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FINITELY BI-NORMAL RELATIONS

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Abstract. In this paper the concept of finitely bi-normal relations as finite extension of bi-normal relations is introduced. A characterization of that relations is obtained.

1. Introductions

The fundamental works of K.A.Zareckii, B.M.Schein and others on regular relations motivated several mathematicians to investigate similar classes of relations, obtained by putting α^{-1}, α^c or $(\alpha^c)^{-1}$ in place of one or both α's on the right side of the regularity equation

$$\alpha = \alpha \circ \beta \circ \alpha$$

(where β is some relation). In 2003. X.-Q. Xu and Y.-M.Lui, in [7], introduced a definition of finitely regular relations so that the relation is finitely regular if and only if its finitely extension is regular. In this article, following concepts of finitely conjugative relations ([1], Jiang Guanghao and Xu Luoshan), finitely dual normal relations ([2], Jiang Guanghao and Xu Luoshan) and finitely quasi-conjugative relations ([5], D.A.Romano and M.Vinčić) introduced in their articles, we introduce and analyze notions of finitely bi-normal relations on sets.

For a relation α on a set X is called *finitely bi-normal* if there exists a relation $\beta^{(<\omega)}$ on $X^{(<\omega)}$ such that

$$\alpha^{(<\omega)} = ((\alpha^{-1})^C)^{(<\omega)} \circ \beta^{(<\omega)} \circ ((\alpha^{-1})^C)^{(<\omega)},$$

where $\sigma^{(<\omega)}$ is an extension of relation σ. We found some necessary and sufficient conditions such that a relation is a finitely bi-normal.

2. Preliminaries

For a set X, we call ρ a binary relation on X, if $\rho \subseteq X \times X$. Let $\mathcal{B}(X)$ be the set of all binary relations on X. For $\alpha, \beta \in \mathcal{B}(X)$, define

$$\beta \circ \alpha = \{(x, z) \in X \times X : (\exists y \in X)((x, y) \in \alpha \wedge (y, z) \in \beta)\}.$$

The relation $\beta \circ \alpha$ is called the composition of α and β. It is well known that $(\mathcal{B}(X), \circ)$ is a semigroup. The latter family, with the composition, is not only a semigroup, but also a monoiod. Namely, $Id_X = \{(x, x) : x \in X\}$ is its identity

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element. For a binary relation α on a set X, define $\alpha^{-1} = \{(x, y) \in X \times X : (y, x) \in \alpha\}$ and $\alpha^C = (X \times X) \setminus \alpha$.

Let A and B be subsets of X. For $\alpha \in \mathcal{B}(X)$, set

$$A\alpha = \{y \in X : (\exists a \in A)((a, y) \in \alpha)\}, \quad \alpha B = \{x \in X : (\exists b \in B)((x, b) \in \alpha)\}.$$

It is easy to see that $A\alpha = \alpha^{-1}A$ holds and $(\alpha^C)^{-1} = (\alpha^{-1})^C$. In particular, we put $a\alpha$ instead of $\{a\}\alpha$ and αb instead of $\alpha\{b\}$.

The following classes of elements in the semigroup $\mathcal{B}(X)$ have been investigated:

– *dually normal* ([2]) if there exists a relation $\beta \in \mathcal{B}(X)$ such that
$$\alpha = (\alpha^C)^{-1} \circ \beta \circ \alpha.$$
– *conjugative* ([1]) if there exists a relation $\beta \in \mathcal{B}(X)$ such that
$$\alpha = \alpha^{-1} \circ \beta \circ \alpha.$$
– *dually conjugative* ([1]) if there exists a relation $\beta \in \mathcal{B}(X)$ such that
$$\alpha = \alpha \circ \beta \circ \alpha^{-1}.$$
– *quasi-regular* ([4]) if there exists a relation $\beta \in \mathcal{B}(X)$ such that
$$\alpha = \alpha^C \circ \beta \circ \alpha.$$

Put $\alpha^1 = \alpha$. Previous description gives equality

$$\alpha = (\alpha^a)^i \circ \beta \circ (\alpha^b)^j$$

for some $\beta \in \mathcal{B}(X)$ where $i, j \in \{-1, 1\}$ and $a, b \in \{1, C\}$. We should investigate all other possibilities since some of possibilities given in the previous equation have been investigated. (See, for example, our article [4], [5] and [6].)

