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EXISTENCE AND STABILITY FOR IMPULSIVE NEUTRAL STOCHASTIC INTEGRODIFFERENTIAL EQUATIONS WITH INFINITE DELAYS

MAMADOU ABDOUL DIOP [1*], KHALIL EZZINBI [2] AND MAHAMAT MAHAMAT ZENE [3]

Abstract. This work presents result on the existence and stability of mild solutions for some stochastic partial differential equations with infinite delay with non-lipschitz coefficients. We use the theory of resolvent operator devolopped in R. Grimmer (1982)[8] to show the existence and stability of the mild solutions. An example is provided to illustrate the results of this work.

1. Introduction and preliminaries

Stochastic neutral differential equations have become an important object of investigation in recent years stimulated by their numerous applications to problems arising in physics, mechanics and other fields. Several authors have established the existence results of mild solutions for these equations (see [7, 18, 19, 3, 1, 11] and references therein).

On the other hand the theory of stochastic functional integrodifferential equations has recently received a lot of attention and now constitutes a significant branch of nonlinear analysis; numerous research papers and monographs have appeared devoted to stochastic functional integrodifferential equations, for example(see [12, 5, 6]).

Also, many evolution processes from fields as diverse as physics, population dynamics, aeronautics, economics, telecommunications and engineering etc., are characterized by the fact that they undergo an abrupt change of state at certain moments of time between intervals of continuous evolution. The duration of these changes are often negligible compared to the total duration of the process acting instantaneously in the form of impulses. The impulses may be deterministic or random.There are lot of papers which investigate the qualitative properties of impulsive stochastic partial neutral functional integrodifferential equations with infinite delays. Investigations of these classes of delay equations essentially differ

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* Corresponding author.

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from once of equations with finite delays. For these reasons the theory of impulsive stochastic partial neutral functional integrodifferential equations(ISDEs) with infinite delays has drawn the attention of researchers in recent years.
In [21], the authors J. Yang *et al.* have studied the stability analysis of impulsive stochastic differential equations with delays. Z. Yang *et al.* .[20] studied the exponential p-th stability of impulsive stochastic differential equations with delays. In [15, 16], R. Sakthivel and J. Luo, investigated the existence and asymptotic stability in p-th moment of mild solutions to ISDEs with and without infinite delays through the fixed point theory.
Inspired by the above mentioned works the main purpose of this paper is to establish the existence and stability results fo the following class of abstract partial neutral impulsive integrodifferential equations with infinite delays:

$$\begin{cases} d\left[x(t)+g(t,x_t)\right] = \left[A[x(t)+g(t,x_t)] + \displaystyle\int_0^t F(t-s)[x(s)+g(s,x_s)]ds\right]dt \\ +a(t,x_t)dt + b(t,x_t)dw(t), \quad t \in J \ := \ [0,T], \\ \qquad \Delta x(t_k) = x(t_k^+) - x(t_k^-) = I_k(x(t_k)), \quad k = 1,2,\cdots,m, \\ \qquad x_0 = \varphi \in \mathcal{B}, \end{cases} \tag{1.1}$$

where, the state $x(.)$ takes values in a separable real Hilbert spaces $\mathbb{H}$ with inner product $(.,.)$ and norm $\|.\|$, A is the infinitesimal generator of a strongly continous semigroup of bounded linear operators $S(t), t \geq 0$ on $\mathbb{H}$, with $D(A) \subset \mathbb{H}$, and $F(t),\ t \in J$ is a closed linear operator on $\mathbb{H}$. The history $x_t \ : \ ]-\infty, 0] \ \to \mathbb{H},\ x_t(\theta) \ = \ x(t+\theta)$, for $t \geq 0$, belongs to some abstract phase space $\mathcal{B}$ which will be described axiomatically in Section 2 . Let $\mathbb{K}$ be another separable Hilbert spaces with inner product $\langle .,. \rangle_{\mathbb{K}}$ and norm $\|.\|_{\mathbb{K}}$. Suppose $\{w(s) : 0 \leq s \leq t\}$ is a given $\mathbb{K}$-valued Wiener process with covariance operator $Q \geq 0$ defined on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ equipped with a normal filtration $\{\mathcal{F}_t\}_{t\geq 0}$ which is generated by the Wiener process w. We are also employing the same notations $\|.\|$ for the norm $\mathcal{L}(\mathbb{K};\mathbb{H})$, where $\mathcal{L}(\mathbb{K};\mathbb{H})$ denotes the space of all bounded linear operator from $\mathbb{K}$ into $\mathbb{H}$. Assume that $g, a : \mathbb{R}_+ \times \mathcal{B} \to \mathbb{H}$ and $b : \mathbb{R}_+ \times \mathcal{B} \to \mathcal{L}_2^0$ are two appropriate mappings specified later. Here $\mathcal{L}_2^0 = \mathcal{L}_2(\mathbb{K}_0;\mathbb{H})$ denotes the space of all Q-HilbertSchmidt operators from $\mathbb{K}_0$ to $\mathbb{H}$ wich will be defined in Section 2. The initial data $\varphi = \{\varphi(t) : -\infty < t \leq 0\}$is an $\mathcal{F}_0$-adapted, $\mathcal{B}$-valued random variable independent of the Wiener process w with finite second moment, $I_k \ : \ \mathcal{B} \ \to \mathbb{H},\ k = 1,2,\cdots,m$ are appropriate functions.The fixed moments of time t_k satisfies $0 < t_1 < \cdots < t_m < T$, $x(t_k^+)$ and $x(t_k^-)$ represent the right and left limits of $x(t)$ at $t = t_k$, respectively, $\Delta x(t_k)$ represents the jump in the state x at time t_k with I_k determining the size of the jump. Our main results concerning (1.1) , relay essentially on techniques employing a strongly continous family of operators $R(t), t \geq 0$ defined on the Hilbert space $\mathbb{H}$ and called their resolvent (the precise definition will be given below). The paper is organized as follows: in Section 2, we recall some preliminaries which are used throughout this paper. In Section 3, we state the existence and uniqueness of a mild solution. In Section 4, we study the stability through the continuous dependence on the initial values. Finally in section 5, an example is given to illustrate our results.

