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DOMAIN OF DIFFERENCE MATRIX OF ORDER ONE IN SOME SPACES OF DOUBLE SEQUENCES

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ABSTRACT. In this study, we define the spaces $\mathcal{M}_u(\Delta), \mathcal{C}_p(\Delta), \mathcal{C}_{0p}(\Delta), \mathcal{C}_r(\Delta)$ and $\mathcal{L}_q(\Delta)$ of double sequences whose difference transforms are bounded , convergent in the Pringsheim's sense, null in the Pringsheim's sense, both convergent in the Pringsheim's sense and bounded, regularly convergent and absolutely $q-$summable, respectively, and also examine some inclusion relations related to those sequence spaces. Furthermore, we show that these sequence spaces are Banach spaces . We determine the alpha-dual of the space $\mathcal{M}_u(\Delta)$ and the $\beta(v)-$dual of the space $\mathcal{C}_\eta(\Delta)$ of double sequences, where $v, \eta \in \{p, bp, r\}$. Finally, we characterize the classes $(\mu : \mathcal{C}_v(\Delta))$ for $v \in \{p, bp, r\}$ of four dimensional matrix transformations, where μ is any given space of double sequences.

1. Introduction and preliminaries

By ω and Ω, we denote the sets of all real valued single and double sequences which are the vector spaces with coordinatewise addition and scalar multiplication. Any vector subspaces of ω and Ω are called as the *single sequence space* and *double sequence space*, respectively. By $\mathcal{M}_u$, we denote the space of all bounded double sequences, that is

$$\mathcal{M}_u := \left\{ x = (x_{mn}) \in \Omega : \|x\|_\infty = \sup_{m,n\in\mathbb{N}} |x_{mn}| < \infty \right\}$$

which is a Banach space with the norm $\|x\|_\infty$; where $\mathbb{N}$ denotes the set of all positive integers. Consider a sequence $x = (x_{mn}) \in \Omega$. If for every $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$ and $l \in \mathbb{R}$ such that $|x_{mn} - l| < \varepsilon$ for all $m, n > n_0$ then we call that the double sequence x is *convergent* in the *Pringsheim's sense* to the limit l and write $p - \lim x_{mn} = l$; where $\mathbb{R}$ denotes the real field. By $\mathcal{C}_p$, we denote the space of all convergent double sequences in the Pringsheim's sense. It is well-known that there are such sequences in the space $\mathcal{C}_p$ but not in the space $\mathcal{M}_u$. Indeed following Boos [5], if we define the sequence $x = (x_{mn})$ by

$$x_{mn} := \begin{cases} n & , \quad m = 1, n \in \mathbb{N}, \\ 0 & , \quad m \geq 2, n \in \mathbb{N}, \end{cases}$$

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then it is trivial that $x \in \mathcal{C}_p \backslash \mathcal{M}_u$, since $p - \lim x_{mn} = 0$ but $\|x\|_\infty = \infty$. So, we can consider the space $\mathcal{C}_{bp}$ of the double sequences which are both convergent in the Pringsheim's sense and bounded, i.e., $\mathcal{C}_{bp} = \mathcal{C}_p \cap \mathcal{M}_u$. A sequence in the space $\mathcal{C}_p$ is said to be *regularly convergent* if it is a single convergent sequence with respect to each index and denote the set of all such sequences by $\mathcal{C}_r$. Also by $\mathcal{C}_{bp0}$ and $\mathcal{C}_{r0}$, we denote the spaces of all double sequences converging to 0 contained in the sequence spaces $\mathcal{C}_{bp}$ and $\mathcal{C}_r$, respectively. Móricz [9] proved that $\mathcal{C}_{bp}, \mathcal{C}_{bp0}, \mathcal{C}_r$ and $\mathcal{C}_{r0}$ are Banach spaces with the norm $\|.\|_\infty$.

Let us consider the isomorphism T defined by Zeltser [14] as

$$\begin{aligned} T \quad : \quad & \Omega \to \omega \\ & x \mapsto z = (z_i) := (x_{\varphi^{-1}(i)}), \end{aligned} \tag{1.1}$$

where $\varphi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is a bijection defined by

$$\begin{aligned} \varphi[(1,1)] &= 1, \\ \varphi[(1,2)] &= 2, \varphi[(2,2)] = 3, \varphi[(2,1)] = 4, \\ &. \\ &. \\ &. \\ \varphi[(1,n)] &= (n-1)^2 + 1, \varphi[(2,n)] = (n-1)^2 + 2, ..., \\ \varphi[(n,n)] &= (n-1)^2 + n, \varphi[(n,n-1)] = n^2 - n + 2, ..., \varphi[(n,1)] = n^2, \\ &. \\ &. \\ &. \end{aligned}$$

Let us consider a double sequence $x = (x_{mn})$ and define the sequence $s = (s_{mn})$ which will be used throughout via x by

$$s_{mn} := \sum_{i=0}^{m} \sum_{j=0}^{n} x_{ij} \tag{1.2}$$

for all $m, n \in \mathbb{N}$. For the sake of brevity, here and in what follows, we abbreviate the summation $\sum_{i=0}^{\infty} \sum_{j=0}^{\infty}$ by $\sum_{i,j}$ and we use this abbreviation with other letters. Let λ be a space of a double sequences, converging with respect to some linear convergence rule $v - \lim : \lambda \to \mathbb{R}$. The sum of a double series $\sum_{i,j} x_{ij}$ with respect to this rule is defined by $v - \sum_{i,j} x_{ij} = v - \lim_{m,n \to \infty} s_{mn}$. Let λ, μ be two spaces of double sequences, converging with respect to the linear convergence rules $v_1 - \lim$ and $v_2 - \lim$, respectively, and $A = (a_{mnkl})$ also be a four dimensional infinite matrix over the real or complex field.

The $\alpha-dual$ λ^α, $\beta(v)-dual$ $\lambda^{\beta(v)}$ with respect to the $v-$convergence for $v \in \{p, bp, r\}$ and the $\gamma-dual$ λ^γ of a double sequence space λ are respectively defined by

$$\begin{aligned}
\lambda^{\alpha} &:= \left\{(a_{ij}) \in \Omega : \sum_{i,j} |a_{ij}x_{ij}| < \infty \quad \text{for all} \quad (x_{ij}) \in \lambda\right\}, \\
\lambda^{\beta(v)} &:= \left\{(a_{ij}) \in \Omega : v - \sum_{i,j} a_{ij}x_{ij} \quad \text{exists for all} \quad (x_{ij}) \in \lambda\right\}, \\
\lambda^{\gamma} &:= \left\{(a_{ij}) \in \Omega : \sup_{k,l\in\mathbb{N}} \left|\sum_{i,j=1}^{k,l} a_{ij}x_{ij}\right| < \infty \quad \text{for all} \quad (x_{ij}) \in \lambda\right\}.
\end{aligned}$$

It is easy to see for any two spaces λ, μ of double sequences that $\mu^{\alpha} \subset \lambda^{\alpha}$ whenever $\lambda \subset \mu$ and $\lambda^{\alpha} \subset \lambda^{\gamma}$. Additionally, it is known that the inclusion $\lambda^{\alpha} \subset \lambda^{\beta(v)}$ holds while the inclusion $\lambda^{\beta(v)} \subset \lambda^{\gamma}$ does not hold, since the $v-$convergence of the sequence of partial sums of a double series does not imply its boundedness.