For all notions and notations which are not explicitly exposed but are used in this article, reader can find them from book [3] and articles [1], [2] and [4], for an example.

In the following definition we introduce a new class of elements in the semigroup $\mathcal{B}(X)$:

Definition 2.1 ([6], Definition 4.2)**.** A relation $\alpha \in \mathcal{B}(X)$ is a *bi-normal* relations on X if there exists a relation $\beta \in \mathcal{B}(X)$ such that

$$\alpha = (\alpha^{-1})^C \circ \beta \circ (\alpha^{-1})^C.$$

In the following proposition we give a characterization of bi-normal relations on set X. Here, for relation α, we use notation α_* to sign the maximal element of the family of relations β such that $(\alpha^{-1})^C \circ \beta \circ (\alpha^{-1})^C \subseteq \alpha$.

Theorem 2.2 ([6], Theorem 4.2)**.** *For a relation $\alpha \in \mathcal{B}(X)$, relation*

$$\alpha_* = (\alpha^C \circ \alpha^C \circ \alpha^C)^C$$

is the maximal element in the family of all relation $\beta \in \mathcal{B}(X)$ such that

$$(\alpha^{-1})^C \circ \beta \circ (\alpha^{-1})^C \subseteq \alpha.$$

In the following proposition we give a description of bi-normal relation.

Theorem 2.3 ([6], Theorem 4.5)**.** *For a relation α on a set X, the following conditions are equivalent:*
(1) *α is a bi-quasiregular relation.*

(2) *For all* $x, y \in X$, *if* $(x, y) \in \alpha$, *there exists* $(u, v) \in X^2$ *such that:*
(a) $(u, x) \in \alpha^C \wedge (v, y) \in (\alpha^{-1})^C$,
(b) $(\forall s, t \in X)((u, s) \in \alpha^C \wedge (v, t) \in (\alpha^{-1})^C \Longrightarrow (s, t) \in \alpha)$.
(3) $\alpha \subseteq (\alpha^{-1})^C \circ \alpha_* \circ (\alpha^{-1})^C$.

3. Finitely bi-normal relations

In this section we introduce the concept of finitely bi-normal relations as a finite extension of bi-normal relation, introduced in the forthcoming article [6], and give a characterization of that relations. For that we need the concept of finite extension of a relation. That notion and associated notation are borrowed from articles [1] and [2]. For any set X, let

$$X^{(<\omega)} = \{F \subseteq X : F \text{ is finite and nonempty }\}.$$

Definition 3.1. ([1], Definition 3.3; [2], Definition 3.4) Let α be a binary relation on a set X. Define a binary relation $\alpha^{(<\omega)}$ on $X^{(<\omega)}$, called the *finite extension* of α, by

$$(\forall F, G \in X^{(<\omega)})((F, G) \in \alpha^{(<\omega)} \Longleftrightarrow G \subseteq F\alpha).$$

From Definition 2.1, we immediately obtain that

$$(\forall F, G \in X^{(<\omega)})((F, G) \in (\alpha^C)^{(<\omega)} \Longleftrightarrow G \subseteq F\alpha^C),$$
$$(\forall F, G \in X^{(<\omega)})((F, G) \in (\alpha^{-1})^{(<\omega)} \Longleftrightarrow G \subseteq F\alpha^{-1} = \alpha F)$$

and

$$(\forall F, G \in X^{(<\omega)})((F, G) \in ((\alpha^{-1})^C)^{(<\omega)} \Longleftrightarrow G \subseteq F(\alpha^C)^{-1} = \alpha^C F)$$

Now, we can introduce the concept of *finitely bi-normal relation*.

Definition 3.2. A relation α on a set X is called *finitely bi-normal* if there exists a relation $\beta^{(<\omega)}$ on $X^{(<\omega)}$ such that

$$\alpha^{(<\omega)} = ((\alpha^{-1})^C)^{(<\omega)} \circ \beta^{(<\omega)} \circ ((\alpha^{-1})^C)^{(<\omega)}.$$

Now we give an essential characterization of finitely bi-normal relations.

