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# SCHWARZ LEMMA AND ITS APPLICATIONS ON THE UNIT DISC

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Abstract. In this article, a refinement of the Schwarz lemma (boundary Schwarz lemma) is presented for a different class. For the analytic function $p(z) = z+b_2z^2+b_3z^3+...$, defined in the unit disc $U$ satisfying $\Re\left(\frac{p(rz)-p(sz)}{(r-s)z}\right) \geq 0$ for $z \in U$, where $r, s \in \mathbb{C}$, $r \neq s$, $|r| \leq 1$, $|s| \leq 1$, we estimate a module of angular derivative at the boundary point 1with $p(r) = p(s)$. The sharpness of these estimates is also proved.

## 1. Introduction and preliminaries

The classic Schwarz lemma provides details regarding the behavior of an analytical function on the unit disc under the minor hypotheses that the function maps the unit disc to itself and the origin to the origin. We will look at the Schwarz lemma in the unit disc for a different class and present the theorems in their theoretical form. It will also be provided a variety of evidence and viewpoints on the findings. In addition, an alternative Schwarz lemma will be presented for the defined function. Circuit applications of Schwarz's lemma are common. Overall, the circuit application of the Schwarz lemma provides a powerful tool for analyzing and designing electronic circuits. It allows engineers to obtain bounds on the behavior of the circuit, understand its limitations, and ensure stability and performance [10, 12]. The well-known Schwarz lemma, in its simplest form, states the following [4]:

**Lemma 1.1.** *Let $g : U \to U$ be an analytic function that fixes the origin 0. Then, for all $z \in U = \{z : |z| < 1\}$, $|g(z)| \leq |z|$ and $|g'(0)| \leq 1$. Furthermore, if $|g(z)| = |z|$ for any $z \neq 0$ or $|g'(0)| = 1$, then $g$ is a rotation: $g(z) = \varepsilon z$ for some constant $\varepsilon$ with $|\varepsilon| = 1$.*

Now, we will determine an upper bound for the first coefficient in the Taylor expansion of the analytic function that forms the class we give below. In addition, a stronger evaluation will be made for this coefficient from above, taking into account the non-zero zeros of the analytical function.

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Let $p(z)=z+b_2z^2+b_3z^3+...$ be an analytic function in $U$, $\Re\left(\frac{p(rz)-p(sz)}{(r-s)z}\right)\geq 0$ for $z\in U$, where $r,s\in\mathbb{C}$, $r\neq s$, $|r|\leq 1$, $|s|\leq 1$. Consider the functions

$$h(z)=\frac{1-w(z)}{1+w(z)},$$

where

$$w(z)=\frac{p(rz)-p(sz)}{(r-s)\,z}=1+b_2\,(r+s)\,z+b_3\left(r^2+rs+s^2\right)z^2+...$$

Here, $h(z)$ is an analytic function in $U$, $|h(z)|<1$ for $z\in U$ and $h(0)=0$. Therefore, by the Schwarz lemma, we obtain

$$h(z)=\frac{1-w(z)}{1+w(z)}=-\frac{b_2\,(r+s)\,z+b_3\,(r^2+rs+s^2)\,z^2+...}{2+b_2\,(r+s)\,z+b_3\,(r^2+rs+s^2)\,z^2+...},$$

$$\frac{h(z)}{z}=-\frac{b_2\,(r+s)+b_3\,(r^2+rs+s^2)\,z+...}{2+b_2\,(r+s)\,z+b_3\,(r^2+rs+s^2)\,z^2+...},$$

$$|h'(0)|=\frac{|b_2|\,|r+s|}{2}\leq 1$$

and

$$|b_2|\leq\frac{2}{|r+s|}.$$

Now, let us show that this last inequality is sharp. Let

$$\frac{p(rz)-p(sz)}{(r-s)\,z}=\frac{1-z}{1+z}.$$

From the expression of the function $w(z)$, we have

$$\frac{p(rz)-p(sz)}{(r-s)\,z}=1+b_2\,(r+s)\,z+b_3\left(r^2+rs+s^2\right)z^2+...,$$