2. Preliminaries

2.1. **Wiener process.** Let $(\Omega, \mathcal{F}, P, \mathbf{F})$ with $(\mathbf{F} = \{\mathcal{F}_t\}_{t\geq 0})$ be a complete filtered probability space satisfying that $\mathcal{F}_0$ contains all P-null sets of $\mathcal{F}$. An $\mathbb{H}$-valued random variable is an $\mathcal{F}$-mesurable function $x(t) : \Omega \to \mathbb{H}$ and the collection of random variables $\mathcal{S} = \{x(t,\omega) : \Omega \to \mathbb{H}|\ t \in J\}$ is called a stochastic process. Generally, we juste write $x(t)$ instead of $x(t,\omega)$ and $x(t) : J \to \mathbb{H}$ in the space of $\mathcal{S}$. Let $\{e_i\}_{i=1}^{\infty}$ be a complete orthormal basis of $\mathbb{K}$. Suppose that $\{\omega(t) : t \geq 0\}$ is a cylindrical $\mathbb{K}$-valued Wiener process with a finite trace nuclear covariance operator $Q \geq 0$, denote $Tr(Q) = \sum_{n=1}^{\infty}\lambda_i = \lambda < \infty$, which satisfaies that $Qe_i = \lambda_i e_i$. So, actually, $w(t) = \sum_{i=1}^{\infty}\sqrt{\lambda_i}\omega_i(t)e_i$, where $\{\omega_i\}_{i=1}^{\infty}$ are mutually independent one-dimensional standard Wiener processes. We assume that $\mathcal{F}_t = \sigma\{w(s); 0 \leq s \leq t\}$ is the σ-algebra generated by ω and $\mathcal{F}_T = \mathcal{F}$.
Let $\mathcal{L}(\mathbb{K};\mathbb{H})$ denote the space of all bounded linear operators from $\mathbb{K}$ into $\mathbb{H}$. For $h_1, h_2 \in \mathcal{L}(\mathbb{K};\mathbb{H})$, we define $(h_1, h_2) = Tr(h_1 Q h_2^*)$, where h_2^* is the adjoint of the operator h_2 and Q is the nuclear operator associated with the Wiener process, where $Q \in L_n^+(K)$, the space of positve nuclear operator in $\mathbb{K}$. For $\psi \in L(\mathbb{K},\mathbb{H})$ we define

$$\|\psi\|_Q^2 = Tr(\psi Q \psi^*) = \sum_{n=1}^{\infty}\left\|\sqrt{\lambda_n}\ \psi e_n\right\|^2.$$

If $\|\psi\|_Q < \infty$, then ψ is called a Q-Hilbert Schmidt operator. Let $L_Q(\mathbb{K},\mathbb{H})$ denote the space of all Q-Hilbert Schmidt operator ψ. The completion $L_Q(\mathbb{K},\mathbb{H})$ of $L(\mathbb{K},\mathbb{H})$ with respect to the topology induced by the norm $\|\cdot\|_Q$ where $\|\psi\|_Q^2 = (\psi,\psi)$ is a Hilbert space with the above norm topology. For more details, we refer the reader to Da Prato and Zabczyk [26].
The collection of all strongly measurable, square integrable, $\mathbb{H}$-valued random variables, denoted by $L_2(\Omega,\mathcal{F},P;\mathbb{H}) \equiv L_2(\Omega,\mathbb{H})$, is a Banach space equiped with norm

$$\|x(\cdot)\|_{L_2} = \left(\mathbb{E}\,\|x(\cdot,\omega)\|^2\right)^{\frac{1}{2}},$$

where the expectation $\mathbb{E}$ is defined by $\mathbb{E}x = \int_\Omega x(w)dP$. Let $C(J, L_2(\Omega,\mathbb{H}))$ be the Banach space of all continuous maps from J into $L_2(\Omega,\mathbb{H})$ satisfying the condition

$$\sup_{t\in J}\mathbb{E}\,\|x(t)\|^2 < \infty.$$

An important subspace is given by $L_2^0(\Omega,\mathbb{H}) = \{f \in L_2(\Omega,\mathbb{H}) : f \text{ is } \mathcal{F}_0 - \text{measurable}\}$.

We say that a function $u : [\nu, \tau] \to H$ is a normalized piecewise continuous function on $[\nu, \tau]$ if u is piecewise continuous and left continuous on $[\nu, \tau]$. We denote by $\mathcal{PC}([\nu, \tau]; \mathbb{H})$ the space formed by the normalized piecewise continuous, $\mathcal{F}_t$-adapted measurable processes from $[\nu, \tau]$ into $\mathbb{H}$. In particular, we introduce the space $\mathcal{PC}$ formed by all $\mathcal{F}_t$-adapted measurable, $\mathbb{H}$-valued stochastic processes $\{u(t) : t \in [0, T]\}$ such that u is continuous at $t \neq t_k$, $u(t_k^-) = u(t_k)$ and $u(t_k^+)$ exists, for $k = 1, \cdots, m$. In this paper, we always assume that $\mathcal{PC}$ is endowed with the norm $\|u\|_{\mathcal{PC}} = \left(\sup_{s\in J} \mathbb{E}\,\|u(s)\|^2\right)^{\frac{1}{2}}$. It is clear that $(\mathcal{PC}, \|\cdot\|_{\mathcal{PC}})$ is a Banach space. Further, let $\mathcal{B}_{\mathcal{T}}$ be a Banach space $\mathcal{B}_{\mathcal{T}}((-\infty, T], L_2)$, the family of all $\mathcal{F}_t$-adapted process $\varphi(t, \omega)$ with almost surely continuous in t for fixed $\omega \in \Omega$ with norm defined for any $\varphi \in \mathcal{B}_{\mathcal{T}}$ by

$$\|\varphi\|_{\mathcal{B}_{\mathcal{T}}} = \left(\sup_{0\le t\le T} \mathbb{E}\,|\varphi|_t^2\right)^{\frac{1}{2}}.$$

2.2. **Partial integrodifferential equations in Banach spaces.** In the present section, we recall some definitions, notations and properties needed in the work.

In what follows , $\mathbb{H}$ is a Banach space, A and $B(t)$ are closed linear operators on $\mathbb{H}$. Y represents the Banach space $D(A)$ equipped with the graph norm defined by

$$|y|_Y := |Ay| + |y| \quad \text{for } \; y \in Y.$$

The notations $C([0, +\infty); Y)$, $\mathcal{L}(Y, \mathbb{H})$ stand for the space of all continuous functions from $[0, +\infty)$ into Y, the set of all bounded linear operators from Y into $\mathbb{H}$, respectively. We consider the following Cauchy problem

$$\begin{cases} v'(t) &= Av(t) + \displaystyle\int_0^t F(t-s)v(s)ds \;\text{ for }\; t \ge 0 \\ v(0) &= v_0 \in \mathbb{H}. \end{cases} \tag{2.1}$$

Definition 2.1. ([8]) A resolvent operator for Eq. (2.1) is a bounded linear operator valued function $R(t) \in \mathcal{L}(\mathbb{H})$ for $t \ge 0$, having the following properties :

(i) $R(0) = I$ and $\|R(t)\| \le Ne^{\beta t}$ for some constants N and β.
(ii) For each $x \in \mathbb{H}$, $R(t)x$ is strongly continuous for $t \ge 0$.
(iii) For $x \in Y$, $R(\cdot)x \in C^1([0, +\infty); \mathbb{H}) \cap C([0, +\infty); Y)$ and

$$\begin{aligned} R'(t)x &= AR(t)x + \int_0^t F(t-s)R(s)xds \\ &= R(t)Ax + \int_0^t R(t-s)F(s)xds \qquad \text{for } \; t \ge 0 \end{aligned}$$

For additional details on resolvent operators, we refer the reader to [13, 8] .The resolvent operators plays an important role to study the existence of solutions and to give a variation of constants formula for non linear systems. We need to know when the linear system (2.1) has a resolvent operator. Theorem 2.2 gives a satisfactory answer to this problem.

In what follows we suppose the following assumptions:

(H1) A is the generator of a strongly continuous semigroup $(S(t))_{t\geq 0}$ on $\mathbb{H}$.