Theorem 3.3. *A relation* α *on a set* X *if a finitely bi-normal relation if and only if for all* $F, G \in X^{(<\omega)}$, *if* $G \subseteq F\alpha$, *then there are* $U, V \in X^{(<\omega)}$, *such that*
(i) $U \subseteq F\alpha^C$, $G \subseteq V\alpha^C$, *and*
(ii) *for all* $S, T \in X^{(<\omega)}$, *if* $U \subseteq S\alpha^C$ *and* $T \subseteq V\alpha^C$ *then* $T \subseteq S\alpha$.

Proof. (1) Let α be a finitely bi-quasiregular relation on set X. Then there is a relation $\beta^{(<\omega)} \subseteq X^{(<\omega)} \times X^{(<\omega)}$ such that $(\alpha^C)^{(<\omega)} \circ \beta^{(<\omega)} \circ (\alpha^C)^{(<\omega)} = \alpha^{(<\omega)}$. For all $(F, G) \in (X^{(<\omega)})^2$, if $G \subseteq F\alpha$, i.e., $(F, G) \in \alpha^{(<\omega)}$, thus

$$(F, G) \in (\alpha^C)^{(<\omega)} \circ \beta^{(<\omega)} \circ (\alpha^C)^{(<\omega)}.$$

Whence there is $(U, V) \in (X^{(<\omega)})^2$ such that

$$(F, U) \in (\alpha^C)^{(<\omega)}, (U, V) \in \beta^{(<\omega)} \text{ and } (V, G) \in (\alpha^C)^{(<\omega)},$$

i.e.,

$$U \subseteq F\alpha^C, G \subseteq V\alpha^C.$$

Hence we get the condition (i).

Now we check the condition (ii). For all $(S,T) \in (X^{(<\omega)})^2$, if $U \subseteq S\alpha^C$ and $T \subseteq V\alpha^C$, i.e., $(S,U) \in (\alpha^C)^{(<\omega)}$ and $(V,T) \in (\alpha^C)^{(<\omega)}$, then by $(U,V) \in \beta^{(<\omega)}$, we have $(S,T) \in (\alpha^C)^{(<\omega)} \circ \beta^{(<\omega)} \circ (\alpha^C)^{(<\omega)}$, i.e., $(S,T) \in \alpha^{(<\omega)}$. Hence $T \subseteq S\alpha$.

(2) Let α be a relation on a set X such that for $F, G \in X^{(<\omega)}$ with $G \subseteq F\alpha$ there are $U, V \in X^{(<\omega)}$ such that conditions (i) and (ii) hold. Define a binary relation $\beta^{(<\omega)} \subseteq X^{(<\omega)} \times X^{(<\omega)}$ by

$$(F,G) \in \beta \Longleftrightarrow (\forall S,T \in X^{(<\omega)})((F \subseteq S\alpha^C \wedge T \cap G\alpha^C \neq \emptyset) \Longrightarrow T \cap S\alpha \neq \emptyset).$$

First, check that (a) $(\alpha^C)^{(<\omega)} \circ \beta^{(<\omega)} \circ (\alpha^C)^{(<\omega)} \subseteq \alpha^{(<\omega)}$ holds. For all $H, W \in X^{(<\omega)}$, if $(H,W) \in (\alpha^C)^{(<\omega)} \circ \beta^{(<\omega)} \circ (\alpha^C)^{(<\omega)}$, then there are $F, G \in X^{(<\omega)}$ with $(H,F) \in (\alpha^C)^{(<\omega)}$, $(F,G) \in \beta^{(<\omega)}$ and $(G,W) \in (\alpha^C)^{(<\omega)}$. Then $F \subseteq H\alpha^C$ and $W \subseteq G\alpha^C$. For all $w \in W$, let $S = H$, $T = \{w\}$. Then $F \subseteq S\alpha^C$ and $G\alpha^C \cap T \neq \emptyset$ because $w \in T$ and $w \in T \subseteq W \subseteq G\alpha^C$. Since $(F,G) \in \beta^{(<\omega)}$, we have that $F \subseteq S\alpha^C \wedge G\alpha^C \cap T \neq \emptyset$ implies $T \cap S\alpha \neq \emptyset$. Hence, $w \in S\alpha$, i.e. $W \subseteq S\alpha$. So, we have $(H,W) = (S,W) \in \alpha^{(<\omega)}$. Therefore, we have $(\alpha^C)^{(<\omega)} \circ \beta \circ (\alpha^C)^{(<\omega)} \subseteq \alpha^{(<\omega)}$.