$$1+b_2\,(r+s)\,z+b_3\left(r^2+rs+s^2\right)z^2+...=\frac{1-z}{1+z},$$

$$b_2\,(r+s)\,z+b_3\left(r^2+rs+s^2\right)z^2+...=\frac{-2z}{1+z},$$

and

$$b_2\,(r+s)+b_3\left(r^2+rs+s^2\right)z+...=\frac{-2}{1-z}.$$

Passing to limit $(z\to 0)$ in the last equality yields

$$|b_2|=\frac{2}{|r+s|}.$$

Moreover, we have

$$\Re\left(\frac{p(rz)-p(sz)}{(r-s)\,z}\right)=\Re\left(\frac{1-z}{1+z}\right)=\frac{\frac{1-z}{1+z}+\frac{1-\overline{z}}{1+\overline{z}}}{2}$$

and

$$\Re\left(\frac{1+z}{1-z}\right)=\frac{1-|z|^2}{|1+z|^2}\geq 0.$$

**Lemma 1.2.** *Let $p(z) = z + b_2z^2 + b_3z^3 + ...$ be an analytic function in $U$, $\Re\left(\frac{p(rz)-p(sz)}{(r-s)z}\right) \geq 0$ for $z \in U$, where $r, s \in \mathbb{C}$, $r \neq s$, $|r| \leq 1$, $|s| \leq 1$. Then we have the inequality*

$$|b_2| \leq \frac{2}{|r+s|}. \tag{1.1}$$

*This result is sharp for the function*

$$\frac{p(rz)-p(sz)}{(r-s)\,z} = \frac{1-z}{1+z}.$$

Now, let $c_1, c_2, ..., c_n$ be the zeros of the function $1 - \frac{p(rz)-p(sz)}{(r-s)z}$ in $U$ that are different from zero. Consider the function

$$\vartheta(z) = \frac{1-w(z)}{1+w(z)} \frac{1}{{}_{k=1}^{n} \frac{z-c_k}{1-\overline{c_k}z}}.$$

Here, $\vartheta(z)$ is an analytic function in $U$, $\vartheta(0) = 0$ and $|\vartheta(z)| < 1$ for $z \in U$. Therefore, $\vartheta(z)$ satisfies the conditions of the Schwarz lemma. Thus, from the Schwarz lemma, we obtain

$$\vartheta(z) = -\frac{b_2\,(r+s)\,z + b_3\,(r^2+rs+s^2)\,z^2 + ...}{2 + b_2\,(r+s)\,z + b_3\,(r^2+rs+s^2)\,z^2 + ...} \frac{1}{{}_{k=1}^{n} \frac{z-c_k}{1-\overline{c_k}z}},$$

$$\frac{\vartheta(z)}{z} = -\frac{b_2\,(r+s) + b_3\,(r^2+rs+s^2)\,z + ...}{2 + b_2\,(r+s)\,z + b_3\,(r^2+rs+s^2)\,z^2 + ...} \frac{1}{{}_{k=1}^{n} \frac{z-c_k}{1-\overline{c_k}z}},$$

$$|\vartheta'(0)| = \frac{|b_2|\,|r+s|}{2{}_{k=1}^{n}\,|c_k|} \leq 1$$

and

$$|b_2| \leq \frac{2}{|r+s|}{}_{k=1}^{n} |c_k|\,.$$

This result is sharp with equality for the function

$$\frac{p(rz)-p(sz)}{(r-s)\,z} = \frac{1 - z{}_{k=1}^{n} \frac{z-c_k}{1-\overline{c_k}z}}{1 + z{}_{k=1}^{n} \frac{z-c_k}{1-\overline{c_k}z}}.$$