(H2) For all $t \geq 0$, $F(t)$ is closed linear operator from $D(A)$ to $\mathbb{H}$, and $F(t) \in \mathcal{L}(Y, \mathbb{H})$. For any $y \in Y$, the map $t \to F(t)y$ is bounded uniformly continuous, differentiable and the derivative $t \to F'(t)y$ is bounded uniformly continuous on $\mathbb{R}^+$

The following theorem gives sufficient conditions ensuring the existence of the resolvent operator for Eq. (2.1)

Theorem 2.2. *([8]) Assume that the assumptions* **(H1)** *and* **(H2)** *hold. Then there existe a unique resolvent operator of the Cauchy problem Eq. (2.1).*

In the following, we give some results on the existence of solutions for the following integrodifferential equation

$$\begin{cases} v'(t) &= Av(t) + \displaystyle\int_0^t F(t-s)v(s)ds + q(t) \quad \text{for} \quad t \geq 0 \\ v(0) &= v_0 \in \mathbb{H}, \end{cases} \tag{2.2}$$

where $q : [0, +\infty[\to \mathbb{H}$ is a continuous function.

Definition 2.3. ([8]) A continuous function $v : [0, +\infty) \to \mathbb{H}$ is said to be a strict solution of equation (2.2) if

(i) $v \in C^1([0, +\infty); \mathbb{H}) \cap C([0, +\infty); Y)$,
(ii) v satisfies (2.2) for $t \geq 0$.

Remark 2.4. From this definition, we deduce that $v(t) \in D(A)$, the function $F(t-s)v(s)$ is integrable, for all $t > 0$ and $s \in [0, +\infty)$.

Theorem 2.5. *([8]) Assume that* ***(H1)-(H2)*** *hold. If v is a strict solution of Eq. (2.2), then the following variation of constants formula holds*

$$v(t) = R(t)v_0 + \int_0^t R(t-s)q(s)ds \quad \textit{for} \quad t \geq 0. \tag{2.3}$$

Accordingly, we make the following definition.

Definition 2.6. ([8]) For $v_0 \in \mathbb{H}$. A function $v : [0, +\infty) \to \mathbb{H}$ is called a mild solution of equation (2.2) if v satisfies the variation of constants formula (2.3).

The next theorem provides sufficient conditions for the regularity of solutions of (2.2).

Theorem 2.7. *([8]) Let $q \in C^1([0, +\infty); \mathbb{H})$ and v be defined by (2.3). If $v_0 \in D(A)$, then v is a strict solution of Eq. (2.2).*

In the whole of this work, we suppose that the phase space is axiomatically defined, we use the approach proposed by Hale and Kato in [9]. To establish the axioms of the phase space $\mathcal{B}$, we follows the terminology used in Hino et al.[10].The axioms of the phase space $\mathcal{B}$ are established for $\mathcal{F}_0$-measurable functions from $]-\infty, 0]$ into $\mathbb{H}$, endowed with a norm wich satisfies the following axioms:

Axiom 2.8. (A1) If $x : (-\infty, T] \to \mathbb{H}$, $T > 0$ is such that $x_0 \in \mathcal{B}$ and $x_{|_{[0,T]}} \in \mathcal{PC}([\nu, \tau]; \mathbb{H})$ then, for every $t \in [0, T]$, the following conditions hold:
(i) $x_t \in \mathcal{B}$,
(ii) $\|x(t)\| \leq L\,\|x_t\|_{\mathcal{B}}$,
(iii) $\|x_t\|_{\mathcal{B}} \leq u(t)\sup_{0\leq s\leq t}\|x(s)\| + v(t)\,\|x_0\|_{\mathcal{B}}$,
where $L > 0$ is a constant ; $u(.), v(.) : [0, +\infty) \to [1, +\infty)$, $u(.)$ is continuous, $v(.)$ is locally bounded, and $L, u(.), v(.)$ are independent of $x(\cdot)$
(A2) The space $\mathcal{B}$ is complete.

The next result is a consequence of the phase space axioms.

Lemma 2.9. *Let $x : (-\infty, 0] \to \mathbb{H}$ be a $\mathcal{F}_t$-adapted measurable process such that the $\mathcal{F}_0$-adapted process $x_0 = \varphi \in L_2^0(\Omega, \mathcal{B})$ and $x_{|_{[0,T]}} \in \mathcal{PC}(J, \mathbb{H})$, then*

$$\|x_s\|_{\mathcal{B}} \leq u_T \mathbb{E}\,\|\phi\|_{\mathcal{B}} + v_T \sup_{0\leq s\leq T} \mathbb{E}\,\|x(s)\| \tag{2.4}$$

where $u_T = \sup_{t\in J}\{u(t)\}$ and $v_T = \sup_{t\in J}\{v(t)\}$.

Remark 2.10. In retarded functional differential equations without impulses, the axioms of the abstract phase space $\mathcal{B}$ include the continuity of the function $t \to x_t$; see for instance [10]. Due to the impulsive effect, this property is not satisfied in impulsive delay systems and, for this reason, has been eliminated in our abstract description of $\mathcal{B}$.

Before starting and proving the main results, we need the following lemma

Lemma 2.11. *(Bihari inequality) ([2]) Let $T > 0$ and $u_0 \geq 0$, $u(t)$, $v(t)$ be continuous functions on $[0, T]$. Let $K : \mathbb{R}^+ \to \mathbb{R}^+$ be a concave, continuous and nondecreasing function such that $K(r) > 0$ for > 0. If*

$$u(t) \leq u_0 + \int_0^t v(t) K\left(u(s)\right) ds \quad \textit{for} \quad 0 \leq t \leq T,$$

then

$$u(t) \leq G^{-1}\left(G(u_0) + \int_0^t v(s) ds\right)$$

for all $t \in [0, T]$ such that

$$G(u_0) + \int_0^t v(s) ds \in Dom(G^{-1}),$$

where $G(r) = \int_1^r \frac{ds}{K(s)}(r \geq 0)$ and G^{-1} is the inverse function of the G. In particular, $u_0 = 0$ and $\int_{0^+} \frac{ds}{K(s)} = +\infty$, then $u(t) = 0$ for all $t \in [0, T]$.

In order to obtain the stability of solutions, we use the following well-known Bihari's inequality

Corollary 2.12. *Let the assumptions of Lemma 2.11 hold and $v(t) \geq 0$ for $t \in [0, T]$. If for all $\epsilon > 0$, there exists $t_1 \geq 0$ such that for $0 \leq u_0 < \epsilon$, $\int_{t_1}^T v(s) ds \leq \int_{u_0}^{\epsilon} \frac{ds}{K(s)}$ holds. Tthen for every $t \in [t_1, T]$, the estimate $u(t) \leq \epsilon$ holds.*

Now, we present the definition of the mild solution of Eq. (1.1).

