Gulf Journal of Mathematics ISSN: 2309-4966
Vol 17, Issue 1 (2024) 58-71 https://doi.org/10.56947/gjom.v17i1.1503

A NOTE ON (λ, μ)- SLANT HANKEL OPERATORS ON $L^2(\mathbb{T})$

SHESH KUMAR PANDEY[1*] AND ANAND PRAKASH MISHRA[2]

Abstract. In the view of existing works on operators (namely, λ- Hankel and λ- Slant Hankel operators etc.) done by Avendano, Datt and Aggrawal, we take up an operator equation $\mu M_z^* X - X M_{z^2} = \lambda X$ and solve it for $\lambda, \mu \in \mathbb{C}$. Its solution yields an important class of slant Hankel operators, (namely, (λ, μ)- slant Hankel operators). After that, we provide certain basic properties of (λ, μ)- slant Hankel operator on $L^2(\mathbb{T})$ of unit circle $\mathbb{T}$. Finally, we come up with a results for (λ, μ)- slant Hankel operator, which is analogous to classical Kronecker theorem for Hankel matrix.

1. Introduction and preliminaries

In the paper, the set of all complex numbers is denoted by $\mathbb{C}$ and the symbols $\mathbb{D}$ and $\mathbb{T}$ are given by the sets $\{z \in \mathbb{C} : |z| < 1\}$ and $\{z \in \mathbb{C} : |z| = 1\}$, respectively. An organized study of slant Toeplitz operators on $L^2(\mathbb{T})$ of unit circle was done by M. C. Ho [15, 14], who not only analyzed several operator theoretic properties of the slant Toeplitz operators on Lebesgue space of unit circle (like algebraic and spectral properties etc) but also some C*-algebraic properties of slant Toeplitz operators. The study of k^{th}-order slant Toeplitz operators on $L^2(\mathbb{T})$ and the compression of k^{th}-order slant Toeplitz operators to the Hardy space was done by Arora and Batra [1], which is an extension of slant Toeplitz operator. For the fundamental results and definitions related to Toeplitz and Hankel operators, one can see [3, 17, 18]. The work of Ho [15, 14] enlights academicians to consider a similar kind of study for the Hankel operator and slant Hankel operators are taken into consideration on $L^2(\mathbb{T})$ in [2]. The theory of Hankel operators has several applications and it is one of the most fascinating and interesting branch in analysis. The study of finite matrices, whose elements are same along each antidiaganals, are done by Hankel [13] in 1861 and such matrices are known as Hankel matrices. A characterization of infinite Hankel matrices of finite rank was first given by Kronecker in 1881. Several generalizations of Hankel operator have been obtained with the passes of times using new techniques and ideas, like, $M_z^* X - X M_{z^2} = 0$ (slant Hankel operators), $U^* X - XU = \lambda X$ (λ-Hankel operators) and $\mu U^* X - XU = \lambda X$ (λ, μ)-Hankel operators (see [2, 6, 7, 8, 9, 10]), where

Date: Received: Oct 17, 2023; Accepted: Apr 12, 2024.

* Shesh Kumar Pandey.

2010 *Mathematics Subject Classification.* Primary 47B35.

Key words and phrases. Toeplitz operator, Hankel operator, Slant Hankel operator, Lebesgue space, Hardy space, λ- Hankel operator, λ- Slant Hankel operator.

M_z and U are multiplication and shift operators respectively. The concept of essentially (λ,μ)- Hankel operator was introduced and linked to λ- Hankel operator in [7]. The theory of Hankel operators has been extensively studied on Dirichlet, Dirichlet type, Bergman and Hardy spaces on certain domains (namely,unit ball in $\mathbb{C}$ or $\mathbb{C}^n$ and symmetric domain etc.) [3, 4, 16, 20, 21].

The operator equations $M_z^*X = XM_z$ and $U^*X = XU$, characterize Hankel operators on $L^2(\mathbb{T})$ and $H^2(\mathbb{T})$, respectively. Here, $U(f)(z) = zf(z)$ for all $f \in H^2(\mathbb{T})$. The references [11, 17, 21] are useful for the basic concepts related to Hankel operators. The work of Barria, Halmos, Sun [5, 19] and others motivate mathematicians to consider different generalizations (see, [2, 3, 4, 6, 8]). The approach initiated by Avendaño [3, 4] gave new insights to consider the operator equation [8] $\lambda M_z^*X = XM_{z^k}$, for $k \geq 2$ and $\lambda \in \mathbb{C}$. Particularly, the value $\lambda = 1$ in preceding expression provides the characterization for the k^{th}-order slant Hankel operators.

The Nehari's theorem [17] states that a Hankel operator is an operator induced by an essentially bounded function ϕ on $\mathbb{T}$, which is written as $H_\phi = PJM_\phi$, where P and J are projection and flip operators respectively. More research is not available in literature on the spectral properties of Hankel operators in the form of function ϕ. But Power [18] has obtained the essential spectrum of Hankel operator H_ϕ with inducing symbol ϕ, where ϕ is essentially bounded piecewise continuous function.

The space $L^2(\mathbb{T})$ represents the set of all functions $f : \mathbb{T} \to \mathbb{C}$, which are Lebesgue measurable with the condition that $\int_{\mathbb{T}} |f|^2 \, d\sigma$ is finite, where $d\sigma$ is Lebesgue measure. Similarly, the space $L^\infty(\mathbb{T})$ denotes the set of all essentially bounded measurable complex valued functions on $\mathbb{T}$. In the form of Fourier series expansion, the elements $f \in L^2(\mathbb{T})$ and $g \in H^2(\mathbb{T})$ can be expressed as

$$f(z) = \sum_{n\in\mathbb{Z}} f_n z^n \text{ with } \sum_{n\in\mathbb{Z}} |f_n|^2 < \infty,$$

and

$$g(z) = \sum_{n\in\mathbb{Z}_+} g_n z^n \text{ with } \sum_{n\in\mathbb{Z}_+} |g_n|^2 < \infty,$$

where $\mathbb{Z} = \{n : n \text{ is an integer}\}$ and $\mathbb{Z}_+ = \{n : n \text{ is a non-negative integer}\}$. It is familiar fact that the inner product on the Hilbert spaces $L^2(\mathbb{T})$ and $H^2(\mathbb{T})$ is given by

$$\langle f, g\rangle = \frac{1}{(2\pi)} \int_0^{2\pi} f(e^{i\theta})\overline{g(e^{i\theta})} d\theta.$$

The orthonormal bases for the Hilbert spaces $L^2(\mathbb{T})$ and $H^2(\mathbb{T})$ are given by the collections $\{e_m : m \in \mathbb{Z}\}$ and $\{e_m : m \in \mathbb{Z}_+\}$, respectively. The basis elements e_m are defined by $e_m(z) = z^m$. The basis elements are usually written as z^m instead of e_m. The Hardy space can also be thought of as a closed subspace of $L^2(\mathbb{T})$ having all those functions f of $L^2(\mathbb{T})$ such that $\langle f, e_n\rangle = 0$, when $n < 0$ (see [12]).