The *$v-$summability domain* $\lambda_A^{(v)}$ of a four dimensional infinite matrix $A = (a_{mnkl})$ in a space λ of a double sequences is defined by

$$\lambda_A^{(v)} = \left\{x = (x_{kl}) \in \Omega : Ax = \left(v - \sum_{k,l} a_{mnkl}x_{kl}\right)_{m,n\in\mathbb{N}} \text{exists and is in} \;\; \lambda\right\}. \tag{1.3}$$

We say, with the notataion (1.3), that A maps the space λ into the space μ if and only if Ax exists and is in μ for all $x \in \lambda$ and denote the set of all four dimensional matrices, transforming the space λ into the space μ, by $(\lambda : \mu)$. It is trivial that for any matrix $A \in (\lambda : \mu)$, $(a_{mnkl})_{k,l\in\mathbb{N}}$ is in the $\beta(v)-$dual $\lambda^{\beta(v)}$ of the space λ for all $m, n \in \mathbb{N}$. An infinite matrix A is said to be *$\mathcal{C}_v-$conservative* if $\mathcal{C}_v \subset (\mathcal{C}_v)_A$. Also by $(\lambda : \mu; p)$, we denote the class of all four dimensional matrices $A = (a_{mnkl})$ in the class $(\lambda : \mu)$ such that $v_2 - \lim Ax = v_1 - \lim x$ for all $x \in \lambda$.

Now, following Zeltser [15] we note the terminology for double sequence spaces. A locally convex double sequence space λ is called a *$DK-$space*, if all of the seminorms $r_{kl} : \lambda \to \mathbb{R}$, $x = (x_{kl}) \mapsto |x_{kl}|$ for all $k, l \in \mathbb{N}$ are continuous. A $DK-$space with a Fréchet topology is called an *$FDK-$space*. A normed $FDK-$space is called a *$BDK-$space*. We record that C_r endowed with the norm $\|.\|_{\infty} : \mathcal{C}_r \to \mathbb{R}$, $x = (x_{kl}) \mapsto \sup_{k,l\in\mathbb{N}} |x_{kl}|$ is a $BDK-$space.

Let us define the following sets of double sequences:

$$\begin{aligned}
\mathcal{M}_u(t) &:= \Big\{(x_{mn}) \in \Omega : \sup_{m,n\in\mathbb{N}} |x_{mn}|^{t_{mn}} < \infty\Big\},\\
\mathcal{C}_p(t) &:= \Big\{(x_{mn}) \in \Omega : \exists l \in \mathbb{C} \ni p - \lim_{m,n\to\infty} |x_{mn} - l|^{t_{mn}} = 0\Big\},\\
\mathcal{C}_{0p}(t) &:= \Big\{(x_{mn}) \in \Omega : p - \lim_{m,n\to\infty} |x_{mn}|^{t_{mn}} = 0\Big\},\\
\mathcal{L}_u(t) &:= \Big\{(x_{mn}) \in \Omega : \sum_{m,n} |x_{mn}|^{t_{mn}} < \infty\Big\},\\
\mathcal{C}_{bp}(t) &:= \mathcal{C}_p(t) \cap \mathcal{M}_u(t) \quad \text{and} \quad \mathcal{C}_{0bp}(t) := \mathcal{C}_{0p}(t) \cap \mathcal{M}_u(t);
\end{aligned}$$

where $t = (t_{mn})$ is the sequence of strictly positive reals t_{mn} for all $m, n \in \mathbb{N}$. In the case $t_{mn} = 1$ for all $m, n \in \mathbb{N}$; $\mathcal{M}_u(t), \mathcal{C}_p(t), \mathcal{C}_{0p}(t), \mathcal{L}_u(t), \mathcal{C}_{bp}(t)$ and $\mathcal{C}_{0bp}(t)$ reduce to the sets $\mathcal{M}_u, \mathcal{C}_p, \mathcal{C}_{0p}, \mathcal{L}_u, \mathcal{C}_{bp}$ and $\mathcal{C}_{0bp}$, respectively. Now, we can summarize the knowledge given in some document related to the double sequence spaces. Gökhan and Çolak [6, 7] have proved that $\mathcal{M}_u(t), \mathcal{C}_p(t)$ and $\mathcal{C}_{bp}(t)$ are complete paranormed spaces of double sequences and gave the alpha-, beta-, gamma-duals of the spaces $\mathcal{M}_u(t)$ and $\mathcal{C}_{bp}(t)$. Quite recently, in her PhD thesis, Zeltser [14] has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [10] have introduced the statistical convergence and statistical Cauchy for double sequences, and gave the relation between statistically convergent and strongly Cesàro summable double sequences. Nextly, Mursaleen [11] and Mursaleen and Edely [12] have defined the almost strong regularity of matrices for double sequences and applied these matrices to establish a core theorem and introduced the $M-$core for double sequences and determined those four dimensional matrices transforming every bounded double sequence $x = (x_{jk})$ into one whose core is a subset of the $M-$core of x. More recently, Altay and Başar [1] have defined the spaces $\mathcal{BS}, \mathcal{BS}(t), \mathcal{CS}_{bp}, \mathcal{CS}_r$ and $\mathcal{BV}$ of double series whose sequence of partial sums are in the spaces $\mathcal{M}_u, \mathcal{M}_u(t), \mathcal{C}_p, \mathcal{C}_{bp}, \mathcal{C}_r$ and $\mathcal{L}_u$, respectively, and also examined some properties of those sequence spaces and determined the alpha-duals of the spaces $\mathcal{BS}, \mathcal{BV}, \mathcal{CS}_{bp}$ and the $\beta(v)-$duals of the spaces $\mathcal{CS}_{bp}$ and $\mathcal{CS}_r$ of double series. Quite recently, Başar and Sever [4] have introduced the Banach space $\mathcal{L}_q$ of double sequences corresponding to the well-known space ℓ_q of absolutely $q-$summable single sequences and examine some properties of the space $\mathcal{L}_q$. Furthermore, they determine the $\beta(v)-$dual of the space and establish that the alpha- and gamma-duals of the space $\mathcal{L}_q$ coincide with the $\beta(v)-$dual; where

$$\begin{aligned}
\mathcal{L}_q &:= \Big\{(x_{ij}) \in \Omega : \sum_{i,j} |x_{ij}|^q < \infty\Big\}, \quad (1 \le q < \infty),\\
\mathcal{CS}_v &:= \big\{(x_{ij}) \in \Omega : (s_{mn}) \in \mathcal{C}_v\big\}.
\end{aligned}$$

Here and after we assume that $v \in \{p, bp, r\}$.