Definition 2.13. A stochastic process $\{x(t) \in \mathcal{B}_{\mathcal{T}}, t \in (-\infty, T]\}$, $(0 < T < \infty)$ is called a mild solution of the Eq. (1.1) if the following conditions hold,

(i) $x(t)$ is $\mathcal{F}_t$-adapted,
(ii) $x(t)$ satisfies the integral equation

$$x(t) = \begin{cases} \varphi(t) \in \mathcal{B}\,(\,]-\infty, 0], \mathbb{H}) \ \text{for} \quad t \in]-\infty, 0] \\ R(t)[\varphi(0) + g(t, \varphi] - g(t, x_t) + \displaystyle\int_0^t R(t-s)a(s, x_s)ds + \int_0^t R(t-s)b(s, x_s)dw(s) \\ + \displaystyle\sum_{0<t_k<t} R(t-t_k)I_k(x(t_k)) \quad \text{a.s} \ \ t \in [0, T]. \end{cases} \tag{2.5}$$

3. Main Results

In this section we discuss the existence and uniqueness of mild solution for the Eq. (1.1). Now we assume the following assumptions:

(H3): $a : \mathbb{R}_+ \times \mathcal{B} \to \mathbb{H}$, $b : \mathbb{R}_+ \times \mathcal{B} \to \mathcal{L}_2^0$ are Borel measurable functions satisfying the following assumptions

(i) there exists $K\ : [0, +\infty[\to [0, +\infty[$ such that K is nondecreasing and concave with $K(0) = 0$, $K(u) > 0$ for $u > 0$ and $\int_{0+} \frac{du}{K(u)} = \infty$.
(ii) $\|a(t, \varphi) - a(t, \psi)\|^2 \vee \|b(t, \varphi) - b(t, \psi)\|^2 \leq K(\|\varphi - \psi\|^2)$, for all $0 \leq t \leq T$ and $\varphi, \psi \in \mathcal{B}$.

(H4) There exists a positive constant L_g, $0 < L_g < 1$ such that for any $\varphi, \psi \in \mathcal{B}$ and for $t \in [0, T]$, we have

$$\|g(t, \varphi) - g(t, \psi)\|^2 \leq L_g \|\varphi - \psi\|^2.$$

(H5) The function $I_k \in C\,(\mathbb{H}, \mathbb{H})$ and there exists some constant h_k such that $|I_k(x) - I_k(y)|^2 \leq h_k |x - y|^2$, for each $\ x, y \in \mathbb{H}$, and $k = 1, 2, \cdots, m$.

(H6) For all $t \in J$, $a(t, 0)$, $b(t, 0)$, $g(t, 0)$, $I_k(0) \in L^2$, for $k = 1, 2, \cdots, m$ and there exists a positive constant k_0 such that

$$\|a(t, 0)\|^2 \vee \|b(t, 0)\|^2 \vee \|g(t, 0)\|^2 \vee \|I_k(0)\|^2 \leq k_0.$$

Let us now introduce the successive approximations to Eq.(2.5) as follows

$$x^0(t) = \begin{cases} \varphi(t) \ \ \text{for} \ \ t \in]-\infty, 0], \\ R(t)\varphi(0) \quad \text{for}\ t \in [0, T], \end{cases}$$

and for $n = 1, 2, \cdots,$

$$x^n(t)=\begin{cases}\varphi(t)\ \text{ for }\ t\in]-\infty,0],\\[2ex] R(t)[\varphi(0)+g(t,\varphi]-g(t,x_t^n)+\displaystyle\int_0^t R(t-s)a(s,x_s^{n-1})ds+\int_0^t R(t-s)b(s,x_s^{n-1})dw(s)\\[2ex] +\displaystyle\sum_{0<t_k<t}R(t-t_k)I_k(x^{n-1}(t_k)),\quad a.s\ t\in[0,T],\ \text{for } n=1,2,\cdots.\end{cases}\tag{3.1}$$

with an arbitrary non-negative initial approximation $x^0\in\mathcal{B}_{\mathcal{T}}$.

Theorem 3.1. *Under assumptions **(H1) - (H6)**, the sequence $\{x^n,\ n\geq 0\}$ is well defined and Eq. (1.1) has unique mild solution x in $\mathcal{B}_{\mathcal{T}}$ provided $\widetilde{Q}=\max\{Q_1,Q_2\}<1$ and*

$$\mathbb{E}\{\sup_{0\leq t\leq T}\|x^n(t)-x(t)\|^2\}\to 0 \text{ as } n\to\infty$$

with an arbitrary non-negative initial approximation $\{x^0\}\in\mathcal{B}_{\mathcal{T}}$.

Proof : Let $x^0\in\mathcal{B}_{\mathcal{T}}$ be a fixed initial approximation of (3.1). Firstly, we can observe that by **(H1) - (H6)**, $\|R(t)\|\leq M$ for some $M\geq 1$ and all $t\in[0,T]$. Then for $n\geq 1$, we have

$$\mathbb{E}\|x^n(t)\|^2\leq 5M^2[E\|\varphi(0)+g(0,\varphi)\|^2]$$

$$+10\mathbb{E}\left[\|g(t,x_t^n)-g(t,0)\|^2+\|g(t,0)\|^2\right]$$

$$+10TM^2\mathbb{E}\int_0^t\left[\left\|a(s,x_s^{n-1})-a(s,0)\right\|^2+\|a(s,0)\|^2\right]ds$$

$$+10TM^2\mathbb{E}\int_0^t\left[\left\|b(s,x_s^{n-1})-b(s,0)\right\|^2+\|b(s,0)\|^2\right]ds$$

$$+10M^2m\mathbb{E}\sum_{k=1}^m\left[\left\|I_k(x^{n-1}(t_k))-I_k(0)\right\|^2+\|I_k(0)\|^2\right].$$

Thus from the above,

$$\mathbb{E}\|x^n(t)\|^2\leq 10M^2[E\|\varphi(0)\|^2+E\|g(0,\varphi)\|^2]$$

$$+10\left[L_g\mathbb{E}\|x^n\|_t^2+k_0\right]$$

$$+10(T+1)M^2\mathbb{E}\int_0^t\left[K(\left\|x^{n-1}\right\|_s^2)+k_0\right]ds$$

$$+10M^2m\sum_{k=1}^m\left[h_k\mathbb{E}\left\|x^{n-1}\right\|_t^2+k_0\right]$$

$$\mathbb{E}\|x^n(t)\|_t^2\leq Q_2+\frac{10M^2(T+1)}{1-Q_1}\mathbb{E}\int_0^t K(\left\|x^{n-1}\right\|_s^2)ds+\frac{10M^2m\sum_{k=1}^m h_k}{1-Q_1}\left\{\mathbb{E}\left\|x^{n-1}\right\|_t^2\right\},$$

where $Q_1 = 10L_g$ and $Q_2 = \frac{10M^2(1+L_g)\mathbb{E}\|\varphi\|^2+10L_gk_0+10T(T+1)M^2k_0+10M^2m\sum_{k=1}^m k_0}{1-Q_1}$.
Since $K(\cdot)$ is concave and $K(0)=0$, we cand find positive constants α and β such that $K(u) \leq \alpha + \beta u$, for all $u \geq 0$, and applying mathematical induction, we get that

$$\begin{aligned}\mathbb{E}\left\|x^n(t)\right\|_t^2 \leq Q_3 + \frac{10M^2(T+1)\beta}{1-Q_1}\int_0^t \frac{(t-s)^{n-1}}{(n-1)!}\mathbb{E}\left\|x^0\right\|_s^2 ds \\ +\frac{10M^2m\sum_{k=1}^m h_k}{1-Q_1}\left\{\frac{T^{n-1}}{(n-1)!}\right\}\mathbb{E}\left\|x^0\right\|_t^2,\end{aligned} \tag{3.2}$$

where, $Q_3 = Q_2 + \frac{10M^2(T+1)T\alpha}{1-Q_1}$. We know that,

$$\mathbb{E}\left\|x^0(t)\right\|^2 \leq M^2\left\|\varphi(0)\right\| = Q_4 \leq \infty. \tag{3.3}$$