We know that the set of all functions $f : \mathbb{D} \to \mathbb{C}$, which are analytic on disk $\mathbb{D}$, with the norm

$$\|f\| := \left(\sup_{0 \leq r < 1} \int_0^{2\pi} |f(re^{it})|^2 dt \right)^{\frac{1}{2}} < \infty,$$

is denoted by $H^2(\mathbb{D})$ and known as Hardy space. Here the normalized Lebesgue measure on $\mathbb{T}$ is denoted by dt. The space $H^2(\mathbb{D})$ is also a Hilbert space, which can be identified with $H^2(\mathbb{T})$ via the radial limits of functions of $H^2(\mathbb{D})$. Therefore, functions (elements of $H^2(\mathbb{T})$) with Fourier coefficients $f_m = 0$, whenever $m < 0$ are termed as analytic function in $L^2(\mathbb{T})$. An element g of $L^2(\mathbb{T})$ is said to be co-analytic if $\bar{g}$ is analytic in $L^2(\mathbb{T})$.

2. (λ, μ)- Slant Hankel Operator

This section of the paper starts with a generalization of slant Hankel operators via an operator equation with the insights given by Barria, Halmos, Sun and Avendaño [3, 4, 5, 19]. We begin our section by listing certain basic definitions and results.

Definition 2.1 (Slant Hankel Operator)**.** [1, 2] For an essentially bounded function ϕ on $\mathbb{T}$, a slant Hankel operator on $L^2(\mathbb{T})$ is defined as $S_\phi = WJM_\phi$, where the operators J, M_ϕ, and W on $L^2(\mathbb{T})$ are given by $J(f)(z) = f(\bar{z})$, $M_\phi(f)(z) = \phi(z)f(z)$, and

$$W(z^n) = \begin{cases} z^{\frac{n}{2}}, & \text{if n is a multiple of 2} \\ 0, & \text{if n is not a multiple of 2,} \end{cases}$$

respectively. Moreover, the operator equation $M_z^* X = XM_{z^2}$ characterizes the slant Hankel operator on the space $L^2(\mathbb{T})$.

Definition 2.2 (λ- slant Hankel operator)**.** [8] A non-zero solution of the operator equation $\lambda M_z^* X = XM_{z^2}$ is known as λ- slant Hankel operator on the Lebesgue space of $\mathbb{T}$. It is induced by an essentially bounded function ϕ on $\mathbb{T}$ and given by $S_{\phi,\lambda} = D_\lambda WJM_\phi$, where $D_\lambda(f(z)) = f(\bar{\lambda}z)$, which is in fact a composition operator on the Lebesgue space. Here, $W, J,$ and M_ϕ are same as defined above.

Now, we provide few results, which is essential to carry out further discussion in this paper. We know that a functional Hilbert space is a Hilbert space H consisting of functions $f : Y \to \mathbb{C}$, where Y is a non-empty set; the structure of Hilbert space on H can be linked to non-empty set Y in the following elementary ways. For $x \in Y$, define a functional $\psi_x : H \to \mathbb{C}$ such that $\psi_x(f) = f(x)$, which is known as evaluation functional. Then it is essential that

(1) The functional ψ_x is linear for each $x \in Y$ that is, if $f, g \in H$ and $\alpha, \beta \in K$, where K is a scalar field, then $\psi_x(\alpha f + \beta g) = \alpha f(x) + \beta g(x) = \alpha\psi_x(f) + \beta\psi_x(g)$ for all $x \in Y$.
(2) The functional ψ_x is bounded for each $x \in Y$, that is, for each $x \in Y$ we have $\gamma_x > 0$, with $|\psi_x(f)| = |f(x)| < \gamma_x\|f\|$ for all $f \in H$.

It is easy to visualize every Hilbert space as a functional Hilbert space in the following way. Let H be a given Hilbert space, take $Y = H$. Let $\tilde{H}$ be a set of all functionals f on $Y(= H)$, which are bounded as well as conjugate linear. Then, there is a natural correspondence $\Phi : H \to \tilde{H}$, given by $\Phi(f) = \tilde{f}$ and $\tilde{f}(h) = \langle f, h\rangle$ for all $h \in X(= H)$, between H and $\tilde{H}$. Next, we list certain results, which are essential for discussion. The following result provides an expression for reproducing kernel in terms of an orthonormal basis of functional Hilbert space H.

Theorem 2.3. [12] *Let $\{e_j\}$ be an orthonormal basis for functional Hilbert space H over Y, then the function K on $Y \times Y$ is defined by*

$$K_y(x) = \sum_n e_n(x)e_n^*(y).$$

Here, $K(x, y) = K_y(x)$ is known as reproducing kernel of H.

Utilizing the above discussion, we can define a function $S_w : \mathbb{D} \to \mathbb{C}$ given by

$$S_w(z) = \sum_m e_m(z)e_m^*(w) = \sum_m z^m \bar{w}^m, \tag{2.1}$$

which acts as a reproducing kernel for $L^2(\mathbb{T})$. The following result is due to Avendaño, which can be written as:

Proposition 2.4. [4] *Let $A \in \mathfrak{B}(H^2)$ with $\|A\| < 1$, where $\mathfrak{B}(H^2)$ is the Hilbert space consisting of bounded linear operators on the Hardy space. Then an operator $X \in \mathfrak{B}(H^2)$ satisfies the operator equation $AX = XU$ if and only if X is compact and has the form $X = \sum_{n=0}^{\infty}(A^n\phi) \otimes e_n$, where $\phi \in H^2(\mathbb{D})$ and an orthonormal basis for $H^2(\mathbb{D})$ is (e_n).*

Now, we are in a position to introduce an operator equation, which gives rise another generalization of slant Hankel operator, namely (λ, μ)- slant Hankel operator.
Let λ, μ be two fixed complex numbers, consider an operator equation

$$\mu M_z^* X - X M_{z^2} = \lambda X. \tag{2.2}$$

Here, we try to solve this operator equation completely, that is, we search for an operator X on $L^2(\mathbb{T})$, which satisfies the operator equation (2.2) for a choice of complex numbers λ, μ. The solution $X \in \mathfrak{B}(L^2)$ of the operator equation (2.2) is known as (λ, μ)- slant Hankel operator. One may consider other symmetric form $\mu M_z^* X - \lambda X M_{z^2} = X$ of operator equation appearing in (2.2). It is trivial to observe that the operator equation $\mu M_z^* X - \lambda X M_{z^2} = X$ has only trivial solution for the case when $\lambda = \mu = 0$. But, on the other hand, equation (2.2) has non-trivial solution under the same condition $\lambda = \mu = 0$. It confirms that operator equation (2.2) has wider range of non-trivial solutions in compared to $\mu M_z^* X - \lambda X M_{z^2} = X$.
The particular choices to the scalars λ and μ give rise operator equations, which characterize different other well-known generalizations of slant Hankel operator.