Then

$$1 + b_2\,(r+s)\,z + b_3\left(r^2+rs+s^2\right)z^2 + ... = \frac{1 - z{}_{k=1}^{n} \frac{z-c_k}{1-\overline{c_k}z}}{1 + z{}_{k=1}^{n} \frac{z-c_k}{1-\overline{c_k}z}},$$

$$b_2\,(r+s)\,z + b_3\left(r^2+rs+s^2\right)z^2 + ... = \frac{-2z{}_{k=1}^{n} \frac{z-c_k}{1-\overline{c_k}z}}{1 + z{}_{k=1}^{n} \frac{z-c_k}{1-\overline{c_k}z}},$$

$$b_2\,(r+s) + b_3\left(r^2+rs+s^2\right)z + ... = \frac{-2{}_{k=1}^{n} \frac{z-c_k}{1-\overline{c_k}z}}{1 + z{}_{k=1}^{n} \frac{z-c_k}{1-\overline{c_k}z}}$$

and

$$|b_2| = \frac{2}{|r+s|}{}_{k=1}^{n} |c_k|\,.$$

Moreover, we have

$$\begin{aligned}\Re\left(\frac{p(rz)-p(sz)}{(r-s)\,z}\right) &= \Re\left(\frac{1-z_{k=1}^{n}\frac{z-c_k}{1-\overline{c_k}z}}{1+z_{k=1}^{n}\frac{z-c_k}{1-\overline{c_k}z}}\right)\\ &= \frac{1-\left|z_{k=1}^{n}\frac{z-c_k}{1-\overline{c_k}z}\right|^2}{\left|1+z_{k=1}^{n}\frac{z-c_k}{1-\overline{c_k}z}\right|^2}\geq 0\end{aligned}$$

**Lemma 1.3.** *Let $p(z)=z+b_2z^2+b_3z^3+\ldots$ be an analytic function in $U$, $\Re\left(\frac{p(rz)-p(sz)}{(r-s)z}\right)\geq 0$ for $z\in U$, where $r,s\in\mathbb{C}$, $r\neq s$, $|r|\leq 1$, $|s|\leq 1$. Also, let $c_1,c_2,\ldots,c_n$ be the zeros of the function $1-\frac{p(rz)-p(sz)}{(r-s)z}$ in $U$ that are different from zero. Then we have the inequality*

$$|b_2|\leq \frac{2}{|r+s|}{}_{k=1}^{n}\,|c_k|\,. \tag{1.2}$$

*The equality in* (1.2) *occurs for the function*

$$\frac{p(rz)-p(sz)}{(r-s)\,z}=\frac{1-z_{k=1}^{n}\frac{z-c_k}{1-\overline{c_k}z}}{1+z_{k=1}^{n}\frac{z-c_k}{1-\overline{c_k}z}}.$$

One of the important results of Schwarz lemma is at the boundary Schwarz lemma. This lemma was first expressed by Unkelbach [14] and later restated by Osserman [8]. This lemma is expressed as follows.

**Lemma 1.4.** *Let $g(z)$ be an analytic function in $U$, $g(0)=0$ and $|g(z)|<1$ for $z\in U$. If $g(z)$ extends continuously to boundary point $1\in\partial U=\{z:|z|=1\}$, and if $|g(1)|=1$ and $g'(1)$ exists, then*

$$|g'(1)|\geq\frac{2}{1+|g'(0)|} \tag{1.3}$$

*and*

$$|g'(1)|\geq 1. \tag{1.4}$$

*Moreover, the equality in* (1.3) *holds if and only if*

$$g(z)=z\frac{z-\sigma}{1-\sigma z}$$

*for some $\sigma\in(-1,0]$. Also, the equality in* (1.4) *holds if and only if $g(z)=ze^{i\theta}$.*

These inequalities and its generalizations have significant uses in the geometric theory of functions and continue to be popular topics in the mathematics literature [1, 2, 3, 5, 6, 7, 8, 9, 10, 11].

The following lemma, known as the Julia-Wolff lemma, is needed in the sequel (see, [13]).