Thus $\sup_{t\in[0,T]}\mathbb{E}\left\|x^0\right\|_t^2 < \infty$, then from (3.2), $\sup_{t\in[0,T]}\mathbb{E}\left\|x^n\right\|_t^2 < \infty$ for all $n = 1, 2, \cdots$ and $t \in [0,T]$. This proove the boundedness of $\{x^n\}$.
We claim that $\{x^n\}$ is Cauchy sequence in $\mathcal{B}_T$. For this, choose $T_1 \in [0, T[$ such that

$$\frac{5M^2(T_1+1)}{1-Q_5}K\left(\frac{(t-s)^n}{n!}Q_9\right) \leq \frac{5M^2(T_1+1)}{1-Q_5}\frac{(t-s)^n}{n!}Q_9, \quad \text{for} \quad \text{all} \ \ 0 \leq t \leq T_1.$$

In the same way as above we have the following estimation

$$\begin{aligned}\mathbb{E}\left\|x^{n+1}(t) - x^n(t)\right\|^2 &\leq 4L_g\mathbb{E}\left\|x^{n+1} - x^n\right\|_t^2 \\ &+4M^2(T_1+1)\int_0^t K(\mathbb{E}\left\|x^n - x^{n-1}\right\|_s^2)ds \\ &+4M^2m\sum_{k=1}^m h_k\mathbb{E}\left\|x^n - x^{n-1}\right\|_t^2.\end{aligned}$$

Thus,

$$\begin{aligned}\mathbb{E}\left\|x^{n+1} - x^n\right\|_t^2 \leq \frac{4M^2(T_1+1)}{1-Q_5}\int_0^t K(\mathbb{E}\left\|x^n - x^{n-1}\right\|_s^2)ds \\ +\frac{4M^2m\sum_{k=1}^m h_k}{1-Q_5}\mathbb{E}\left\|x^n - x^{n-1}\right\|_t^2,\end{aligned} \tag{3.4}$$

where, $Q_5 = 4L_g$. Moreover,

$$\begin{aligned}\left\|x^1 - x^0\right\|^2 = &\left\|R(t)g(0,\varphi) - [g(t,x_t^1) - g(t,x_t^0)] - g(t,x_t^0) + \int_0^t R(t-s)a(s,x_s^0)ds\right. \\ &\left. + \int_0^t R(t-s)b(s,x_s^0)dw(s) + \sum_{0<t_k<t} R(t-t_k)I_k(x^0(t_k))\right\|^2.\end{aligned}$$

Then, we get that

$$\mathbb{E}\left\|x^1-x^0\right\|_t^2 \leq Q_6+\frac{12\left(L_g+M^2 m \sum_{k=1}^m h_k\right)}{1-Q_7}\mathbb{E}\left\|x^0\right\|_t^2 + \frac{12M^2(T_1+1)}{1-Q_7}\int_0^t K(\mathbb{E}\left\|x^0\right\|_s^2)ds,$$

where

$$Q_7=6L_g \text{ and } Q_6=\frac{12M^2T_1^2k_0+12T_1M^2k_0+12M^2m\sum_{k=1}^m k_0}{1-Q_7}.$$

Taking the supremum over t and from (3.3), we obtain that

$$\sup_{t\in[0,T_1]}\mathbb{E}\left\|x^1-x^0\right\|_t^2 \leq Q_8+\frac{12M^2(T_1+1)}{1-Q_7}\int_0^t K(Q_4)ds \leq Q_9, \tag{3.5}$$

where

$$Q_8=Q_6+\frac{12(L_g+M^2m\sum_{k=1}^m h_k}{1-Q_7}E\left\|x^0\right\|_t^2.$$

Applying mathematical induction in (3.4) and from (3.5), we obtain that

$$\begin{aligned}\sup_{t\in[0,T_1]}\mathbb{E}\left\|x^{n+1}-x^n\right\|_t^2 &\leq \frac{4M^2(T_1+1)}{1-Q_5}\int_0^t K\left(\frac{(t-s)^n}{n!}\sup_{t\in[0,T_1]}\mathbb{E}\left\|x^1-x^0\right\|_s^2\right)ds \\ &\quad+\frac{4M^2m\sum_{k=1}^m h_k}{1-Q_5}\left\{\frac{T_1^n}{n!}\right\}\sup_{t\in[0,T_1]}\mathbb{E}\left\|x^1-x^0\right\|_t^2 \\ &\leq \frac{4M^2(T_1+1)}{1-Q_5}\int_0^t K\left(\frac{(t-s)^n}{n!}Q_9\right)ds \\ &\quad+\frac{4M^2m\sum_{k=1}^m h_k}{1-Q_5}\left\{\frac{T_1^n}{n!}\right\}Q_9 \\ &\leq \frac{4M^2(T_1+1)}{1-Q_5}\int_0^t \frac{(t-s)^n}{n!}Q_9 ds \\ &\quad+\frac{4M^2m\sum_{k=1}^m h_k}{1-Q_5}\left\{\frac{T_1^n}{n!}\right\}Q_9 \\ &\leq Q_{10}\frac{T_1^n}{n!},\quad n\geq 0,\quad t\in[0,T_1].\end{aligned}$$

where

$$Q_{10}=\frac{4M^2Q_9}{1-Q_5}\left[\frac{T_1(T_1+1)}{n+1}+m\sum_{k=1}^m h_k\right].$$

Note that for any $m>n\geq 1$, we have,

$$\sup_{t\in[0,T_1]}\mathbb{E}\left\|x^m(t)-x^n(t)\right\|^2 \leq \sum_{r=n}^{+\infty}\sup_{t\in[0,T_1]}\mathbb{E}\left\|x^{r+1}-x^r\right\|_t^2.$$

$$\leq \sum_{r=n}^{+\infty}\left(Q_{10}\frac{T_1^r}{r!}\right)$$

$$\to 0 \text{ as } n\to\infty. \tag{3.6}$$

This shows that $(x^n)_{n\geq 0}$ is Cauchy in $\mathcal{B}_T$. Then the standard Borel-Cantelli lemma argument can be used to show that $x^n(t)\to x(t)$ uniformly in t on $[0,T_1]$. Hence $x(t)$ is a solution of Eq.(1.1) in the interval $[0,T_1]$. By using the same method as Theorem (6) in [14], the existence of solution of (1.1) on $[0,T]$ can be obtained by iteration. Now, we prove the uniqueness of the solution of Eq. (2.5). Let $x_1, x_2\in\mathcal{B}_T$ be two solutions to (1.1)on some interval $]-\infty,T]$. Then, for $t\in]-\infty,0]$, the uniqueness is obvious and for $0\leq t\leq T$, we have

$$\mathbb{E}\left\|x_1(t)-x_2(t)\right\|^2 \leq 5\left(L_g+M^2m\sum_{k=1}^{m}h_k\right)\mathbb{E}\left\|x_1-x_2\right\|_t^2$$
$$+4M^2(T+1)\int_0^t K(\mathbb{E}\left\|x_1-x_2\right\|_s^2)ds.$$