(1) The pair $(\lambda, \mu) = (0, \lambda)$ gives $\lambda M_z^* X - X M_{z^2} = 0$, which characterizes λ-slant Hankel operator.
(2) The pair $(\lambda, \mu) = (\lambda, 1)$ gives $M_z^* X - X M_{z^2} = \lambda X$, which characterizes another form λ- slant Hankel operator.
(3) The pair $(\lambda, \mu) = (0, 1)$ gives $M_z^* X - X M_{z^2} = 0$, which characterizes slant Hankel operator.

Now, we take up an operator equation $\mu M_{z^2} X - X M_z^* = \lambda X$ and provide the solution of this operator equation.

Theorem 2.5. *The operator equation $\mu M_{z^2} X - X M_z^* = \lambda X$ has only trivial solution for any choice of complex numbers λ and μ.*

Proof. The operator equation $\mu M_{z^2} X - X M_z^* = \lambda X$ can be re-written as $(\mu M_{z^2} - \lambda I)X = X M_z^*$. Let $f \in L^2(\mathbb{T})$ is in the kernel of $(\mu M_{z^2} - \lambda I)$ and, let f can be expressed as $f(z) = \sum_{m\in\mathbb{Z}} a_m z^m$. Then, we get

$$\begin{aligned} (\mu M_{z^2} - \lambda I) f(z) &= 0 \\ (\mu M_{z^2} - \lambda I)\left(\sum_{m\in\mathbb{Z}} a_m z^m\right) &= 0 \\ \sum_{m\in\mathbb{Z}} \mu a_m z^{m+2} &= \sum_{m\in\mathbb{Z}} \lambda a_m z^m. \end{aligned}$$

On replacing m by $m-2$ on the left hand side of preceding expression we get,

$$\sum_{m\in\mathbb{Z}} \mu a_{m-2} z^m = \sum_{m\in\mathbb{Z}} \lambda a_m z^m.$$

On comparing the coefficients of z^m in the above expression, we have

$$\lambda a_m = \mu a_{m-2}. \tag{2.3}$$

Now, we may consider different cases, namely,
Case I: When $\lambda = 0$ but $\mu \neq 0$, then the equation (2.3) gives that $a_{m-2} = 0$ for all $m \in \mathbb{Z}$. It implies $a_m = 0$ for all $m \in \mathbb{Z}$ and hence $f = 0$.
Case II: When $\lambda \neq 0$ but $\mu = 0$, then the equation (2.3) yields that $a_m = 0$ for all $m \in \mathbb{Z}$, which gives $f = 0$.
Case III: When both $\lambda \neq 0$ and $\mu \neq 0$, then iteratively, equation (2.3) reduces to $a_{2k} = c^k a_0$, for $k \in \mathbb{Z}$ and $a_{2l+1} = c^l a_1$, for $l \in \mathbb{Z}$, where $c = \frac{\mu}{\lambda}$. Since $f \in L^2(\mathbb{T})$ therefore $\sum_{m\in\mathbb{Z}} |a_m|^2 < \infty$. This implies that the series

$$\sum_{m\in\mathbb{Z}} |a_{2m}|^2 \quad \text{and} \quad \sum_{m\in\mathbb{Z}} |a_{2m+1}|^2$$

are convergent. It is possible only if $a_0 = 0$ and $a_1 = 0$, which yield that $a_m = 0$ for all $m \in \mathbb{Z}$. Thus, in the view of all above cases, we have $\ker(\mu M_{z^2} - \lambda I) = \{0\}$. Now, by Theorem 2.1 of [4], we conclude that $X = 0$. In the case when $(\lambda, \mu) = (0, 0)$, the operator equation $\mu M_{z^2} X - X M_z^* = \lambda X$ reduces to $X M_z^* = 0$. It trivially gives that $X = 0$. □

Now, we move towards our main aim and consider the operator equation $\mu M_z^* X - XM_{z^2} = \lambda X$. As we know that for $\lambda = 0$, the equation $\mu M_z^* X - XM_{z^2} = \lambda X$ provides μ- slant Hankel operator for $\mu \in \mathbb{C}$, which is already discussed by Datt and Aggarwal [8]. If we take $\lambda = 0$ and $\mu = 0$ in the preceding operator equation then we have only trivial solution. In the next result, we search for the solution of operator equation

$$\mu M_z^* X - XM_{z^2} = \lambda X, \tag{2.4}$$

when $\mu = 0$.

Theorem 2.6. *If $\mu = 0$, then the operator equation (2.4) has only trivial solution for any choice of complex number λ.*

Proof. Under the assumption, the operator equation (2.4) can be written as $XM_{z^2} + \lambda X = 0$, that is,

$$X(M_{z^2} + \lambda I) = 0. \tag{2.5}$$

Since, we know that the spectrum of multiplication operator M_{z^2} is $\{z \in \mathbb{C} : |z| = 1\}$. Therefore, for $|\lambda| \neq 1$, the operator $(M_{z^2} + \lambda I)$ is an invertible operator and hence the equation (2.5) yields that X is the zero operator. In the case when $|\lambda| = 1$, equation (2.5) can be re-written as $X = cXM_{\bar{z}^2}$, where $c = -\lambda$. It gives that $X(z^n) = cX(z^{n-2})$ for all $n \in \mathbb{Z}$. Iteratively, this provides relations in terms of $1(= e_0)$ and $z(= e_1)$, that is,

$$X(z^{2k}) = c^k X(1) \quad \text{and} \quad X(z^{2k+1}) = c^k X(z), \text{ for } k \in \mathbb{Z}.$$

If possible assume that at least one of $X(1)$ and $X(z)$ is non-zero. Without lose of generality, consider $X(z) \neq 0$, therefore there exist at least one non-zero Fourier coefficient a_k of $X(z)$ for some $k \in \mathbb{Z}$. If we take a sequence of elements appearing in the k^{th} row of the matrix of X. Then it does not converge to 0, because this sequence contains a subsequence (namely, having elements appearing at odd positions in the k^{th} row of the matrix of X), which does not converge to 0. It is a contradiction to the fact that the operator X is bounded. This implies that both functions $X(1)$ and $X(z)$ are zero. Therefore, $X(z^n) = 0$, for $n \in \mathbb{Z}$ and hence $X = 0$. This completes the proof. □

Next, we investigate a result, which is analogous to Proposition 2.4, for the Lebesgue space $L^2(\mathbb{T})$.

Lemma 2.7. *Let $A \in \mathfrak{B}(L^2(\mathbb{T}))$ with $\|A\| < 1$ and $\phi \in L^\infty(\mathbb{T})$ such that $\|\phi\|_\infty < 1$. Then the operator equation $AX = XM_\phi$ has only trivial solution.*

Proof. The given operator equation can be written as $X = AXM_{\bar{\phi}}$, which gives that

$$\begin{aligned} \|X\| &= \|AXM_{\bar{\phi}}\| \\ &\leq \|A\|\|X\|\|M_{\bar{\phi}}\| \\ &< \|X\|. \end{aligned}$$

It means that $\|X\| = 0$ and $X = 0$. □

An immediate consequence of the preceding result is the following.