The double difference matrix $\Delta = (\delta_{mnkl})$ of order one is defined by

$$\delta_{mnkl} := \begin{cases} (-1)^{m+n-k-l} & , \quad m-1 \leq k \leq m, \quad n-1 \leq l \leq n, \\ 0 & , \quad \text{otherwise} \end{cases}$$

for all $m, n, k, l \in \mathbb{N}$. Define the sequence $y = (y_{mn})$ as the $\Delta-$transform of a sequence $x = (x_{mn})$, i.e.,

$$y_{mn} = (\Delta x)_{mn} = x_{mn} - x_{m,n-1} - x_{m-1,n} + x_{m-1,n-1} \tag{1.4}$$

for all $m, n \in \mathbb{N}$. Additionally, a direct calculation gives the inverse $\Delta^{-1} = S = (s_{mnkl})$ of the matrix Δ as follows:

$$s_{mnkl} := \begin{cases} 1 & , \quad 0 \leq k \leq m, \quad 0 \leq l \leq n, \\ 0 & , \quad \text{otherwise} \end{cases}$$

for all $m, n, k, l \in \mathbb{N}$.

In the present paper, we introduce the new double difference sequence spaces $\mathcal{M}_u(\Delta), \mathcal{C}_p(\Delta), \mathcal{C}_{0p}(\Delta)$ and $\mathcal{L}_q(\Delta)$, that is,

$$\begin{aligned} \mathcal{M}_u(\Delta) &:= \Big\{(x_{mn}) \in \Omega : \sup_{m,n \in \mathbb{N}} |y_{mn}| < \infty\Big\}, \\ \mathcal{C}_p(\Delta) &:= \Big\{(x_{mn}) \in \Omega : \exists l \in \mathbb{C} \ni p - \lim_{m,n \to \infty} |y_{mn} - l| = 0\Big\}, \\ \mathcal{C}_{0p}(\Delta) &:= \Big\{(x_{mn}) \in \Omega : p - \lim_{m,n \to \infty} |y_{mn}| = 0\Big\}, \\ \mathcal{L}_q(\Delta) &:= \left\{(x_{ij}) \in \Omega : \sum_{m,n} |y_{mn}|^q < \infty\right\}, \quad (1 \leq q < \infty). \end{aligned}$$

By $\mathcal{C}_{bp}(\Delta)$ and $\mathcal{C}_r(\Delta)$, we denote the sets of all the $\Delta-$transforms convergent and bounded, and the $\Delta-$transforms regularly convergent double sequences. One can easily see that the spaces $\mathcal{M}_u(\Delta), \mathcal{C}_p(\Delta), \mathcal{C}_{0p}(\Delta), \mathcal{C}_{bp}(\Delta), \mathcal{C}_r(\Delta)$ and $\mathcal{L}_q(\Delta)$ are the domain of the double difference matrix Δ in the spaces $\mathcal{M}_u, \mathcal{C}_p, \mathcal{C}_{0p}, \mathcal{C}_{bp}$, $\mathcal{C}_r$ and $\mathcal{L}_q$, respectively.

2. Some new double difference sequence spaces

In the present section, we deal with the sets $\mathcal{M}_u(\Delta), \mathcal{C}_p(\Delta), \mathcal{C}_{0p}(\Delta), \mathcal{C}_{bp}(\Delta), \mathcal{C}_r(\Delta)$ and $\mathcal{L}_q(\Delta)$ consisting of the double sequences whose $\Delta-$transforms of order one are in the spaces $\mathcal{M}_u, \mathcal{C}_p, \mathcal{C}_{0p}, \mathcal{C}_{bp}$, $\mathcal{C}_r$ and $\mathcal{L}_q$, respectively.

Theorem 2.1. *The sets $\mathcal{M}_u(\Delta), \mathcal{C}_p(\Delta)$, $\mathcal{C}_{0p}(\Delta), \mathcal{C}_{bp}(\Delta), \mathcal{C}_r(\Delta)$ and $\mathcal{L}_q(\Delta)$ are the linear spaces with the coordinatewise addition and scalar multiplication, and $\mathcal{M}_u(\Delta), \mathcal{C}_p(\Delta)$, $\mathcal{C}_{0p}(\Delta), \mathcal{C}_{bp}(\Delta), \mathcal{C}_r(\Delta)$ and $\mathcal{L}_q(\Delta)$ are the Banach spaces with the norms*

$$\|x\|_{\mathcal{M}_u(\Delta)} = \sup_{m,n \in \mathbb{N}} \big|x_{mn} + x_{m-1,n-1} - x_{m,n-1} - x_{m-1,n}\big|, \tag{2.1}$$

$$\|x\|_{\mathcal{L}_q(\Delta)} = \left[\sum_{m,n} |x_{mn} + x_{m-1,n-1} - x_{m,n-1} - x_{m-1,n}|^q\right]^{1/q}, \quad (1 \leq q < \infty). \tag{2.2}$$

Proof. The first part of the theorem is a routine verification. So, we omit the detail.

Since the proof may be given for the spaces $\mathcal{M}_u(\Delta), \mathcal{C}_p(\Delta)$, $\mathcal{C}_{0p}(\Delta), \mathcal{C}_{bp}(\Delta)$ and $\mathcal{C}_r(\Delta)$, to avoid the repetition of the similar statements, we prove the theorem only for the space $\mathcal{L}_q(\Delta)$.

It is obvious that $\|x\|_{\mathcal{L}_q(\Delta)} = \|y\|_q$, where $\|.\|_q$ is the norm on the space $\mathcal{L}_q$. Let $x^{(r)} = \{x_{jk}^{(r)}\}$ be a Cauchy sequence in $\mathcal{L}_q(\Delta)$. Then, $\{y^{(r)}\}_{r\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{L}_q$, where $y^{(r)} = \{y_{mn}^{(r)}\}_{m,n=0}^{\infty}$ with

$$y_{mn}^{(r)} = x_{mn}^{(r)} + x_{m-1,n-1}^{(r)} - x_{m,n-1}^{(r)} - x_{m-1,n}^{(r)}$$

for all $m, n, r \in \mathbb{N}$. Then, for a given $\varepsilon > 0$, there is a positive integer $N = N(\varepsilon)$ such that

$$\|y^{(r)} - y^{(s)}\|_q = \left\{ \sum_{m,n} \left|y_{mn}^{(r)} - y_{mn}^{(s)}\right|^q \right\}^{1/q} < \varepsilon \tag{2.3}$$

for all $r, s > N$. Therefore $\left|y_{mn}^{(r)} - y_{mn}^{(s)}\right| < \varepsilon$, i.e. $\{y_{mn}^{(r)}\}_{r\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{C}$, and hence converges in $\mathbb{C}$. Say,

$$\lim_{r\to\infty} y_{mn}^{(r)} = y_{mn}. \tag{2.4}$$

Using these infinitely many limits, we define the sequence $y = (y_{mn})_{m,n=0}^{\infty}$. Then, we get by (2.4) that

$$\lim_{r\to\infty} \|y_{mn}^{(r)} - y_{mn}\|_{\mathcal{L}_q(\Delta)} = \lim_{r\to\infty} \left\{ \sum_{m,n} \left|y_{mn} - y_{mn}^{(r)}\right|^q \right\}^{1/q} = 0. \tag{2.5}$$