**Lemma 1.5** (Julia-Wolff lemma)**.** *Let $g$ be an analytic function in $U$, $g(0)=0$ and $g(U)\subset U$. If, in addition, the function $g$ has an angular limit $g(1)$ at $1\in\partial U$, $|g(1)|=1$, then the angular derivative $g'(1)$ exists and $1\leq|g'(1)|\leq\infty$.*

## 2. Main Results

In this section, we discuss different versions of the boundary Schwarz lemma.

**Theorem 2.1.** *Let* $p(z) = z + b_2 z^2 + b_3 z^3 + ...$ *be an analytic function in* $U$, $\Re\left(\frac{p(rz)-p(sz)}{(r-s)z}\right) \geq 0$ *for* $z \in U$, *where* $r, s \in \mathbb{C}$, $r \neq s$, $|r| \leq 1$, $|s| \leq 1$. *Assume that, for* $1 \in \partial U$, $g$ *has an angular limit* $p(1)$ *at the points* $1$ *with* $p(r) = p(s)$. *Then, we have the inequality*

$$|rp'(r) - sp'(s)| \geq \frac{|r-s|}{2}. \tag{2.1}$$

*The equality in* (2.1) *occurs for the function*

$$\frac{p(rz) - p(sz)}{(r-s)\,z} = \frac{1-z}{1+z}.$$

*Proof.* Let

$$h(z) = \frac{1-w(z)}{1+w(z)}.$$

Here, since the function $h(z)$ satisfies the conditions of Lemma 1.4, we take

$$1 \leq |h'(1)| = \frac{2\,|w'(1)|}{|1+w(1)|^2} = \frac{2\,|rp'(r) - sp'(s)|}{|r-s|}$$

and

$$|rp'(r) - sp'(s)| \geq \frac{|r-s|}{2}.$$

Now, we shall show that the inequality (2.1) is sharp. Let

$$\frac{p(rz) - p(sz)}{(r-s)\,z} = \frac{1-z}{1+z}.$$

Then

$$\frac{(rp'(rz) - sp'(sz))\,(r-s)\,z - (r-s)\,(p(rz) - p(sz))}{((r-s)\,z)^2} = \frac{-2}{(1+z)^2}$$

and for $z = 1$

$$|rp'(r) - sp'(s)| = \frac{|r-s|}{2}.$$

□

The inequality (2.1) can be strengthened from below by taking into account, $b_2 = \frac{p''(0)}{2}$, the second coefficient of the expansion of the function $p(z) = z + b_2 z^2 + b_3 z^3 + ....$

**Theorem 2.2.** *Under the same assumptions as in Theorem 2.1, we have*

$$|rp'(r) - sp'(s)| \geq \frac{2\,|r-s|}{2 + |r+s|\,|b_2|}. \tag{2.2}$$

*The inequality* (2.2) *is sharp with extremal function*

$$\frac{p(rz) - p(sz)}{(r-s)\,z} = \frac{1-z^2}{1+2az+z^2},$$

*where* $0 < a = \frac{|b_2||r+s|}{2} \leq 1$.

*Proof.* Let the function $h(z)$ be as given above. From the Lemma 1.4, we obtain

$$\frac{2}{1+|h'(0)|} \leq |h'(1)| = \frac{2\,|rp'(r) - sp'(s)|}{|r-s|}.$$

Since

$$|h'(0)| = \frac{|b_2|\,|r+s|}{2},$$

we take

$$\frac{2}{1+\frac{|b_2||r+s|}{2}} \leq \frac{2\,|rp'(r) - sp'(s)|}{|r-s|}$$

and

$$|rp'(r) - sp'(s)| \geq \frac{2\,|r-s|}{2+|r+s|\,|b_2|}.$$

Let's show the equality of this expression. Let

$$\frac{p(rz)-p(sz)}{(r-s)\,z} = \frac{1-z^2}{1+2az+z^2}.$$

Then

$$\frac{(rp'(rz) - sp'(sz))\,(r-s)\,z - (r-s)\,(p(rz)-p(sz))}{((r-s)\,z)^2}$$
$$= \frac{-2z\,(1+2az+z^2) - (2a+2z)\,(1-z^2)}{(1+2az+z^2)^2}$$