Corollary 2.8. *If* $|\lambda|+|\mu|<1$. *Then the only solution of the operator equations* $\mu M_z^* X - XM_{z^2} = \lambda X$ *and* $\mu M_z X - XM_{z^2} = \lambda X$ *is the zero solution.*

Proof. Let X be an operator on $L^2(\mathbb{T})$, which is bounded and satisfies the operator equation $\mu M_z^* X - XM_{z^2} = \lambda X$ and $\mu M_z X - XM_{z^2} = \lambda X$, that is, $(\mu M_z^* - \lambda I)X = XM_{z^2}$ and $(\mu M_z - \lambda I)X = XM_{z^2}$. Since $\|\mu M_z^* - \lambda I\| < |\lambda| + |\mu| < 1$. Similarly, $\|\mu M_z - \lambda I\| < 1$. Therefore, in the view of preceding result, we can conclude that $X = 0$ is the only solution for both the equations. □

It is easy to see that the zero operator satisfies both the operator equations $\mu M_z^* X - XM_{z^2} = \lambda X$ and $\mu M_z X - XM_{z^2} = \lambda X$ for any choice of μ and λ. In the next result, we look for pairs (λ, μ) and (α, β) of complex numbers for which a (λ, μ)- slant Hankel operator is also a (α, β)- slant Hankel operator.

Theorem 2.9. *A* (λ, μ)*- slant Hankel operator* X *is also a* (α, β)*- slant Hankel operator if and only if* $X = 0$.

Proof. Let X be an operator on $L^2(\mathbb{T})$, which is (λ, μ) as well as (α, β) slant Hankel operator. Then $\mu M_z^* X - XM_{z^2} = \lambda X$ and $\beta M_z^* X - XM_{z^2} = \alpha X$. On subtracting the preceding expression, we get that $\delta M_z^* X = \gamma X$, where $\delta = \mu - \beta$ and $\gamma = \lambda - \alpha$. It gives that $(M_z^* - \frac{\gamma}{\delta} I)X = 0$. Now, we consider two cases.
Case: I When $|\gamma| \neq |\delta|$, that is, $|\lambda - \alpha| \neq |\mu - \beta|$. Then we consider three subcases.
Subcase: (i) When $\gamma = 0$ but $\delta \neq 0$, equation reduces to $M_z^* X = 0$. It gives that X=0.
Subcase: (ii) When $\gamma \neq 0$ but $\delta = 0$. Then, it is obvious that X=0.
Subcase: (iii) When $\gamma \neq 0$ but $\delta \neq 0$. Then, $|\frac{\gamma}{\delta}| \neq 1$. Therefore, the operator $(M_z^* - \frac{\gamma}{\delta} I)$ is invertible operator and hence $X = 0$.
Case: II When $|\gamma| = |\delta| \neq 0$, that is, $|\lambda - \alpha| = |\mu - \beta| \neq 0$. In this case, the equation $\delta M_z^* X = \gamma X$ can be re-written as $X^* M_z = cX^*$, where $c = \overline{\left(\frac{\gamma}{\delta}\right)}$. This implies that $X^*(z^n) = c^n X^*(1)$ for all $n \in \mathbb{Z}$. By utilizing the technique used in Theorem 2.6, we obtain $X^* = 0$ and hence $X = 0$. The converse part of the theorem is trivial. This completes the proof. □

Now, we discuss the specific property (namely, closedness) of the collection $(\lambda, \mu) - SHank(L^2)$ of all (λ, μ)- slant Hankel operators on $L^2(\mathbb{T})$.

Theorem 2.10. *The collection* $(\lambda, \mu) - SHank(L^2)$ *is a linear subspace of* $\mathfrak{B}(L^2(\mathbb{T}))$*, which is uniform closed subspace of* $\mathfrak{B}(L^2(\mathbb{T}))$.

Proof. Let X_1 and X_2 be any two (λ, μ)- slant Hankel operator on $L^2(\mathbb{T})$ and $\alpha, \beta \in \mathbb{C}$. Then, $\mu M_z^* X_1 - X_1 M_{z^2} = \lambda X_1$ and $\mu M_z^* X_2 - X_2 M_{z^2} = \lambda X_2$. Now, consider the operator $\mu M_z^*(\alpha X_1 + \beta X_2) - (\alpha X_1 + \beta X_2)M_{z^2}$, which can be written as

$$\begin{aligned} \mu M_z^*(\alpha X_1 + \beta X_2) - (\alpha X_1 + \beta X_2)M_{z^2} & \\ &= \alpha(\mu M_z^* X_1 - X_1 M_{z^2}) + \beta(\mu M_z^* X_2 - X_2 M_{z^2}) \\ &= \alpha\lambda X_1 + \beta\lambda X_2 \\ &= \lambda(\alpha X_1 + \beta X_2). \end{aligned}$$

It immediately gives that $\alpha X_1+\beta X_2$ is a (λ,μ)- slant Hankel operator and hence the collection $(\lambda,\mu)-SHank(L^2)$ is a subspace of $\mathfrak{B}(L^2(\mathbb{T}))$.
Now, let $(X_m)_{m\geq 0}$ is a sequence of (λ,μ)- slant Hankel operators on $L^2(\mathbb{T})$ converging uniformly to an operator X. Then the operator norm $\|X_m - X\| \to 0$. Consider the norm

$$\begin{aligned}\|\mu M_z^* X_m - X_m M_{z^2} - \mu M_z^* X + X M_{z^2}\| &= \|\mu M_z^*(X_m - X) - (X_m - X)M_{z^2}\| \\ &\leq |\mu|\|M_z^*\|\|X_m - X\| + \|X_m - X\|\|M_{z^2}\|.\end{aligned}$$

Since the multiplication operator is a bounded operator and $\|X_m - X\| \to 0$. Therefore, the above expression gives that $\|(\mu M_z^* X_m - X_m M_{z^2}) - (\mu M_z^* X - X M_{z^2})\| \to 0$, that is $(\mu M_z^* X_m - X_m M_{z^2}) \to (\mu M_z^* X - X M_{z^2})$ in uniform topology of $\mathfrak{B}(L^2(\mathbb{T}))$. It yields that $\lambda X_m \to \mu M_z^* X - X M_{z^2}$, but we know that $\lambda X_m \to \lambda X$ in uniform topology of $\mathfrak{B}(L^2(\mathbb{T}))$. Therefore, by the uniqueness of limit, we have $\mu M_z^* X - X M_{z^2} = \lambda X$. This implies that $X \in (\lambda,\mu)-SHank(L^2)$ and hence the collection $(\lambda,\mu)-SHank(L^2)$ is uniformly closed subspace of $\mathfrak{B}(L^2(\mathbb{T}))$. □

In the next result, we provide a unitary operator, which depicts the connection amongst (λ,μ)- slant Hankel operators corresponding to different pairs (λ,μ) of unimodular complex numbers.