Thus,

$$E\left\|x_1-x_2\right\|_t^2 \leq \frac{4M^2(T+1)}{1-Q_{11}}\int_0^t K(\mathbb{E}\left\|x_1-x_2\right\|_s^2)ds,$$

where,

$$Q_{11}=4\left(L_g+M^2m\sum_{k=1}^{m}h_k\right).$$

Thus, Bihari's inequality yields that

$$\sup_{t\in[0,T]}\mathbb{E}\left\|x_1-x_2\right\|_t^2=0,\quad 0\leq t\leq T.$$

Thus $x_1(t)=x_2(t)$, for all $0\leq t\leq T$. Therefore, for all $-\infty<t\leq T$, $x_1(t)=x_2(t)$ a.s. This achieve the proof of Theorem 3.1.

4. Stability in the mean square

In this section, we mean in this stability that small changes in the initial conditions lead to small changes in the solutions over a given finite time interval.

Definition 4.1. A mild solution $x(t)$ of the Eq. (1.1) with initial value ϕ is said to be stable in the mean square if for all $\epsilon>0$, there exists $\delta>0$ such that

$$\mathbb{E}\left\|x(t)-\widehat{x}(t)\right\|_t^2\leq\epsilon,\text{ when } \mathbb{E}\left\|\phi-\widehat{\phi}\right\|_t^2\leq\delta,\text{ for all } t\in[0,T]. \tag{4.1}$$

where $\widehat{x}(t)$ is another mild solution of the system (1.1) with initial $\widehat{\phi}$.

Theorem 4.2. *Let $x(t)$ and $y(t)$ the mild solution of the equation Eq.(1.1) with initial values φ_1 and φ_2 respectively. If the assumption of the theorem 3.1 are satisfied, then the mild solution of the Eq.(1.1) is stable in the mean square.*

Proof: Let, $x(t)$ and $y(t)$ be two mild solutions of equation (1.1) with initial values φ_1 and φ_2 respectively. Then for $0 \leq t \leq T$

$$x(t) - y(t) = R(t)\left([\varphi_1(0) - \varphi_2(0)] + [g(0,\varphi_1) - g(0,\varphi_2)]\right) - [g(t,x_t) - g(t,y_t)]$$
$$+ \int_0^t R(t-s)[a(s,x_s) - a(s,y_s)]ds + \int_0^t R(t-s)[b(s,x_s) - b(s,y_s)]dw(s)$$
$$+ \sum_{0<t_k<t} R(t-s)[I_k(x(t_k)) - I_k(y(t_k))].$$

Eestimating as before, we get that

$$\mathbb{E}\left\|x(t) - y(t)\right\|^2 \leq 6M^2(1+L_g)\mathbb{E}\left\|\varphi_1 - \varphi_2\right\|^2$$
$$+6\left(L_g + M^2 m\sum_{k=1}^{m} h_k\right)\mathbb{E}\left\|x - y\right\|_t^2$$
$$+6M^2(T+1)\int_0^t K(\mathbb{E}\left\|x-y\right\|_s^2)ds.$$

Thus

$$\mathbb{E}\left\|x - y\right\|_t^2 \leq \frac{6M^2(1+L_g)}{1-Q_{12}}\mathbb{E}\left\|\varphi_1 - \varphi_2\right\|^2$$
$$+\frac{7M^2(T+1)}{1-Q_{12}}\int_0^t K(\mathbb{E}\left\|x-y\right\|_s^2)ds,$$

where, $\quad Q_{12} = 6\left(L_g + M^2 m\sum_{k=1}^{m} h_k\right).$
Let $K_1(u) = \frac{6M^2(T+1)}{1-Q_{10}}K(u)$, for K is a concave increasing function from $\mathbb{R}^+$ to $\mathbb{R}^+$ such that $K(0) = 0$, $K(u) > 0$ for $u > 0$ and $\int_{0^+} \frac{du}{K(u)} = +\infty$. Then, $K_1(u)$ is obviously concave function from $\mathbb{R}^+$ to $\mathbb{R}^+$ such that $K_1(0) = 0$, $K_1(u) \geq K(u)$ for $0 \leq u \leq 1$ and $\int_{0^+} \frac{du}{K_1(u)} = +\infty$. For any $\epsilon > 0$, $\epsilon_1 = \frac{1}{2}\epsilon$, we have $\lim_{s\to 0}\int_s^{\epsilon_1} \frac{du}{K_1(u)} = \infty$. Then, there is positive constant $\delta < \epsilon_1$ such that $\int_\delta^{\epsilon_1} \frac{du}{K_1(u)} \geq T$.
From Corollary 2.12, let

$$u_0 = \frac{6M^2(1+L_g)}{1-Q_{12}}E\left\|\varphi_1 - \varphi_2\right\|^2$$
$$u(t) = E\left\|x-y\right\|_t^2,\ v(t) = 1$$

when $u_0 \leq \delta \leq \epsilon_1$, we have

$$\int_{u_0}^{\epsilon_1} \frac{du}{K_1(u)} \geq \int_{\delta}^{\epsilon_1} \frac{du}{K_1(u)} \geq T = \int_0^T v(s)ds.$$

Then for any $t \in [0,T]$, the estimate $u(t) \geq \epsilon_1$ holds. This completes the proof.

Remark 4.3. If $m = 0$ in (1.1), then the system behave as neutral stochastic partial integrodifferential equation with infinte delay of the form

$$\begin{cases} d[x(t)+g(t,x_t)] = A[x(t)+g(t,x_t)]dt + \left[\int_0^t F(t-s)x(s)ds + a(s,x_s)\right] dt \\ +b(t,x_t)dw(t), \quad 0 \le t \le T, \\ x_0 = \varphi \in C^b_{B_0}(]-\infty,0],\mathbb{H}). \end{cases} \tag{4.2}$$

By applying Theorem 3.1 under the hypotheses **(H1) - (H6)**, the system (4.2) guarantees the existence and uniqueness of the mild solution.