and for $z=1$

$$\frac{|rp'(r) - sp'(s)|}{|r-s|} = \frac{1}{1+a}.$$

For $a = \frac{|b_2||r+s|}{2} \leq 1$, we have

$$|rp'(r) - sp'(s)| = \frac{|r-s|}{1+\frac{|b_2||r+s|}{2}} = \frac{2\,|r-s|}{2+|b_2|\,|r+s|}.$$

□

By including the $b_3$ coefficient in the Taylor expansion of the $p(z)$ function, we strengthen the above theorem.

**Theorem 2.3.** *Let* $p(z) = z + b_2z^2 + b_3z^3 + \ldots$ *be an analytic function in* $U$, $\Re\left(\frac{p(rz)-p(sz)}{(r-s)z}\right) \geq 0$ *for* $z \in U$, *where* $r, s \in \mathbb{C}$, $r \neq s$, $|r| \leq 1$, $|s| \leq 1$. *Assume that, for* $1 \in \partial U$, $g$ *has an angular limit* $p(1)$ *at the points* $1$ *with* $p(r) = p(s)$. *Then, we have the inequality*

$$|rp'(r) - sp'(s)| \geq \frac{|r-s|}{2}\left(1 + \frac{2\,(2-|b_2|\,|r+s|)^2}{4-(|b_2|\,|r+s|)^2 + \left|2b_3\,(r^2+rs+s^2) - b_2^2\,(r+s)^2\right|}\right). \tag{2.3}$$

*This result is sharp with the function*

$$\frac{p(rz)-p(sz)}{(r-s)\,z}=\frac{1-z^2}{1+z^2}.$$

*Proof.* Let $h(z)$ take the form indicated above. and $d(z)=z$. By the maximum principle, for each $z\in U$, we have the inequality $|h(z)|\leq|d(z)|$. Therefore, we take

$$\begin{aligned} q(z) &= \frac{h(z)}{d(z)}=\frac{1}{z}\left(\frac{1-w(z)}{1+w(z)}\right) \\ &= \left(-\frac{b_2\,(r+s)\,z+b_3\,(r^2+rs+s^2)\,z^2+...}{2+b_2\,(r+s)\,z+b_3\,(r^2+rs+s^2)\,z^2+...}\right)\frac{1}{z} \\ &= -\frac{b_2\,(r+s)+b_3\,(r^2+rs+s^2)\,z+...}{2+b_2\,(r+s)\,z+b_3\,(r^2+rs+s^2)\,z^2+...} \end{aligned}$$

is an analytic function in $U$ and $|q(z)|<1$ for $z\in U$. In particular, we have

$$|q(0)|=\frac{|b_2|\,|r+s|}{2}\leq 1 \tag{2.4}$$

and

$$|q'(0)|=\frac{\left|2b_3\,(r^2+rs+s^2)-b_2^2\,(r+s)^2\right|}{4}.$$

The auxiliary function

$$\Theta(z)=\frac{q(z)-q(0)}{1-\overline{q(0)}q(z)}$$

is analytic in $U$, $\Theta(0)=0$, $|\Theta(z)|<1$ for $|z|<1$ and $|\Theta(1)|=1$ for $1\in\partial U$. From Lemma 1.4, we obtain

$$\begin{aligned} \frac{2}{1+|\Theta'(0)|} &\leq |\Theta'(1)|=\frac{1-|q(0)|^2}{\left|1-\overline{q(0)}q(1)\right|^2}\,|q'(1)| \\ &\leq \frac{1+|q(0)|}{1-|q(0)|}\left\{|h'(1)|-|d'(1)|\right\} \\ &= \frac{2+|b_2|\,|r+s|}{2-|b_2|\,|r+s|}\left(\frac{2\,|rp'(r)-sp'(s)|}{|r-s|}-1\right). \end{aligned}$$