Theorem 2.11. *For a given $X \in (\lambda,\mu)-SHank(L^2)$, the operator $D_{\frac{1}{\bar{\alpha}\beta\lambda\bar{\mu}}} X D_{\frac{1}{\sqrt{\bar{\alpha}\lambda}}}$ is (α,β)- slant Hankel operator, where α,β,λ and μ are unimodular complex numbers and D_a is a unitary diagonal operator given by $D_a(e_n) = a^n e_n$ for all $n \in \mathbb{Z}$.*

Proof. With the help of definition of diagonal and multiplication operator, it is easy to show that $D_{\frac{1}{a}} M_z^*(e_n) = \frac{1}{a^{n-1}} e_{n-1} = a M_z^*\left(\frac{1}{a^n} e_n\right) = a M_z^* D_{\frac{1}{a}}(e_n)$ and $M_{z^2} D_{\frac{1}{a}}(e_n) = \frac{1}{a^n} e_{n+2} = a^2 \frac{1}{a^{n+2}} e_{n+2} = a^2 D_{\frac{1}{a}} M_{z^2}(e_n)$. It yields that $D_{\frac{1}{a}} M_z^* = a M_z^* D_{\frac{1}{a}}$ and $M_{z^2} D_{\frac{1}{a}} = a^2 D_{\frac{1}{a}} M_{z^2}$.
Now consider the expression $D_{\frac{1}{\bar{\alpha}\beta\lambda\bar{\mu}}}(\mu M_z^* X - X M_{z^2}) D_{\frac{1}{\sqrt{\bar{\alpha}\lambda}}}$. In the view of above relation, the expression can be written as

$$\begin{aligned} D_{\frac{1}{\bar{\alpha}\beta\lambda\bar{\mu}}}(\mu M_z^* X - X M_{z^2}) D_{\frac{1}{\sqrt{\bar{\alpha}\lambda}}} &= D_{\frac{1}{\bar{\alpha}\beta\lambda\bar{\mu}}}(\mu M_z^* X) D_{\frac{1}{\sqrt{\bar{\alpha}\lambda}}} - D_{\frac{1}{\bar{\alpha}\beta\lambda\bar{\mu}}}(X M_{z^2}) D_{\frac{1}{\sqrt{\bar{\alpha}\lambda}}} \\ &= \bar{\alpha}\lambda\left(\beta M_z^* D_{\frac{1}{\bar{\alpha}\beta\lambda\bar{\mu}}} X D_{\frac{1}{\sqrt{\bar{\alpha}\lambda}}} - D_{\frac{1}{\bar{\alpha}\beta\lambda\bar{\mu}}} X D_{\frac{1}{\sqrt{\bar{\alpha}\lambda}}} M_{z^2}\right).\end{aligned}$$

Since X is a (λ,μ)- slant Hankel operator therefore $\mu M_z^* X - X M_{z^2} = \lambda X$. Pre and post-multiplication to the equation $\mu M_z^* X - X M_{z^2} = \lambda X$ by operators $D_{\frac{1}{\bar{\alpha}\beta\lambda\bar{\mu}}}$ and $D_{\frac{1}{\sqrt{\bar{\alpha}\lambda}}}$ respectively gives that $\bar{\alpha}\lambda(\beta M_z^* Y - Y M_{z^2}) = \lambda Y$, where $Y = D_{\frac{1}{\bar{\alpha}\beta\lambda\bar{\mu}}} X D_{\frac{1}{\sqrt{\bar{\alpha}\lambda}}}$. Consequently, we obtain $\beta M_z^* Y - Y M_{z^2} = \alpha Y$. Thus the result follows. □

In the following result, it can be seen that the class $(\lambda,\mu)-SHank(L^2)$ is invariant under the composition of multiplication operator.

Theorem 2.12. *Let X be a (λ,μ)- slant Hankel operator and M be a multiplication operator on $L^2(\mathbb{T})$. Then, $XM \in (\lambda,\mu)-SHank(L^2)$.*

Proof. Let X and M be a (λ,μ)- slant Hankel operator and a multiplication operator on $L^2(\mathbb{T})$, repectively. We know that $M_\phi M_\psi = M_\psi M_\phi$ and $\mu M_z^* X - X M_{z^2} =$

λX. Therefore, in the view of preceding relation, expression $\mu M_z^*(XM)-(XM)M_{z^2}$ can be written as

$$\begin{aligned}\mu M_z^*(XM)-(XM)M_{z^2} &= (\mu M_z^*X - XM_{z^2})M \\ &= \lambda XM.\end{aligned}$$

Similarly, we obtain that $\mu M_z^*(MX)-(MX)M_{z^2} = M(\mu M_z^*X - XM_{z^2}) = \lambda MX$. These expressions trivially gives that XM and MX are (λ,μ)- slant Hankel operators. □

Next, we try to deduce a spectral property of (λ,μ)- slant Hankel operator, which utilizes following result.

Theorem 2.13. *Every (λ,μ)- slant Hankel operator X is neither invertible nor essentially invertible.*

Proof. Let $X \in (\lambda,\mu) - SHank(L^2)$. If $X = 0$ then it is obvious that X is neither invertible nor essentially invertible. Now, assume that $X \neq 0$, it satisfies $XM_{z^2} = (\mu M_z^* - \lambda I)X$. It yields that M_{z^2} and $(\mu M_z^* - \lambda I)$ are similar operators if X is invertible, which is not true. Hence X can not be invertible.
Next assume that X is essentially invertible then the expression $XM_{z^2} = (\mu M_z^* - \lambda I)X$ gives that the operators M_{z^2} and $(\mu M_z^* - \lambda I)$ are essentially similar operators under the assumption that X is essentially invertible. Obviously, It is not possible. Therefore X can not be essentially invertible. Hence the result follows. □

An immediate consequence of preceding result is the following.

Corollary 2.14. *For a (λ,μ)- slant Hankel operator X, the complex number 0 is a common point in both spectrum $\sigma(X)$ and essential spectrum $\sigma_\epsilon(X)$ of X.*

It is trivial to observe that the adjoint of an operator $X \in (\lambda,\mu) - SHank(L^2)$ is a (α,β)- slant Hankel operator if it holds the condition $XM_{\bar{\beta}z+z^2} + [M_{\mu\bar{z}+\bar{z}^2} - (\bar{\alpha} - \lambda)I]X = 0$. The following result provides invariant subspaces (given by (λ,μ)- slant Hankel operator) for multiplication operator with specific inducing symbols.