5. Application

In this section, we propose to study the following model that is given by a stochastic partial differential equation involving infinite delay

$$\begin{cases} \frac{\partial}{\partial t}[u(t,\xi)+g(t,u(t-\tau,\xi))] = \frac{\partial^2}{\partial \xi^2}[u(t,\xi)+g(t,u(t-\tau,\xi))] \\ \qquad + \int_0^t f(t-s)\frac{\partial^2}{\partial \xi^2}[u(s,\xi)+g(s,u(s-\tau,\xi))]\,ds \\ \qquad +a(t,u(t-\tau,\xi))dt + b(t,u(t-\tau,\xi))dw(t) \text{ for } t \neq t_k \text{ and } t \ge 0 \\ \Delta u(t_k) = u(t_k^+) - x(t_k^-) = I_k(u(t_k)) \text{ for } t = t_k, \quad k = 1,2,\cdots, \\ u(t,0)+g(t,u(t-\tau,0)) = 0 \quad \text{for} \quad t \ge 0, \\ u(t,\pi)+g(t,u(t-\tau,\pi)) = 0 \quad \text{for } t \ge 0, \\ u(\theta,\xi) = u_0(\theta,\xi) \quad \text{for} \quad \theta \in]-\infty,0] \quad \text{and} \quad \xi \in [0,\pi], \end{cases} \tag{5.1}$$

where $w(t)$ denotes a $\mathbb{R}$-valued Brownian motion.
The functions $f : \mathbb{R}^+ \to \mathbb{R}, g, a, b : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ $u_0 :]-\infty,0] \times [0,\pi] \to \mathbb{R}$ are continuous and $u_0(\cdot) \in L^2(0,\pi)$, is $\mathcal{F}_0$-measurable and satisfies $E\,\|\beta_0\|^2 < \infty$.
Let $\mathbb{H} = L^2(0,\pi)$ with the norm $\|.\|$ and
$e_n := \sqrt{\frac{2}{\pi}}\sin(nx)$, $(n = 1,2,3,\cdots)$ denote the completed orthonormal basis in $\mathbb{H}$. Let $w(t) := \sum_{n=1}^{\infty}\sqrt{\lambda_n}\beta_n(t)e_n(\lambda_n > 0)$, where $\beta_n(t)$ are one dimensional standard Brownian motion mutually independent on a usual complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge 0}, \mathbb{P})$.
Define $A : \mathbb{H} \to \mathbb{H}$ by $A = \frac{\partial^2}{\partial z^2}$, with domain $D(A) = H^2(0,\pi) \cap H^1_0(0,\pi)$.
Then $Ah = -\sum_{n=1}^{\infty} n^2 < h, e_n > e_n$, $h \in D(A)$, where e_n, $n = 1,2,3,\cdots$, is also the orthonormal set of eigenvectors of A. It is well-known that A is the

infinitesimal generator of a strongly continuous semigroup on $\mathbb{H}$, thus $(\mathbf{H1})$ is true .
Let $F : D(A) \subset \mathbb{H} \to \mathbb{H}$ be the operator defined by
$F(t)(z) = f(t)Az$ for $t \geq 0$ and $z \in D(A)$.
Let $\gamma > 0$, define the phase space
$\mathcal{B} = \left\{\varphi \in C((-\infty, 0], \mathbb{H}) \ : \ \lim_{\theta\to-\infty} \ e^{\theta\gamma}\varphi(\theta) \text{ exists in } \mathbb{H}\right\}$
and let $\|\varphi\|_{\mathcal{B}} = \sup_{\theta\in(-\infty,0]} \left\{e^{\gamma\theta}\|\varphi\|_{L_2}\right\}$. Then $(\mathcal{B}, \| \ .\|_{\mathcal{B}})$ is a Banach space and satisfies (A1)-(A2) with $L = 1, u(t) = e^{-\gamma t}, \ v(t) = \max\left\{1, e^{-\gamma t}\right\}$. Therefore, for $(t, \varphi) \ \in J \times \mathcal{B}$, where $\varphi(\theta)(\xi) = \varphi(\theta, \xi), \ \ (\theta, \xi) \ \in (-\infty, 0] \times [0, \pi]$, let
$x(t)(\xi) = u(t, \xi)$ and define the functions $g, a \ : \ J \times \mathcal{B} \ \to \mathbb{H}$ and
$b \ : \ J \times \mathcal{B} \ \to \mathcal{L}_2^0(\mathbb{H}, \mathbb{H})$ for the infinite delay as follows:
$g(t, \phi)(\xi) = \int_{-\infty}^0 k_1(\theta)\phi(\theta)(\xi)d\theta,$
$a(t, \phi)(\xi) = \int_{-\infty}^0 k_2(t, \xi, \theta)G_1(\varphi(\theta))d\theta,$
$b(t, \phi)(\xi) = \int_{-\infty}^0 k_3(t, \xi, \theta)G_2(\varphi(\theta))d\theta,$
where

(i) the functions $k_1(\theta) \geq 0$ is continuous in $(-\infty, 0]$ and satisfies
$\int_{-\infty}^0 k_1^2(\theta)d\theta < \infty, \quad L_g = \left(-\frac{1}{2\gamma}\int_{-\infty}^0 k_1^2(\theta)d\theta\right);$

(ii) the functions $k_2, \ k_3$ are continuous in $J \times [0, \pi] \times (-\infty, 0]$ and satisfy
$\int_{-\infty}^0 k_2^2(t, \xi, \theta)d\theta = p_2(t, \xi) < \infty, \quad \left(\int_0^\pi p_2^2(t, \xi)d\xi\right) < 1,$
$\int_{-\infty}^0 k_3^2(t, \xi, \theta)d\theta = p_3(t, \xi) < \infty, \quad \left(\int_0^\pi p_3^2(t, \xi)d\xi\right) < 1,$

(iii) the functions $G_i, \ i = 1, 2$ is continuous in $J \times [0, \pi] \times (-\infty, 0)$ and satisfies
$0 \leq G_1(x_1(\theta, \xi)) - G_1(x_2(\theta, \xi)) \leq K_a^{\frac{1}{2}}(\|x_1(\theta, .) - x_2(\theta, .)\|_{L^2}^2),$
$0 \leq G_2(x_1(\theta, \xi)) - G_2(x_2(\theta, \xi)) \leq K_b^{\frac{1}{2}}(\|x_1(\theta, .) - x_2(\theta, .)\|_{L^2}^2),$
for $(\theta, \xi) \in (-\infty, 0] \times [0, \pi]$, where
$K_a(.), K_b(.), K_a^{\frac{1}{2}}(.), K_b^{\frac{1}{2}}(.) \ : \ [0, \infty[\ \to (0, \infty)$ are nondecreasing and concave .

Under the above assumptions, we can rewritten Eq.(5.1) as the abstract form of Eq.(1.1).

$$
\begin{cases}
d\left[x(t) + g(t, x_t)\right] = \left[A[x(t) + g(t, x_t)] + \displaystyle\int_0^t F(t-s)[x(t) + g(t, x_t)]ds\right] dt \\
+a(t, x_t)dt + b(t, x_t)dw(t), \quad t \in J \ := \ [0, T], \\
\Delta x(t_k) = x(t_k^+) - x(t_k^-) = I_k(x(t_k)), \quad k = 1, 2, \cdots, m, \\
\qquad x_0 = \varphi \in \mathcal{B}.
\end{cases}
\tag{5.2}
$$