Since

$$\begin{aligned} |\Theta'(0)| &= \frac{|q'(0)|}{1-|q(0)|^2}=\frac{\frac{\left|2b_3\left(r^2+rs+s^2\right)-b_2^2(r+s)^2\right|}{4}}{1-\left(\frac{|b_2||r+s|}{2}\right)^2} \\ &= \frac{\left|2b_3\,(r^2+rs+s^2)-b_2^2\,(r+s)^2\right|}{4-(|b_2|\,|r+s|)^2}, \end{aligned}$$

we obtain

$$\frac{2}{1+\frac{\left|2b_3\left(r^2+rs+s^2\right)-b_2^2(r+s)^2\right|}{4-(|b_2||r+s|)^2}}\leq\frac{2+|b_2||r+s|}{2-|b_2||r+s|}\left(\frac{2|rp'(r)-sp'(s)|}{|r-s|}-1\right),$$

$\frac{2(2-|b_2||r+s|)^2}{4-(|b_2||r+s|)^2+\left|2b_3(r^2+rs+s^2)-b_2^2(r+s)^2\right|} \leq \frac{2|rp'(r)-sp'(s)|}{|r-s|}-1$

and

$|rp'(r)-sp'(s)| \geq \frac{|r-s|}{2}\left(\frac{2(2-|b_2||r+s|)^2}{4-(|b_2||r+s|)^2+\left|2b_3(r^2+rs+s^2)-b_2^2(r+s)^2\right|}\right).$

Now, let us show that this last inequality is sharp. Let

$$\frac{p(rz)-p(sz)}{(r-s)\,z}=\frac{1-z^2}{1+z^2}.$$

Then

$$|rp'(r)-sp'(s)|=|r-s|\,.$$

On the other hand, we take

$$1+b_2\,(r+s)\,z+b_3\left(r^2+rs+s^2\right)z^2+...=\frac{1-z^2}{1+z^2},$$

$$\begin{aligned} b_2\,(r+s)\,z+b_3\left(r^2+rs+s^2\right)z^2+... &= \frac{1-z^2}{1+z^2}-1 \\ &= -\frac{2z^2}{1+z^2}. \end{aligned}$$

Therefore, we have

$$b_2\,(r+s)+b_3\left(r^2+rs+s^2\right)z+...=-\frac{2z}{1+z^2}.$$

Passing to limit $(z\to 0)$ in the last equality yields $b_2=0$. Similarly, using straightforward calculations, we take $(r^2+rs+s^2)\,b_3=-2$. So, we obtain

$\frac{|r-s|}{2}\left(1+\frac{2(2-|b_2||r+s|)^2}{4-(|b_2||r+s|)^2+\left|2b_3(r^2+rs+s^2)-b_2^2(r+s)^2\right|}\right)=|r-s|\,.$ □

If we include the non-zero zeros of the $1-\frac{p(rz)-p(sz)}{(r-s)z}$ function, we obtain the following theorem.

**Theorem 2.4.** *Let $p(z)=z+b_2z^2+b_3z^3+...$ be an analytic function in $U$, $\Re\left(\frac{p(rz)-p(sz)}{(r-s)z}\right)\geq 0$ for $z\in U$, where $r,s\in\mathbb{C}$, $r\neq s$, $|r|\leq 1$, $|s|\leq 1$. Assume that, for $1\in\partial U$, $g$ has an angular limit $p(1)$ at the points $1$ with $p(r)=p(s)$. Also, let $c_1,c_2,...,c_n$ be zeros of the function $1-\frac{p(rz)-p(sz)}{(r-s)z}$ in $U$ that are different from zero. Then,*

$$|rp'(r)-sp'(s)|\geq\frac{|r-s|}{2}\left(1+\sum_{k=1}^{n}\frac{1-|c_k|^2}{|1-c_k|^2}+\frac{2\alpha^2}{\beta+\gamma}\right). \tag{2.5}$$