Theorem 2.15. *If X is a $(\lambda,\mu)-$ slant Hankel operator. Then kernel $\ker X$ of X and closure $\overline{RanX}$ of range of X are invariant subspaces of M_{z^2} and M_z^* respectively.*

Proof. Let X be a $(\lambda,\mu)-$ slant Hankel operator. Then, it satisfies $\mu M_z^*X - XM_{z^2} = \lambda X$, which can be written as $(\mu M_z^* - \lambda I)X = XM_{z^2}$. Now let $f \in \ker X$, then preceding expression gives $XM_{z^2}(f) = (\mu M_z^* - \lambda I)X(f) = 0$. It implies that $M_{z^2}(f) \in \ker X$ and hence $M_{z^2}(\ker X) \subseteq \ker X$.
For another part, let X satisfies $\mu M_z^*X - XM_{z^2} = \lambda X$. Taking adjoint on the both side of the preceding equation, we get

$$\bar{\mu}X^*M_z - M_{z^2}^*X^* = \bar{\lambda}X^*,$$

which can be re-written as $\bar{\mu}X^*M_z = (M_{z^2}^* + \bar{\lambda}I)X^*$. Using the same technique as in the part first, we get $M_z(\ker X^*) \subseteq \ker X^*$. Since we know that $\ker(X^*)^\perp = \overline{RanX}$, therefore we conclude that $\overline{RanX}$ is an invariant subspace of M_z^*. □

We now return to our main aim of investigation. In the following result, we search for complex numbers λ and μ for which the equation $\mu M_z^*X - XM_{z^2} = \lambda X$ admits solution (mainly, non-trivial solution).

Theorem 2.16. *The operator X is a non-zero solution of operator equation $\mu M_z^*X - XM_{z^2} = \lambda X$ if $|\mu| \neq 0$, and $|\lambda| < 1+|\mu|$. Moreover, it has only trivial solution for the following cases:*

(1) $|\mu| < 1$ *and* $|\lambda| > 1+|\mu|$;
(2) $|\mu| > 1$ *and* $|\lambda| > 1+|\mu|$.

Proof. We know that for given $\mu \neq 0$, $\lambda \in \mathbb{C}$ and $\omega \in \mathbb{D}$, the complex number $\mu\omega - \lambda \in \mathbb{D}$ if and only if $|\lambda| < 1+|\mu|$. Therefore, under the assumption $|\mu| \neq 0$, and $|\lambda| < 1+|\mu|$, it is always possible to choose a complex number $\omega \in \mathbb{D}$ such that the complex number $\mu\omega-\lambda$ is again in open unit disc $\mathbb{D}$. Using the function S_ω, given by (2.1), define an operator $X = S_{\bar\omega} \otimes S_{\sqrt{\mu\omega-\lambda}}$, which is of rank one operator. As we know that $T(f\otimes h)S = (Tf)\otimes(S^*h)$ for each $f,h \in L^2(\mathbb{T})$. Therefore, the expression $\mu M_z^*X - XM_{z^2}$ reduces to

$$\begin{aligned}\mu M_z^*X - XM_{z^2} &= \mu M_z^*(S_{\bar\omega}\otimes S_{\sqrt{\mu\omega-\lambda}}) - (S_{\bar\omega}\otimes S_{\sqrt{\mu\omega-\lambda}})M_{z^2}\\ &= \mu(M_z^*S_{\bar\omega}\otimes S_{\sqrt{\mu\omega-\lambda}}) - (S_{\bar\omega}\otimes M_{z^2}^*S_{\sqrt{\mu\omega-\lambda}}).\end{aligned}$$

One can easily verify that $M_z^*S_{\bar\omega} = \omega S_{\bar\omega}$ and $M_{z^2}^*S_{\bar\omega} = \omega^2 S_{\bar\omega}$ for $\omega \in \mathbb{D}$. Utilizing this fact, the preceding expression reduces to

$$\begin{aligned}\mu M_z^*X - XM_{z^2} &= \mu\omega(S_{\bar\omega}\otimes S_{\sqrt{\mu\omega-\lambda}}) - (\mu\omega-\lambda)(S_{\bar\omega}\otimes S_{\sqrt{\mu\omega-\lambda}})\\ &= \lambda(S_{\bar\omega}\otimes S_{\sqrt{\mu\omega-\lambda}}) = \lambda X.\end{aligned}$$

It gives that X is a non-trivial solution of $\mu M_z^*X - XM_{z^2} = \lambda X$ if $|\mu| \neq 0, |\lambda| < 1+|\mu|$. In particular, X is a non-zero operator satisfying $\mu M_z^*X - XM_{z^2} = \lambda X$ under the condition $0 < |\mu| < 1, |\lambda| < 2|\mu|$. Now, we consider the cases:
Case: 1 Let λ,μ be two complex numbers such that $|\mu| < 1$ and $|\lambda| > 1+|\mu|$. Define an operator $T_\mu : \mathfrak{B}(L^2(\mathbb{T})) \to \mathfrak{B}(L^2(\mathbb{T}))$ by

$$T_\mu(X) = \mu M_z^*X - XM_{z^2},$$

for each $X \in \mathfrak{B}(L^2(\mathbb{T}))$. From the preceding discussion, it is clear that each complex number λ satisfying $|\lambda| < 2|\mu|$ is an eigen value of the operator T_μ. Therefore, the set $\{z\in\mathbb{C} : |\lambda| < 2|\mu|\}$ is contained in spectrum $\sigma(T_\mu) \subseteq \{z \in \mathbb{C} : |\lambda| < 1+|\mu|\}$ of T_μ. It implies that complex numbers satisfying $|\lambda| > 1+|\mu|$ is not in $\sigma(T_\mu)$ and hence $\mu M_z^*X - XM_{z^2} = \lambda X$ has only trivial solution.
Case: 2 Let $|\mu| > 1$ and $|\lambda| > 1+|\mu|$. Again, for $|\mu| > 1$, define an operator $T_{\frac{1}{\mu}} : \mathfrak{B}(L^2(\mathbb{T})) \to \mathfrak{B}(L^2(\mathbb{T}))$ defined as $T_{\frac{1}{\mu}}(Y) = M_z^*Y - \frac{1}{\mu}YM_{z^2}$ for each $Y \in \mathfrak{B}(L^2(\mathbb{T}))$. Since $\|T_{\frac{1}{\mu}}\| \leq 1 + \frac{1}{|\mu|}$, therefore

$$\sigma(T_{\frac{1}{\mu}}) \subseteq \{z \in \mathbb{C} : |z| \leq 1 + \frac{1}{|\mu|}\}.$$

Under the assumption $|\lambda| > 1+|\mu|$, that is, $|\frac{\lambda}{\mu}| > 1 + \frac{1}{|\mu|}$, the operator $\left(T_{\frac{1}{\mu}} - \frac{\lambda}{\mu}I\right)$ is invertible. Hence, we conclude that $\mu M_z^*Y - YM_{z^2} = \lambda Y$ again has only trivial solution. □

An immediate consequence of the Theorem 2.16 can be derived as follows.

Corollary 2.17. *The equation $\mu M_z^* X - X M_{z^2} = \lambda X$ has only trivial solution under the condition $|\mu| < 1$, $|\lambda| \geq 2$ as well as $|\mu| > 1$, $|\lambda| \geq 2|\mu|$.*

Proof. Let $\mu \in \mathbb{C}$ be such that $|\mu| < 1$, then $|\lambda| \geq 2$ gives $|\lambda| \geq 2 > 1 + |\mu|$. Similarly, for $|\mu| > 1$, the relation $|\lambda| \geq 2|\mu|$ gives that $|\lambda| \geq 2|\mu| > 1 + |\mu|$. Thus, the complex numbers μ and λ satisfy the conditions of case (1) and case (2) of the above theorem, which yields the trivial solution of considered operator equation. □

Using the same technique as in the preceding corollary, the following result can be derived from the above theorem.