Now, we have to show that $y \in \mathcal{L}_q$. Since $y^{(r)} = \{y_{mn}^{(r)}\}_{m,n=0}^{\infty} \in \mathcal{L}_q$ and by (2.5)

$$\left\{ \sum_{m,n} |y_{mn}|^q \right\}^{1/q} \leq \left\{ \sum_{m,n} |y_{mn} - y_{mn}^{(r)}|^q \right\}^{1/q} + \left\{ \sum_{m,n} |y_{mn}^{(r)}|^q \right\}^{1/q} < \infty,$$

which shows that $y = (y_{mn})_{m,n=0}^{\infty} \in \mathcal{L}_q$, i.e. $x = (x_{jk}) \in \mathcal{L}_q(\Delta)$. Since $\{x^{(r)}\}_{r\in\mathbb{N}}$ was arbitrary Cauchy sequence in $\mathcal{L}_q(\Delta)$, the double difference sequence space $\mathcal{L}_q(\Delta)$ is complete. This completes the proof of the theorem. □

Theorem 2.2. *The space $\lambda(\Delta)$ is linearly isomorphic to the space λ, where λ denotes any of the spaces $\mathcal{M}_u, \mathcal{C}_p, \mathcal{C}_{0p}, \mathcal{C}_{bp}, \mathcal{C}_r$ and $\mathcal{L}_q$.*

Proof. We show here that $\mathcal{M}_u(\Delta)$ is linearly isomorphic to $\mathcal{M}_u$. Consider the transformation T from $\mathcal{M}_u(\Delta)$ to $\mathcal{M}_u$ defined by $x = (x_{jk}) \mapsto y = (y_{mn})$. Then, clearly T is linear and injective. Now, define the sequence $x = (x_{jk})$ by

$$x_{jk} = \sum_{m=0}^{j} \sum_{n=0}^{k} y_{mn} \tag{2.6}$$

for all $j,k \in \mathbb{N}$. Suppose that $y \in \mathcal{M}_u$. Then, since

$$\begin{aligned}\|x\|_{\mathcal{M}_u(\Delta)} &= \sup_{j,k\in\mathbb{N}} \left| \sum_{m=0}^{j} \sum_{n=0}^{k} y_{mn} \right| \\ &= \sup_{j,k\in\mathbb{N}} |y_{jk}| = \|y\|_\infty < \infty,\end{aligned}$$

$x = (x_{jk})$ defined by (2.6) is in the space $\mathcal{M}_u(\Delta)$. Hence, T is surjective and norm preserving. This completes the proof of the theorem. □

Now, we give some inclusion relations between the double difference sequence spaces.

Theorem 2.3. *$\mathcal{M}_u$ is the subspace of the space $\mathcal{M}_u(\Delta)$.*

Proof. Let us take $x = (x_{mn}) \in \mathcal{M}_u$. Then, there exists an K such that

$$\sup_{m,n\in\mathbb{N}} |x_{mn}| \leq K$$

for all $m,n \in \mathbb{N}$, one can observe that

$$\begin{aligned}\left|(\Delta x)_{mn}\right| &= \left|x_{mn} - x_{m,n-1} - x_{m-1,n} + x_{m-1,n-1}\right| \\ &\leq |x_{mn}| + |x_{m,n-1}| + |x_{m-1},n| + |x_{m-1,n-1}|. \end{aligned} \tag{2.7}$$

Then, we see by taking supremum over $m,n \in \mathbb{N}$ in (2.7) that $\|x\|_\infty \leq 4K$, i.e., $x \in \mathcal{M}_u(\Delta)$.

Now, we see that the inclusion is strict. Let $x = (x_{mn})$ be defined by

$$x_{mn} = mn$$

for all $m,n \in \mathbb{N}$. Then the sequence is in $x \in \mathcal{M}_u(\Delta)\backslash\mathcal{M}_u$. This completes the proof of the theorem. □

Lemma 2.4. [1, Theorem 1.2] *$\mathcal{L}_u \subset \mathcal{BS} \subset \mathcal{M}_u$ strictly hold.*

Lemma 2.5. [1, Theorem 2.9] *Let $v \in \{p, bp, r\}$. Then, the inclusion $\mathcal{BV} \subset \mathcal{C}_v$ and $\mathcal{BV} \subset \mathcal{M}_u$ strictly hold.*

Combining Lemma 2.4, Lemma 2.5 and Theorem 2.3, we get the following corollaries.

Corollary 2.6. *The inclusion $\mathcal{L}_u \subset \mathcal{BS} \subset \mathcal{M}_u(\Delta)$ strictly hold.*

Corollary 2.7. *The inclusion $\mathcal{BV} \subset \mathcal{M}_u(\Delta)$ strictly holds.*

Theorem 2.8. *The inclusion $\mathcal{L}_q \subset \mathcal{L}_q(\Delta)$ strictly holds; where $1 \leq q < \infty$.*

Proof. The prove the validity of the inclusion $\mathcal{L}_q \subset \mathcal{L}_q(\Delta)$ for $1 \leq q < \infty$, it suffices to show the existence of a number $K > 0$ such that

$$\|x\|_{\mathcal{L}_q(\Delta)} \leq K\|x\|_{\mathcal{L}_q}$$

for every $x \in \mathcal{L}_q$. Let $x \in \mathcal{L}_q$ and $1 \leq q < \infty$. Then, we obtain

$$\begin{aligned} \|x\|_{\mathcal{L}_q(\Delta)} &= \left\{ \sum_{m,n} |x_{mn} - x_{m,n-1} - x_{m-1,n} + x_{m-1,n-1}|^q \right\}^{1/q} \\ &\leq 4\|x\|_{\mathcal{L}_q}. \end{aligned}$$

This shows that the inclusion $\mathcal{L}_q \subset \mathcal{L}_q(\Delta)$ holds.

Additionally, since the sequence $x = (x_{mn})$ defined by

$$x_{mn} := \begin{cases} 1 & , \quad n = 0 \\ 0 & , \quad \text{otherwise} \end{cases}$$

for all $m, n \in \mathbb{N}$ is in $\mathcal{L}_q(\Delta)$ but not in $\mathcal{L}_q$, as asserted. This completes the proof. □

Theorem 2.9. *The following statements hold:*

(i) $\mathcal{C}_p$ is the subspace of the space $\mathcal{C}_p(\Delta)$.
(ii) $\mathcal{C}_{0p}$ is the subspace of the space $\mathcal{C}_{0p}(\Delta)$.
(iii) $\mathcal{C}_{bp}$ is the subspace of the space $\mathcal{C}_{bp}(\Delta)$.
(iv) $\mathcal{C}_r$ is the subspace of the space $\mathcal{C}_r(\Delta)$.

