \\
&\qquad + (a_4 + 1) d(Tu, Tfx_n).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\frac{1}{s}d(Tu, Tfu) &\leq d(Tu, Tfx_n) + d(Tfx_n, Tfu) \\
&\leq d(Tu, Tfx_n) + a_1 d(Tx_n, Tu) + a_2 d(Tx_n, Tfx_n) \\
&\qquad + a_3 d(Tu, Tfu) + a_4 d(Tx_n, Tfu) + a_5 d(Tu, Tfx_n) \\
&\leq d(Tu, Tfx_n) + a_1 d(Tx_n, Tu) + a_2 d(Tx_n, Tfx_n) \\
&\qquad + a_3 d(Tu, Tfu) + s a_4 d(Tx_n, Tu) + s a_4 d(Tu, Tfu) \\
&= d(Tu, Tfx_n) + (a_1 + s a_4) d(Tu, Tx_n) + a_2 d(Tx_n, Tfx_n) \\
&\qquad + (a_3 + s a_4) d(Tu, Tfu).
\end{aligned}$$

Adding up the last two inequalities, we have

$$\begin{aligned}
&\left[\frac{2}{s} - (a_3 + s a_4) - (a_2 + s a_5)\right] d(Tu, Tfu) \\
&\leq (2a_1 + s a_4) d(Tu, Tx_n) + (a_2 + a_3) d(Tx_n, Tfx_n) + (2 + a_4) d(Tu, Tfx_n) \\
&\to 0, \text{ as } n \to \infty. \qquad (2.15)
\end{aligned}$$

Since it follows from (2.13) that

$$\begin{aligned}
\frac{2}{s} - (a_3 + s a_4) - (a_2 + s a_5) &= \frac{2 - s(a_2 + a_3) - s^2(a_4 + a_5)}{s} \\
&> \frac{2sa_1 + a_2 + a_3 + s(a_4 + a_5)}{s} > 0,
\end{aligned}$$

then (2.15) implies that $d(Tu, Tfu) = 0$, that is $Tu = Tfu$. Hence, u is a fixed point of f. The uniqueness of fixed point is obvious by (2.14). □

Corollary 2.12. *Taking special values for constants a_i, we obtain fixed point results under b-metric versions of T-Banach, T-Kannan and T-Chatterjea conditions:*

- *T-Banach condition: $d(Tfx, Tfy) \leq a_1 d(Tx, Ty)$ for all $x, y \in X$ if $a_1 < \frac{1}{s}$ ($a_2 = a_3 = a_4 = a_5 = 0$);*
- *T-Kannan condition: $d(Tfx, Tfy) \leq a_2 d(Tx, Tfx) + a_3 d(Ty, Tfy)$ for all $x, y \in X$ if $a_2 + a_3 < \frac{2}{s+1}$ ($a_1 = a_4 = a_5 = 0$);*

• *T-Chatterjea condition:* $d(Tfx, Tfy) \leq a_2 d(Tx, Tfx) + a_3 d(Ty, Tfy)$ *for all* $x, y \in X$ *if* $a_4 + a_5 < \frac{2}{s+s^2}$ $(a_1 = a_2 = a_3 = 0)$.

Finally, we present an example showing that Corollary 2.12 is a proper extension of the results from [3] and [10].

Example 2.13. Let $X = [0, 1]$, $d(x, y) = |x - y|$, $fx = \frac{x}{2}, T(x) = x^2$. Then

$$d(Tfx, Tfy) \leq \frac{1}{4} d(Tx, Ty), \text{ for all } x, y \in X$$

while for $x = 1, y = 0$ we have

$$d(f1, f0) = \frac{1}{2} > \frac{1}{4} = \frac{1}{4} d(1, 0),$$

that is, f is a T-Banach contraction which is not a usual Banach contraction.

Also,

$$d(Tfx, Tfy) \leq \frac{1}{3}(d(Tx, Tfx) + d(Ty, Tfy)), \text{ for all } x, y \in X,$$

but for $x = 1, y = 0$ we get

$$d(f1, f0) = 1 > \frac{1}{3} = \frac{1}{3}(d(1, f1) + d(0, f0)),$$

which means that f is a T-Kannan contraction which is not a Kannan contraction from [10].

Open problem. It is not yet known if the result of Corollary 2.12 for T-Banach contraction can be proved for $\frac{1}{s} \leq a_1 < 1$, $s > 1$, with or without some additional condition on T or the space (X, d).

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