Corollary 2.18. *The equation $\mu M_z^* X - X M_{z^2} = \lambda X$ has only non-trivial solution under the following condition.*

(1) $|\mu| < 1$ *and* $|\lambda| \leq 2|\mu|$
(2) $|\mu| > 1$ *and* $|\lambda| \leq 2$

Proof. Let $\mu \in \mathbb{C}$ be such that $|\mu| < 1$, then $|\lambda| \leq 2|\mu|$ gives $|\lambda| \leq 2|\mu| < 1 + |\mu|$. Next, consider a complex number μ such that $|\mu| > 1$, the relation $|\lambda| \leq 2$ gives that $|\lambda| \leq 2 < 1 + |\mu|$. This shows that the complex numbers μ and λ satisfy the condition of the Theorem 2.16. Therefore, the operator equation $\mu M_z^* X - X M_{z^2} = \lambda X$ has non-trivial solution. □

Now, we discuss a result related to (λ, μ)- slant Hankel operator analogous to classical Kronecker Theorem for Hankel operator, which says that "A matrix is a Hankel matrix with finite rank if and only if it represents a Hankel operator H_ϕ, whose inducing function is a rational function". Here, we are relaxing the boundedness condition on X satisfying $\mu M_z^* X - X M_{z^2} = \lambda X$.

Theorem 2.19. *The matrix representation of (λ, μ)- slant Hankel operator X, relative to orthonormal basis $\{e_n\}$ of the Lebesgue space $L^2(\mathbb{T})$, is of finite rank if and only if the functions Xe_0 and Xe_1 are rational functions.*

Proof. Assume that $X \in (\lambda, \mu) - SHank(L^2)$. Then $\mu M_z^* X - X M_{z^2} = \lambda X$, which can be re-written as $(\mu M_z^* - \lambda I)X = X M_{z^2}$. Using it repeatedly, we have

$$\begin{aligned} X(e_{2n}) &= X M_{z^2}^n (e_0) \\ &= (\mu M_z^* - \lambda I) X M_{z^2}^{n-1} (e_0) \\ &= (\mu M_z^* - \lambda I)^2 X M_{z^2}^{n-2} (e_0) \\ &\quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ &= (\mu M_z^* - \lambda I)^n X (e_0). \end{aligned}$$

Similarly, we can obtain that $X(e_{2n+1}) = X M_{z^2}^n (e_1) = (\mu M_z^* - \lambda I)^n X(e_1)$. It implies that X can be determined by functions $Xe_0 = \phi_0$ and $Xe_1 = \phi_1$ in a unique way. Utilizing the above relation, the expression $\langle X(e_{2m}), e_n \rangle$ can be written as

$$\begin{aligned}\langle X(e_{2m}), e_n\rangle &= \langle XM^m_{z^2}(e_0), e_n\rangle \\ &= \langle(\mu M^*_z - \lambda I)^m X(e_0), e_n\rangle \\ &= \langle X(e_0), (\bar{\mu} M_z - \bar{\lambda} I)^m e_n\rangle \\ &= \sum_{k=0}^{m} \binom{m}{k} (-\lambda)^{m-k} \mu^k \langle X(e_0), M^k_z(e_n)\rangle \\ &= \sum_{k=0}^{m} \binom{m}{k} (-\lambda)^{m-k} \mu^k \langle X(e_0), e_{n+k}\rangle.\end{aligned}$$

In a similar way, we obtain that $\langle X(e_{2m+1}), e_n\rangle = \sum_{k=0}^{m} \binom{m}{k}(-\lambda)^{m-k}\mu^k \langle X(e_1), e_{n+k}\rangle$. The relations $X(e_{2n}) = (\mu M^*_z - \lambda I)^n X(e_0)$ and $X(e_{2n+1}) = (\mu M^*_z - \lambda I)^n X(e_1)$ give that the odd and even columns of the matrix representation of X relative to orthonormal basis are the co-ordinate vectors corresponding to $(\mu M^*_z - \lambda I)^n X(e_1)$ and $(\mu M^*_z - \lambda I)^n X(e_0)$, respectively. Therefore, the rank of a matrix representation of X is $m = N + M < \infty$ if and only if there exists $m + 1$ constants $a_p, 0 \leq p \leq N$ and $b_q, 0 \leq q \leq M - 1$, not all zero such that

$$\sum_{p=0}^{N} a_p(\mu M^*_z - \lambda I)^p \phi_0 + \sum_{q=0}^{M-1} b_q(\mu M^*_z - \lambda I)^q \phi_1 = 0. \tag{2.6}$$

A simple calculation gives that

$$\begin{aligned}\sum_{p=0}^{N} a_p(\mu M^*_z - \lambda I)^p \phi_0 &= \sum_{p=0}^{N} a_p \sum_{k=0}^{p} \binom{p}{k} (-\lambda)^{p-k} \mu^k M^{*k}_z \phi_0 \\ &= \sum_{k=0}^{N} \sum_{p=k}^{N} a_p \binom{p}{k} (-\lambda)^{p-k} \mu^k M^{*k}_z \phi_0. \qquad (2.7)\end{aligned}$$

In the similar fashion, we get that

$$\begin{aligned}\sum_{q=0}^{M-1} b_q(\mu M^*_z - \lambda I)^q \phi_1 &= \sum_{q=0}^{M-1} b_q \sum_{k=0}^{q} \binom{q}{k} (-\lambda)^{q-k} \mu^k M^{*k}_z \phi_1 \\ &= \sum_{k=0}^{M-1} \sum_{q=k}^{M-1} b_q \binom{q}{k} (-\lambda)^{q-k} \mu^k M^{*k}_z \phi_1. \qquad (2.8)\end{aligned}$$

Using the equations (2.7) and (2.8), we have

$$\sum_{p=0}^{N} a_p(\mu M^*_z - \lambda I)^p \phi_0 + \sum_{q=0}^{M-1} b_q(\mu M^*_z - \lambda I)^q \phi_1 = \sum_{k=0}^{N} c_k M^{*k}_z \phi_0 + \sum_{k=0}^{M-1} e_k M^{*k}_z \phi_1 \tag{2.9}$$

where $c_k = \sum_{p=k}^{N} a_p \binom{p}{k}(-\lambda)^{p-k}\mu^k$ and $e_k = \sum_{q=k}^{M-1} b_q \binom{q}{k}(-\lambda)^{q-k}\mu^k$. With the help of equations (2.6) and (2.9), we conclude that $c_p = e_q = 0$ for all $0 \leq p \leq N$ and $0 \leq q \leq M - 1$ if and only if $a_p = b_q = 0$ for all $0 \leq p \leq N$ and $0 \leq q \leq M - 1$.