Proof. We only prove that the inclusion $\mathcal{C}_p \subset \mathcal{C}_p(\Delta)$ holds. Let us take $x \in \mathcal{C}_p$. Then, for a given $\varepsilon > 0$, there exists an $n_x(\varepsilon) \in \mathbb{N}$ such that

$$|x_{mn} - l| < \frac{\varepsilon}{4}$$

for all $m, n > n_x(\varepsilon)$. Then,

$$\begin{aligned} |(\Delta x)_{mn}| &= \left|x_{mn} - x_{m,n-1} - x_{m-1,n} + x_{m-1,n-1}\right| \\ &\leq |x_{mn} - l| + |x_{m,n-1} - l| + |x_{m-1,n} - l| + |x_{m-1,n-1} - l| \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon \end{aligned}$$

for sufficiently large m, n which means that $p-\lim(\Delta x)_{mn} = 0$. Hence, $x \in \mathcal{C}_p(\Delta)$ that is to say that $\mathcal{C}_p \subset \mathcal{C}_p(\Delta)$ holds, as expected.

Now, we see that the inclusion is strict. Let $x = (x_{mn})$ be defined by

$$x_{mn} = (m+1)(n+1)$$

for all $m, n \in \mathbb{N}$. It is easy to see that

$$p - \lim(\Delta x)_{mn} = 1.$$

But

$$\lim_{m,n\to\infty} (m+1)(n+1)$$

which does not tend to a finite limit. Hence $x \notin \mathcal{C}_p$. This completes the proof. □

Lemma 2.10. [1, Theorem 2.3] *$\mathcal{CS}_p$ is the subspace of $\mathcal{C}_p$.*

Combining Lemma 2.10 and Theorem 2.9, we get the following corollary.

Corollary 2.11. *The inclusion $\mathcal{CS}_p \subset \mathcal{C}_p(\Delta)$ strictly holds.*

3. The alpha- and beta-duals of the new spaces of double sequences

In this section, we determine the alpha-dual of the space $\mathcal{M}_u(\Delta)$ and the $\beta(r)-$dual of the space $\mathcal{C}_r(\Delta)$, and $\beta(\vartheta)-$dual of the space $\mathcal{C}_\eta(\Delta)$ of double sequences, $\vartheta, \eta \in \{p, bp, r\}$. Although the alpha-dual of a space of double sequences is unique, its beta-dual may be more than one with respect to $\vartheta-$convergence.

Theorem 3.1. $\{\mathcal{M}_u(\Delta)\}^\alpha = \mathcal{L}_u$.

Proof. Let $x = (x_{kl}) \in \mathcal{M}_u(\Delta)$ and $z = (z_{kl}) \in \mathcal{L}_u$. Hence, there is a sequence $y = (y_{ij}) \in \mathcal{M}_u$ related with $x = (x_{kl})$ from Theorem 2.2 and there is a positive real number K such that $|y_{ij}| \leq \frac{K}{(k+1)(l+1)}$ for all $i, j \in \mathbb{N}$. So we use the relation (2.6)we have that,

$$\sum_{k,l} |z_{kl} x_{kl}| = \sum_{k,l} \left| z_{kl} \sum_{i=0}^{k} \sum_{j=0}^{l} y_{ij} \right| \leq K \sum_{k,l} |z_{kl}| < \infty$$

so $z \in \{\mathcal{M}_u(\Delta)\}^\alpha$, that is

$$\mathcal{L}_u \subset \{\mathcal{M}_u(\Delta)\}^\alpha. \tag{3.1}$$

Conversely, suppose that $z = (z_{kl}) \in \{\mathcal{M}_u(\Delta)\}^\alpha$, that is $\sum_{k,l} |z_{kl} x_{kl}| < \infty$ for all $x = (x_{kl}) \in \mathcal{M}_u(\Delta)$. If $z = (z_{kl}) \notin \mathcal{L}_u$, then $\sum_{k,l} |z_{kl}| = \infty$. Further, if we choose $y = (y_{kl})$ such that

$$y_{kl} := \begin{cases} \dfrac{1}{(k+1)(l+1)} & , \quad 0 \leq i \leq k, \quad 0 \leq j \leq l \\ \qquad\qquad 0 & , \quad \text{otherwise} \end{cases}$$

for all $k, l \in \mathbb{N}$. Then, $y \in \mathcal{M}_u$ but

$$\sum_{k,l} |z_{kl} x_{kl}| = \sum_{k,l} \left| z_{kl} \sum_{i=0}^{k} \sum_{j=0}^{l} \frac{1}{(k+1)(l+1)} \right| = \sum_{k,l} |z_{kl}| = \infty.$$

Hence, $z \notin \{\mathcal{M}_u(\Delta)\}^\alpha$,this is a contradiction. So, we have the following inclusion,

$$\{\mathcal{M}_u(\Delta)\}^\alpha \subset \mathcal{L}_u. \tag{3.2}$$

Hence, from the inclusions (3.1) and (3.2) we get

$$\{\mathcal{M}_u(\Delta)\}^\alpha = \mathcal{L}_u.$$

□

Now, we may give the $\beta-$duals of the spaces with respect to the $\vartheta-$convergence using the technique in [2] and [3] for the single sequences.

The conditions for a 4-dimensional matrix to transform the spaces $\mathcal{C}_{bp}, \mathcal{C}_r$ and $\mathcal{C}_p$ into the space $\mathcal{C}_{bp}$ are well known (see for example [8, 16]).

Lemma 3.2. *The matrix $A = (a_{mnij})$ is in $(\mathcal{C}_r : \mathcal{C}_\vartheta)$ if and only if the following conditions hold:*

$$\sup_{m,n\in\mathbb{N}} \sum_{i,j} |a_{mnij}| < \infty, \tag{3.3}$$

$$\exists\ v \in \mathbb{C} \ni \vartheta - \lim_{m,n\to\infty} \sum_{i,j} a_{mnij} = v, \tag{3.4}$$

$$\exists\ (a_{ij}) \in \Omega \ni \vartheta - \lim_{m,n\to\infty} a_{mnij} = a_{ij} \quad \text{for all } i,j \in \mathbb{N}, \tag{3.5}$$

$$\exists\ u^{j_0} \in \mathbb{C} \ni \vartheta - \lim_{m,n\to\infty} \sum_{i} a_{mnij_0} = u^{j_0} \text{ for fixed } j_0 \in \mathbb{N}, \tag{3.6}$$

$$\exists\ v_{i_0} \in \mathbb{C} \ni \vartheta - \lim_{m,n\to\infty} \sum_{j} a_{mni_0j} = v_{i_0} \text{ for fixed } i_0 \in \mathbb{N}. \tag{3.7}$$

Lemma 3.3. *The matrix $A = (a_{mnij})$ is in $(\mathcal{C}_{bp} : \mathcal{C}_\vartheta)$ if and only if the conditions (3.3)-(3.5) of Lemma 3.2 hold, and*

$$\vartheta - \lim_{m,n\to\infty} \sum_{i} |a_{mnij_0} - a_{ij_0}| = 0 \text{ for each fixed } j_0 \in \mathbb{N}, \tag{3.8}$$

$$\vartheta - \lim_{m,n\to\infty} \sum_{j} |a_{mni_0j} - a_{i_0j}| = 0 \text{ for each fixed } i_0 \in \mathbb{N}. \tag{3.9}$$