Moreover, if f is bounded and C^1 function such that f' is bounded and uniformly continuous, then **(H2)** is satisfied and hence, by Theorem 2.2, Eq. (5.2) has a resolvent operator $(R(t))_{t\geq 0}$ on $\mathbb{H}$.
By assumptions (i),(ii) and (iii) we have

$$
\begin{aligned}
\|g(t,\phi_1)-g(t,\phi_2)\|^2_{\mathbb{H}} &= \int_0^\pi \left(\int_{-\infty}^0 k_1(\theta)\left(\phi_1(\theta)(\xi)-\phi_2(\theta)(\xi)\right)d\theta\right)^2 d\xi \\
&\leq \left(\int_{-\infty}^0 k_1^2(\theta)d\theta\right)\left(\int_0^\pi\int_{-\infty}^0 (\phi_1(\theta)(\xi)-\phi_2(\theta)(\xi))^2 d\theta d\xi\right) \\
&\leq \left(-\frac{1}{2\gamma}\int_{-\infty}^0 k_1^2(\theta)d\theta\right)\sup_{\theta\in]-\infty,0]}\{e^{2\gamma\theta}\|\phi\|^2_{L^2}\} = L_g(\|\phi_1-\phi_2\|^2_{\mathcal{B}}).
\end{aligned}
$$

$$
\begin{aligned}
\|a(t,\phi_1)-a(t,\phi_2)\|^2_{\mathbb{H}} &= \int_0^\pi \left(\int_{-\infty}^0 k_2(t,\xi,\theta)\left(G_1(\phi_1(\theta))-G_1(\phi_2(\theta))\right)d\theta\right)^2 d\xi \\
&\leq \int_0^\pi \left(\int_{-\infty}^0 k_2(t,\xi,\theta)K_a^{\frac{1}{2}}(\|\phi_1(\theta,.)-\phi_2(\theta,.)\|^2_{L^2})d\theta\right)^2 d\xi \\
&\leq \int_0^\pi \left(\int_{-\infty}^0 k_2(t,\xi,\theta)K_a^{\frac{1}{2}}(e^{2\alpha\theta}\|\phi_1(\theta,.)-\phi_2(\theta,.)\|^2_{L^2})d\theta\right)^2 d\xi \\
&\leq \left(\int_0^\pi p_2^2(t,\xi)d\xi\right)K_a(\|\phi_1-\phi_2\|^2_{\mathcal{B}}).
\end{aligned}
$$

In the same way we obtain the following estimation

$$\|b(t,\phi_1)-b(t,\phi_2)\|^2_{\mathbb{H}} \leq K_b(\|\phi_1-\phi_2\|^2_{\mathcal{B}}).$$

The next results as consequence of Theorem 3.1 and Theorem 4.2 respectively.

Proposition 5.1. *Assume that the hypothesis **(H1)-(H5)** hold. Then there exists a mild solution u of the system (5.1) provided*

$$\tilde{Q} = \max\{Q_1, Q_5\} < 1.$$

is satisfied.

Proposition 5.2. *Assume that the conditions of Proposition 5.1 hold. Then the mild solution u of the Eq.(5.2) is stable in the quadratic mean.*

References

1. J. Bao, Z. Hou, *Existence of mild solutions to stochastic neutral partial functional differential equations with non-Lipschitz coefficients*, Computers and Mathematics with Applications **59** (2010), 207-214.
2. I. Bihari, *A generalization of a lemma of Bellman and its application to uniqueness problem of differential equations*, Acta Math. Acad. Sci. Hungar **7** (1956), 71-94.

3. G. Cao, K. He and X. Zhang, *Successive approximations of infinite dimensional SDES with jump*, Stoch. Syst. **5** (2005), 609-619.
4. G. Da Prato and J. Zabczyk, Stochastic Equations in Infinite Dimensions, Cambridge University, Press Cambridge 1992.
5. M. A. Diop, K. Ezzinbi and M. Lo, *Existence and uniqueness of mild solutions solutions to neutral stochastic partial functional integro-differential equations with non-Lipschitz coefficiennts*, International Journal of Mathematics and Mathematical Sciences, (2012), Article ID748590, 12 pages, doi :10.1155/2012/748590.
6. M. A. Diop, K. Ezzinbi and M. Lo, *Mild solutions to neutral stochastic partial functional integro-differential equations with non-Lipschitz coefficients*, Afrika Mathematica,(2014), doi :10.1007/s13370-012-0089-3.
7. T.E. Govindan, *Almost sure exponential stability for stochastic neutral partial functional differential equations*, Stochastics **77** (2005), 139154.
8. R. Grimmer, *Resolvent Operators for Integral Equations in a Banach space*, Transactions of the American Mathematical Society **273(1)** (1982), 333-349.
9. J.K. Hale and J. Kato, *Phase spaces for retarded equations with infinite delay*, Funkcial. Ekvac. **21** (1978), 11-41.
10. Y. Hino, S. Murakami and T. Naito, Functional differential equations with Infinite Delay, in: Lecture Notes in Mathematics, Springer-Verlag, Berlin, Vol. 1473 (1991).
11. F. Jiang, Y.Shen, *A note on the existence and uniqueness of mild solutions to neutral stochastic partial functional differential equations with non-Lipschitz coefficients*, Computers and Mathematics with Applications **61** (2011), 15901594.
12. A. Lin , Y. Ren and N.Xia, *On neutral impulsive stochastic integro-differential equations with infinite delays via fractional operators*, Mathematical and Computer Modelling **51** (2010), 413-424.
13. J. Pruss, Evolutionary Integral Equations and Applications, Birkhuser (1993).
14. Y. Ren and N. Xia, *Existence, uniqueness and stability of the solutions to neutral stochastic functional differential equations with infinite delay*, Applied Mathematics and Computations **210** (2009), 72-79.
15. R. Sakthivel and J. Luo, *Asymptotic Stability of Nonlinear Impulsive Stochastic Differential equations*, Statistics and probability Letters **79** (2009), 1219-1223.
16. R. Sakthivel and J. Luo, *Asymptotic stability of impulsive stochastic partial differential equations with infinite delays*, J. Math. Anal. Appl. **356** (2009), 1-6.
17. B. Schmalfuss, *Attractors for autonomous and random dynamical systems perturbed by impulses*, Discrtete and continuous Dynamical Systems **9(3)** (2003), 727-744.
18. T. Taniguchi, *Successive approximations of solutions of stochastic differential equations*, J.Differential Equations **96** (1992), 152-169.
19. J. Turo, *Successive approximations of solutions to stochastic functional differential equations*, Period. Math.Hungar **30** (1995), 87-96.
20. Z. Yang, D. Xu and L. Xiang, *Exponential p-stability of impulsive stochastic differential equations with delays*, Physics Letter A, **356** (2006), 129-137.
21. J. Yang, S. Zhong and W. Luo, *Mean square stability analysis of impulsive stochastic differential equations with delays*, J. Comput. Appl. Math. **216** (2008), 474-483.

[1] Département de Mathématiques, Université Gaston Berger de Saint-Louis, UFR SAT, B.P 234, Saint-Louis, Sénégal.
E-mail address: mamadou-abdoul.diop@ugb.edu.sn

[2] Département de Mathématiques, Université Cadi Ayyad, Faculté des Sciences Semlalia, , B.P 2390 Marrakech, Morocco
E-mail address: ezzinbi@uca.ma

[3] Département de Mathématiques, Université Gaston Berger de Saint-Louis, UFR SAT, B.P 234, Saint-Louis, Sénégal.
E-mail address: mht.mhtzene@gmail.com