This implies that the set $S = \{(\mu M_z^* - \lambda I)^p \phi_0, (\mu M_z^* - \lambda I)^q \phi_1 : 0 \leq p \leq N$ and $0 \leq q \leq M-1\}$ is linearly independent then if the set $S_0 = \{M_z^{*p}\phi_0, M_z^{*q}\phi_1 : 0 \leq p \leq N$ and $0 \leq q \leq M-1\}$ is linearly independent and vice-versa.

A matrix having the vectors $\{M_z^{*p}\phi_0\}$ and $\{M_z^{*q}\phi_1\}$ as their columns is a Hankel operator with inducing function $(\phi_0 + \phi_1)$. As a consequence, we conclude that the Hankel operator induced by $(\phi_0 + \phi_1)$ is of finite rank operator then the rank of X is finite and vice-versa. Using the classical theorem of Kronecker for Hankel operator, we obtain that $X(e_0) = \phi_0$ and $X(e_1) = \phi_1$ are rational functions. □

3. Conclusion

Motivated by the solution of operator equation $U^*XU = \lambda$, proposed by Barria and Halmos [5] and solved by Sun [19], Avendano; Datt and Aggarwal considered certain generalizations of Hankel operator (namely, λ- Hankel operator [3]; (λ, μ)-Hankel operator [6, 7], λ- Slant Hankel operator [8] respectively). Therefore, in this paper, we have consider operator equation $\mu M_z^* X - X M_{z^2} = \lambda X$, which is one of the important generalizations of Hankel operator. It can be seen as an immediate generalization of λ- Slant Hankel operator. Here, we have solved equation $\mu M_z^* X - X M_{z^2} = \lambda X$ and discussed basic properties of solution (known as (λ, μ)- slant Hankel operator) of considered operator equation. Particular values to the parameters λ, μ appearing in our study will give rise certain existing famous results already obtained by mathematician working in this field. For example, If we take $\lambda = 0$, then our study reduces to the study of μ-slant Hankel operator and if we take $\lambda = 0$ and $\mu = 1$, then it will provide results related to slant Hankel operator.

Acknowledgement. The suggestions given by refree are gratefully acknowledged, which assist us to shape the paper in the present form.

References

1. Arora, S. C. and Batra, R. (2005). Generalized slant Toeplitz operators on H^2. Math. Nachr., 278(4), 347-355. https://doi.org/10.1002/mana.200310244
2. Arora, S. C., Batra, R. and Singh, M. P. (2006). Slant Hankel Operators. Archivum Mathematicum (BRNO) Tomus, 42, 125-133. https://www.emis.de/journals/AM/06-2/am1279.pdf
3. Avendano, R. A. M. (2000). Hankel Operators and Generalizations. Ph. D. Thesis, University of Toronto, Canada. https://tspace.library.utoronto.ca/bitstream/1807/13466/1/NQ53781.pdf
4. Avendano, R. A. M. (2002). A Generalization of Hankel Operators. Journal of Functional Analysis, 190, 418–446. https://doi.org/10.1006/jfan.2001.3869
5. Barria, J. and Halmos, P. R. (1982). Asymptotic Toeplitz Operators. Trans. Amer. Math. Soc., 273(2), 621-630. https://doi.org/10.2307/1999932
6. Datt, G. and Aggarwal, R. (2013). On a Generalization of Hankel Operators via Operator Equations. Extracta Mathematicae, 28(2), 197- 211. https://matematicas.unex.es/~extracta/Vol-28-2/28J2Datt.pdf
7. Datt, G. and Aggarwal, R. (2014). Essentially (λ, μ)- Hankel Operator. Functional Analysis, Approximation and Computation, 6(1), 35-40. http://operator.pmf.ni.ac.rs/www/pmf/publikacije/faac/2014/2014-6-1/faac-6-1-3.pdf

8. Datt, G. and Aggarwal, R. (2016). A Note on the Operator Equation Generalizing the Notion of Slant Hankel Operators. Anal. Theory Appl., 32(4), 387-395. https://doi.org/10.4208/ata.2016.v32.n4.6
9. Datt, G. and Mittal, A. (2017). Essentially Generalized λ-Slant Hankel Operators. Gulf Journal of Mathematics, 5(3), 70-78. https://doi.org/10.56947/gjom.v5i3.110
10. Datt, G. and Porwal, D. K. (2013). On Weighted Slant Hankel operators. Filomat, 27(2), 227–243. https://doi.org/10.2298/FIL1302227D
11. Gu, C., Łanucha, B. and Michalska, M. (2018). Characterizations of Asymmetric Truncated Toeplitz and Hankel Operators. Complex Analysis and Operator Theory. https://doi.org/10.1007/s11785-018-0783-8
12. Halmos, P. R. (1979), Hilbert Space Problem Book. Springer-Verlag New York. https://csclub.uwaterloo.ca/~pbarfuss/Paul%20R.%20Halmos%20-%20A%20Hilbert%20Space%20Problem%20Book%20(1982)%20%5B978-1-4684-9330-6%5D.pdf
13. Hankel, H. (1861). Uber eine besondere Classe der Symmetrischen Determinanten. (Leipziger) Dissertation, Gottingen.
14. Ho, M. C. (1996). Spectral properties of slant Toeplitz operators. Ph.D. thesis, Purdue-University, Indiana. https://docs.lib.purdue.edu/dissertations/AAI9713523/
15. Ho, M. C. (1997). Spectra of slant Toeplitz operators with continuous symbol. Michigan Math. J., 44, 157-166. https://doi.org/10.1307/mmj/1029005627
16. Łanucha, B. and Michalska, M.(1997). Compressions of k^{th}- order slant Toeplitz operators to the model spaces. Luthinian Mathematical Journal, 44, 157-166. https://doi.org/10.1007/s10986-021-09548-3
17. Peller, V. (2003). Hankel operators and applications. Springer-Verlag, New York.
18. Power, S. C. (1979). C^*-Algebras Generated by Hankel Operators and Toeplitz Operators. Journal of Functional Analysis, 31(1), 52-68. https://doi.org/10.1016/0022-1236(79)90097-1
19. Sun, S. H. (1984). On the operator equation $U^*TU = \lambda T$. Kexue Tongbao (English ed.), 29(3), 298-299. https://doi.org/10.1006/jfan.1997.3110
20. Verma, R. (2022). Rationalization of Toeplitz-Hankel Operators . Gulf Journal of Mathematics, 12(2), 86-96. https://doi.org/10.56947/gjom.v12i2.801
21. Zheng, D. (1997). Toeplitz operators and Hankel operators on the Hardy space of the unit sphere. Journal of Functional Analysis, 149, 1-24. https://doi.org/10.1006/jfan.1997.3110

[1] Department of Mathematics, A. N. D. K. P. G. College, Babhnan, Gonda, U. P., 271313 (affiliated to Dr. Rammanohar L. A. University, Ayodhya, Uttar Pradesh, India, 224001)

Email address: sheshkumar.1992@gmail.com

[2] Dr. Rammanohar Lohia Avadh University, Ayodhya, Uttar Pradesh, India, 224001.

Email address: truanand@gmail.com