Lemma 3.4. *The matrix $A = (a_{mnij})$ is in $(\mathcal{C}_p : \mathcal{C}_\vartheta)$ if and only if the conditions (3.3)-(3.5) of Lemma 3.2 hold, and*

$$\forall\ i \in \mathbb{N}\ \exists\ J \in \mathbb{N} \ni a_{mnij} = 0 \text{ for } j > J \text{ for all } m,n \in \mathbb{N}, \tag{3.10}$$

$$\forall\ j \in \mathbb{N}\ \exists\ I \in \mathbb{N} \ni a_{mnij} = 0 \text{ for } i > I \text{ for all } m,n \in \mathbb{N}. \tag{3.11}$$

Theorem 3.5. *Define the sets*

$$\begin{aligned} F_1 &= \left\{ a = (a_{ij}) \in \Omega : \sum_{i,j} (i+1)(j+1)|a_{ij}| < \infty \right\}, \\ F_2 &= \left\{ a = (a_{ij}) \in \Omega : r - \lim_{m,n\to\infty} \sum_{i}^{m} \sum_{p=i}^{m} \sum_{q=j_0}^{n} a_{pq} \text{ exists for each fixed } j_0 \right\}, \\ F_3 &= \left\{ a = (a_{ij}) \in \Omega : r - \lim_{m,n\to\infty} \sum_{j}^{m} \sum_{p=i_0}^{m} \sum_{q=j}^{n} a_{pq} \text{ exists for each fixed } i_0 \right\}. \end{aligned}$$

Then, $\{\mathcal{C}_r(\Delta)\}^{\beta(r)} = F_1 \cap F_2 \cap F_3$.

Proof. Let $x = (x_{ij}) \in \mathcal{C}_r(\Delta)$. Then, there exists a sequence $y = (y_{mn}) \in \mathcal{C}_r$. Consider the following equality

$$\begin{aligned} z_{mn} &= \sum_{i=0}^{m}\sum_{j=0}^{n} a_{ij}x_{ij} = \sum_{i=0}^{m}\sum_{j=0}^{n}\left(\sum_{p=0}^{i}\sum_{q=0}^{j} y_{pq}\right)a_{ij} \\ &= \sum_{i=0}^{m}\sum_{j=0}^{n}\left(\sum_{p=i}^{m}\sum_{q=j}^{n} a_{pq}\right)y_{ij} \\ &= \sum_{i=0}^{m}\sum_{j=0}^{n} b_{mnij}y_{ij} = (By)_{ij} \end{aligned}$$

for all $m,n \in \mathbb{N}$. Hence we can define the four-dimensional matrix $B = (b_{mnij})$ as following

$$b_{mnij} := \begin{cases} \sum_{p=i}^{m}\sum_{q=j}^{n} a_{pq} & , \quad 0 \le i \le m, \quad 0 \le j \le n, \\ 0 & , \quad \text{otherwise.} \end{cases} \tag{3.12}$$

Thus we see that $ax = (a_{mn}x_{mn}) \in \mathcal{CS}_r$ whenever $x = (x_{mn}) \in \mathcal{C}_r(\Delta)$ if and only if $z = (z_{mn}) \in \mathcal{C}_r$ whenever $y = (y_{mn}) \in \mathcal{C}_r$. This means that $a = (a_{mn}) \in \{\mathcal{C}_r(\Delta)\}^{\beta(r)}$ if and only if $B \in (\mathcal{C}_r : \mathcal{C}_r)$. Therefore, we consider the following equality and equation

$$\begin{aligned} \sup_{m,n\in\mathbb{N}} \sum_{i=0}^{m}\sum_{j=0}^{n} |b_{mnij}| &\le \sup_{m,n\in\mathbb{N}} \sum_{i=0}^{m}\sum_{j=0}^{n}\left(\sum_{p=i}^{m}\sum_{q=j}^{n} |a_{pq}|\right) \\ &= \sup_{m,n\in\mathbb{N}} \sum_{i=0}^{m}\sum_{j=0}^{n}\left(\sum_{p=0}^{i}\sum_{q=0}^{j} |a_{ij}|\right) \\ &= \sup_{m,n\in\mathbb{N}} \sum_{i=0}^{m}\sum_{j=0}^{n}(i+1)(j+1)|a_{ij}|, \end{aligned} \tag{3.13}$$

$$\begin{aligned} r - \lim_{m,n\to\infty} \sum_{i,j} b_{mnij} &= r - \lim_{m,n\to\infty} \sum_{i=0}^{m}\sum_{j=0}^{n}\left(\sum_{p=0}^{i}\sum_{q=0}^{j} a_{pq}\right) \\ &= \sum_{i,j}\left(\sum_{p=0}^{i}\sum_{q=0}^{j} a_{pq}\right). \end{aligned} \tag{3.14}$$

Then, we derive from the condition (3.3)-(3.5) that

$$\sum_{i,j}(i+1)(j+1)|a_{ij}| < \infty. \tag{3.15}$$

Further, from Lemma 3.2 conditions (3.6) and (3.7),

$$r - \lim_{m,n\to\infty} \sum_{i}^{m} b_{mnij_0} = r - \lim_{m,n\to\infty} \sum_{i}^{m}\sum_{p=i}^{m}\sum_{q=j_0}^{n} a_{pq} \tag{3.16}$$

exists for each fixed $j_0 \in \mathbb{N}$ and

$$r-\lim_{m,n\to\infty}\sum_{j}^{n} b_{mni_0j} = r-\lim_{m,n\to\infty}\sum_{i}^{m}\sum_{p=i_0}^{m}\sum_{q=j}^{n} a_{pq} \tag{3.17}$$

exists for each fixed $i_0 \in \mathbb{N}$. This show that $\{\mathcal{C}_r(\Delta)\}^{\beta(r)} = F_1 \cap F_2 \cap F_3$ which completes the proof. $\square$

Now, we may give our theorem exhibiting the $\beta(\vartheta)$-dual of the series space $\mathcal{C}_\eta(\Delta)$ in the case $\eta, \vartheta \in \{p, bp, r\}$, without proof.

Theorem 3.6. $\{\mathcal{C}_\eta(\Delta)\}^{\beta(\vartheta)} = \{a = (a_{mn}) \in \Omega : B = (b_{mnij}) \in (\mathcal{C}_\eta : \mathcal{C}_\vartheta)\}$ *where* $B = (b_{mnij})$ *is defined by (3.12).*

4. Characterization of some classes of four dimensional matrices

In the present section, we characterize the matrix transformations from the space $\mathcal{C}_r(\Delta)$ to the double sequence space $\mathcal{C}_\vartheta$. Although the theorem characterizing the class $(\mu : \mathcal{C}_\vartheta(\Delta))$ are stated and proved, the necessary and sufficient conditions on a four dimensional matrix belonging to the classes $(\mathcal{C}_r : \mathcal{C}_r(\Delta))$ and $(\mathcal{C}_{bp} : \mathcal{C}_{bp}(\Delta))$ also given without proof.