Second, check that (b) $\alpha^{(<\omega)} \subseteq (\alpha^C)^{(<\omega)} \circ \beta^{(<\omega)} \circ (\alpha^C)^{(<\omega)}$ holds. For all $H, W \in X^{(<\omega)}$, if $(H,W) \in \alpha^{(<\omega)}$ (i.e., $W \subseteq H\alpha$), there are $A, B \in X^{(<\omega)}$ such that:

(i') $A \subseteq H\alpha^C$, $W \subseteq B\alpha^C$, and

(ii') for all $S, T \in X^{(<\omega)}$, if $A \subseteq S\alpha^C$ and $T \subseteq B\alpha^C$, then $T \subseteq S\alpha$.

Now, we have to show that $(A,B) \in \beta^{(<\omega)}$. Let be for all $(C,D) \in (X^{(<\omega)})^2$ the following $A \subseteq C\alpha^C$ and $D \cap B\alpha^C \neq \emptyset$ hold. From $D \cap B\alpha^C \neq \emptyset$ follows that there exists an element $d \in D \cap B\alpha^C (\neq \emptyset)$. So, $d \in D$ and $d \in B\alpha^C$. Put $S = C$ and $T = \{d\}$. Then, by (ii'), we have

$$(A \subseteq S\alpha^C \wedge T = \{d\} \subseteq B\alpha^C) \Longrightarrow \{d\} = T \subseteq S\alpha,$$

i.e. $\emptyset \neq \{d\} \cap S\alpha = T \cap S\alpha$. Therefore, $(A,B) \in \beta^{(<\omega)}$ by definition of $\beta^{(<\omega)}$. Finally, for $(H,A) \in (\alpha)^{(<\omega)}$, $(A,B) \in \beta^{(<\omega)}$ and $(B,W) \in (\alpha^C)^{(<\omega)}$ follows that $(H,W) \in (\alpha^C)^{(<\omega)} \circ \beta \circ (\alpha^C)^{(<\omega)}$.

By assertion (a) and (b), finally we have $\alpha^{(<\omega)} = (\alpha^C)^{(<\omega)} \circ \beta^{(<\omega)} \circ (\alpha^C)^{(<\omega)}$ □

In particular, if we put $F = \{x\}$ and $G = \{y\}$ in the previous theorem, we conclude the following corollary.

Corollary 3.4. *Let α be a relation on a set X. Then α is a finitely bi-notmal relation on X if and only if for all elements $x, y \in X$ such that $(x,y) \in \alpha$ there are finite subsets $U, V \in X^{(<\omega)}$ such that*

(1^0) $(\forall u \in U)((x,u) \in \alpha^C) \wedge (\exists v \in V)((v,y) \in \alpha^C)$, and

(2^0) for all $S \in X^{(<\omega)}$ and $t \in X$ holds

$$(U \subseteq S\alpha^C \wedge (\exists v \in V)((v,t) \in \alpha^C)) \Longrightarrow (\exists s \in S)((s,t) \in \alpha)\ .$$

Proof. Let α be a finitely bi-normal relation on X and let x, y be elements of X such that $(x, y) \in \alpha$. If we put $F = \{x\}$ and $G = \{y\}$ in Theorem 2.1 then there exist finite U and V of $X^{(<\omega)}$ such that conditions (1^0) and (2^0) hold.

In the opposite direction for all elements $x, y \in X$ such that $(x, y) \in \alpha$ there are U and V of $X^{(<\omega)}$ such that conditions (1^0) and (2^0) hold. Define binary relation $\beta^{(<\omega)} \subseteq X^{(<\omega)} \times X^{(<\omega)}$ by

$$(A, B) \in \beta^{(<\omega)} \Longleftrightarrow (\forall S \in X^{(<\omega)})(\forall t \in X)((A \subseteq S\alpha^C \wedge t \in B\alpha^C) \Longrightarrow t \in S\alpha).$$

The proof that the equality $(\alpha^C)^{(<\omega)} \circ \beta^{(<\omega)} \circ (\alpha^C)^{(<\omega)} = \alpha^{(<\omega)}$ holds is the same as in the Theorem 3.3. So, the relation α is a finitely bi-normal. □

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