Theorem 4.1. $A = (a_{mnkl}) \in (\mathcal{C}_r(\Delta) : \mathcal{C}_\vartheta)$ *if and only if the following conditions hold:*

$$\sup_{m,n}\sum_{k,l}\left|\sum_{p=k}^{\infty}\sum_{q=l}^{\infty} a_{mnpq}\right| < \infty, \tag{4.1}$$

$$\vartheta-\lim_{s,t\to\infty}\sum_{i=0}^{s}\sum_{p=i}^{s}\sum_{p=j_0}^{t} a_{mnpq} - \textit{exists for fixed } j_0, \tag{4.2}$$

$$\vartheta-\lim_{s,t\to\infty}\sum_{j=0}^{t}\sum_{p=i_0}^{s}\sum_{p=j}^{t} a_{mnpq} - \textit{exists for fixed } i_0, \tag{4.3}$$

$$\vartheta-\lim_{m,n}\sum_{p=k}^{\infty}\sum_{q=l}^{\infty} a_{mnpq} = a_{kl} \textit{ for all } k,l \in \mathbb{N}, \tag{4.4}$$

$$\exists u^{l_0} \in \mathbb{C} \ni \vartheta-\lim_{m,n}\sum_{k}\sum_{p=k}^{\infty}\sum_{q=l_0}^{\infty} a_{mnpq} = u^{l_0} \quad \textit{for fixed } l_0 \in \mathbb{N}, \tag{4.5}$$

$$\exists v_{k_0} \in \mathbb{C} \ni \vartheta-\lim_{m,n}\sum_{l}\sum_{p=k_0}^{\infty}\sum_{q=l}^{\infty} a_{mnpq} = v_{k_0} \quad \textit{for fixed } k_0 \in \mathbb{N}, \tag{4.6}$$

$$\exists v \in \mathbb{C} \ni \vartheta-\lim_{m,n}\sum_{k,l}\sum_{p=k}^{\infty}\sum_{q=l}^{\infty} a_{mnpq} = v. \tag{4.7}$$

Proof. Let us take any $x = (x_{mn}) \in \mathcal{C}_r(\Delta)$ and define the sequence $y = (y_{kl})$ by

$$y_{kl} = x_{kl} - x_{k-1,l} - x_{k,l-1} + x_{k-1,l-1}; \quad (k,l \in \mathbb{N}).$$

Then, $y = (y_{kl}) \in \mathcal{C}_r$ by Theorem 2.2. Now, for the (s,t)th rectangular partial sum of the series $\sum_{j,k} a_{mnjk}x_{jk}$, we derive that

$$\begin{aligned}
(Ax)_{mn}^{[s,t]} &= \sum_{j=0}^{s}\sum_{k=0}^{t} a_{mnjk}x_{jk} \\
&= \sum_{j=0}^{s}\sum_{k=0}^{t}\left(\sum_{p=0}^{j}\sum_{q=0}^{k} y_{pq}\right)a_{mnjk} \\
&= \sum_{j=0}^{s}\sum_{k=0}^{t}\left(\sum_{p=j}^{s}\sum_{q=k}^{t} a_{mnpq}\right)y_{jk} \qquad (4.8)
\end{aligned}$$

for all $m,n,s,t \in \mathbb{N}$. Define the matrix $B_{mn} = (b_{mnjk}^{[s,t]})$ by

$$b_{mnjk}^{[s,t]} := \begin{cases} \displaystyle\sum_{p=j}^{s}\sum_{q=k}^{t} a_{mnpq} & , \quad 0 \le j \le s, \quad 0 \le k \le t, \\ 0 & , \quad \text{otherwise}. \end{cases} \qquad (4.9)$$

Then, the equality (4.8) may be rewritten as

$$(Ax)_{mn}^{[s,t]} = (B_{mn}y)_{[s,t]}. \qquad (4.10)$$

Then, the convergence of the rectangular partial sums $(Ax)_{mn}^{[s,t]}$ in the regular sense for all $m,n \in \mathbb{N}$ and for all $x \in \mathcal{C}_r(\Delta)$ is equivalent of saying that $B_{mn} \in (\mathcal{C}_r : \mathcal{C}_\vartheta)$. Hence, the following conditions

$$\sum_{k,l}(k+1)(l+1)|a_{mnkl}| < \infty, \qquad (4.11)$$

$$\vartheta - \lim_{s,t\to\infty}\sum_{i=0}^{s}\sum_{p=i}^{s}\sum_{p=j_0}^{t} a_{mnpq} - \text{exists for fixed } j_0, \qquad (4.12)$$

$$\vartheta - \lim_{s,t\to\infty}\sum_{j=0}^{t}\sum_{p=i_0}^{s}\sum_{p=j}^{t} a_{mnpq} - \text{exists for fixed } i_0 \qquad (4.13)$$

must be satisfied for every fixed $m,n \in \mathbb{N}$. In this case,

$$\vartheta - \lim_{s,t\to\infty} b_{mnjk}^{[s,t]} = \sum_{p=j}^{\infty}\sum_{q=k}^{\infty} a_{mnpq},$$

$$\vartheta - (Ax)_{mn}^{[s,t]} = r - \lim(B_{mn}y)$$

hold. Thus, we derive from the two-sided implication "Ax is in C_r whenever $x \in \mathcal{C}_r(\Delta)$ if and only if $B = \left(\sum_{p=j}^{\infty}\sum_{q=k}^{\infty} a_{mnpq}\right)_{mn} \in (\mathcal{C}_r : \mathcal{C}_\vartheta)$", we have Lemma

3.2 that

$$\sup_{m,n}\sum_{k,l}\left|\sum_{p=k}^{\infty}\sum_{q=l}^{\infty}a_{mnpq}\right|<\infty, \tag{4.14}$$

$$\vartheta-\lim_{m,n}\sum_{p=k}^{\infty}\sum_{q=l}^{\infty}a_{mnpq}=a_{kl}\text{ for all }k,l\in\mathbb{N}, \tag{4.15}$$

$$\exists u^{l_0}\in\mathbb{C}\ni\vartheta-\lim_{m,n}\sum_{k}\sum_{p=k}^{\infty}\sum_{q=l_0}^{\infty}a_{mnpq}=u^{l_0}\quad\text{for fixed }l_0\in\mathbb{N}, \tag{4.16}$$

$$\exists v_{k_0}\in\mathbb{C}\ni\vartheta-\lim_{m,n}\sum_{l}\sum_{p=k_0}^{\infty}\sum_{q=l}^{\infty}a_{mnpq}=v_{k_0}\quad\text{for fixed }k_0\in\mathbb{N}, \tag{4.17}$$

$$\exists v\in\mathbb{C}\ni\vartheta-\lim_{m,n}\sum_{k,l}\sum_{p=k}^{\infty}\sum_{q=l}^{\infty}a_{mnpq}=v. \tag{4.18}$$

Now, from the conditions (4.11)-(4.18), we have that $A=(a_{mnkl})\in(\mathcal{C}_r(\Delta):\mathcal{C}_v)$ if and only if the conditions (4.1)-(4.7) hold. This completes the proof. $\square$

Theorem 4.2. *Suppose that the elements of the four dimensional infinite matrices $E=(e_{mnkl})$ and $F=(f_{mnkl})$ are connected with the relation*

$$f_{mnkl}=\sum_{i=m-1}^{m}\sum_{j=n-1}^{n}(-1)^{m+n-i-j}e_{ijkl} \tag{4.19}$$

for all $k,l,m,n\in\mathbb{N}$ and μ be any given space of double sequences. Then, $E\in(\mu:\mathcal{C}_\vartheta(\Delta))$ if and only if $F\in(\mu:\mathcal{C}_\vartheta)$.

Proof. Let $x=(x_{kl})\in\mu$ and consider the following equality with (4.19)

$$\sum_{i=m-1}^{m}\sum_{j=n-1}^{n}\sum_{k=s-1}^{s}\sum_{l=t-1}^{t}(-1)^{m+n-i-j}e_{ijkl}x_{kl}=\sum_{k=s-1}^{s}\sum_{l=t-1}^{t}f_{mnkl}x_{kl} \tag{4.20}$$

for all $m,n,s,t\in\mathbb{N}$. By letting $s,t\to\infty$ in (4.20) one can derive that

$$\sum_{i=m-1}^{m}\sum_{j=n-1}^{n}(-1)^{m+n-i-j}(Ex)_{ij}=(Fx)_{mn}\quad\text{for all}\quad m,n\in\mathbb{N}. \tag{4.21}$$

Therefore, it is seen by (4.21) that $Ex\in\mathcal{C}_\vartheta(\Delta)$ if and only if $Fx\in\mathcal{C}_\vartheta$ whenever $x\in\mu$. This step completes the proof. $\square$

Of course, Theorem 4.2 has several consequences depending on the choice of the sequence space μ. Prior to giving some results as an application of this idea, we need the following lemmas:

Lemma 4.3. [8, 13, 15] *$A = (a_{mnkl}) \in (\mathcal{C}_r : \mathcal{C}_r)$ if and only if*

$$\sup_{m,n\in\mathbb{N}} \sum_{k,l} |a_{mnkl}| < \infty, \tag{4.22}$$

$$\exists\ (a_{kl}) \in \Omega \ni r - \lim_{m,n\to\infty} a_{mnkl} = a_{kl} \quad \text{for each } k,l \in \mathbb{N}, \tag{4.23}$$

$$\exists\ v \in \mathbb{C} \ni r - \lim_{m,n\to\infty} \sum_{k,l} a_{mnkl} = v, \tag{4.24}$$

$$\exists\ u^{l_0}, v_{k_0} \in \mathbb{C} \ni r - \lim_{m,n\to\infty} \sum_{k} a_{mnkl_0} = u^{l_0} \ \text{and} \tag{4.25}$$

$$r - \lim_{m,n\to\infty} \sum_{l} a_{mnk_0l} = v_{k_0} \ \text{for any } k_0, l_0 \in \mathbb{N}.$$

Lemma 4.4. [8, 13, 15] *$A = (a_{mnkl}) \in (\mathcal{C}_{bp} : \mathcal{C}_{bp})$ if and only if*

$$\sup_{m,n\in\mathbb{N}} \sum_{k,l} |a_{mnkl}| < \infty, \tag{4.26}$$

$$\exists\ (a_{kl}) \in \Omega \ni bp - \lim_{m,n\to\infty} a_{mnkl} = a_{kl} \quad \text{for each } k,l \in \mathbb{N}, \tag{4.27}$$

$$\exists\ v \in \mathbb{C} \ni bp - \lim_{m,n\to\infty} \sum_{k,l} a_{mnkl} = v, \tag{4.28}$$

$$bp - \lim_{m,n\to\infty} \sum_{k} |a_{mnkl_0} - a_{kl_0}| = 0 \quad \text{and} \tag{4.29}$$

$$bp - \lim_{m,n\to\infty} \sum_{l} |a_{mnk_0l} - a_{k_0l}| = 0 \quad \text{for any} \quad k_0, l_0 \in \mathbb{N}.$$

Corollary 4.5. *Suppose that the relation (4.19) holds between the elements of the four dimensional infinite matrices $E = (e_{mnkl})$ and $F = (f_{mnkl})$. Then, the following statements hold:*

(i) $E = (e_{mnjk}) \in (\mathcal{C}_r : \mathcal{C}_r(\Delta))$ if and only if the conditions (4.22)-(4.25) hold with f_{mnkl} instead of a_{mnkl}.

(ii) $E = (e_{mnjk}) \in (\mathcal{C}_{bp} : \mathcal{C}_{bp}(\Delta))$ if and only if the conditions (4.26)-(4.29) hold with f_{mnkl} instead of a_{mnkl}.

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References

1. Altay, B., Başar, F., Some new spaces of double sequences, J. Math. Anal. Appl., **309**(1) (2005), 70-90.
2. Altay, B., Başar, F., Matrix mappings on the space $bs(p)$ and its $\alpha-, \beta-$ and $\gamma-$duals, Aligarh Bull. Math., **21** (2002), 79-91.
3. Altay, B., Başar, F., On the space of sequences of $p-$bounded variation and related matrix mappings, Ukrainian Math. J., **55**(1) (2003), 136-147.
4. Başar, F., Sever, Y., The space $\mathcal{L}_q$ of double sequences, Math. J. Okayama Univ., **51** (2009), 149-157.

5. Boss, J., Classical and Modern Methods in Summability, Oxford University Press, Newyork, 2000.
6. Gökhan, A., Çolak, R., The double sequence spaces $c_2^P(p)$ and $c_2^{BP}(p)$, Appl. Math. Comput., **157**(2) (2004), 491-501.
7. Gökhan, A., Çolak, R., Double sequence space $\ell_2^{\infty}(p)$, Appl. Math. Comput., **160** (2005), 147-153.
8. Hamilton, H.J., Transformations of multiple sequences, Duke Math. J., **2** (1936), 29-60.
9. Mòricz, F., Extensions of the spaces c and c_0 from single to double sequences, Acta Math. Hungar., **57** (1991), 129-136.
10. Mursaleen, M., Edely, O.H.H., Statistical convergence of double sequences, J. Math. Anal. Appl., **288**(1) (2003), 223-231.
11. Mursaleen, M., Almost strongly regular matrices and a core theorem for double sequences, J. Math. Anal. Appl., **293**(2) (2004), 523-531.
12. Mursaleen, M., Edely, O.H.H., Almost convergence and a core theorem for double sequences, J. Math. Anal. Appl., **293**(2) (2004), 532-540.
13. Robison, G.M., Divergent double sequences and series, Amer. Math. Soc. Trans., **28** (1926), 50-73.
14. Zeltser, M., Investigation of double sequence spaces by soft and hard analitic methods, Dissertationes Mathematicae Universtaties Tartuensis **25**, Tartu University Press, Univ. of Tartu, Faculty of Mathematics and Computer Science, Tartu, 2001.
15. Zeltser, M., On conservative matrix methods for double sequence spaces, Acta Math. Hung., **95**(3) (2002), 225-242.
16. Zeltser, M., Mursaleen, M., Mohiuddine, S.A., On almost conservative matrix mathods for double sequence spaces, Publ. Math. Debrecen, **75** (2009), 387-399.

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