$\alpha=2\prod_{k=1}^{n}|c_k|^2-|r+s|\,|b_2|\,,\ \beta=\left(2\prod_{k=1}^{n}|c_k|\right)^2-(|r+s|\,|b_2|)^2\,,$

$\gamma = \prod_{k=1}^{n}|c_k|\left|2b_3(r^2+rs+s^2)-b_2^2(r+s)^2+2b_2(r+s)\sum_{k=1}^{n}\frac{1-|c_k|^2}{c_k}\right|$. This result is sharp with the function

$$\frac{p(rz)-p(sz)}{(r-s)\,z}=\gamma\frac{1-z^2\prod_{k=1}^{n}\frac{z-c_k}{1-\overline{c_k}z}}{1+z^2\prod_{k=1}^{n}\frac{z-c_k}{1-\overline{c_k}z}},$$

*where $c_1, c_2, ..., c_n$ are positive real numbers..*

*Proof.* Assume that the function $h(z)$ is as stated above and $c_1, c_2, ..., c_n$ be zeros of the function $1-\frac{p(rz)-p(sz)}{(r-s)z}$ in $U$ that are different from zero. Also, consider the function

$$B(z)=z\prod_{k=1}^{n}\frac{z-c_k}{1-\overline{c_k}z}.$$

By the maximum principle for each $z\in U$, we have

$$|h(z)|\le|B(z)|.$$

Consider the function

$$G(z)=\frac{h(z)}{B(z)}=\left(\frac{1-w(z)}{1+w(z)}\right)\frac{1}{z\prod_{k=1}^{n}\frac{z-c_k}{1-\overline{c_k}z}}$$

$$=\left(-\frac{b_2(r+s)z+b_3\left(r^2+rs+s^2\right)z^2+...}{2+b_2(r+s)z+b_3(r^2+rs+s^2)z^2+...}\right)\frac{1}{z\prod_{k=1}^{n}\frac{z-c_k}{1-\overline{c_k}z}}$$

$$=-\frac{b_2(r+s)+b_3\left(r^2+rs+s^2\right)z+...}{2+b_2(r+s)z+b_3(r^2+rs+s^2)z^2+...}\frac{1}{\prod_{k=1}^{n}\frac{z-c_k}{1-\overline{c_k}z}}.$$

In particular, we have

$$|G(0)|=\frac{|r+s|\,|b_2|}{2}\frac{1}{\prod_{k=1}^{n}|c_k|}$$

and

$$|G'(0)|=\frac{\left|2b_3(r^2+rs+s^2)-b_2^2(r+s)^2+2b_2(r+s)\sum_{k=1}^{n}\frac{1-|c_k|^2}{c_k}\right|}{4\prod_{k=1}^{n}|c_k|}.$$

The auxiliary function

$$T(z)=\frac{G(z)-G(0)}{1-\overline{G(0)}G(z)}$$

is analytic in $U$, $|T(z)|<1$ for $|z|<1$ and $T(0)=0$. For $1\in\partial U$, we take $|T(1)|=1$.

From Lemma 1.4, we obtain

$$\frac{2}{1+|T'(0)|} \leq |T'(1)| = \frac{1-|G(0)|^2}{\left|1-\overline{G(0)}G(1)\right|}|G'(1)| \leq \frac{1+|G(0)|}{1-|G(0)|}\left(|h'(1)|-|B'(1)|\right).$$

It can be seen that

$$|T'(0)| = \frac{|G'(0)|}{1-|G(0)|^2}$$

and

$$|T'(0)| = \frac{\frac{\left|2b_3(r^2+rs+s^2)-b_2^2(r+s)^2+2b_2(r+s)\sum\limits_{k=1}^{n}\frac{1-|c_k|^2}{c_k}\right|}{4\prod\limits_{k=1}^{n}|c_k|}}{1-\left(\frac{|r+s||b_2|}{2}\frac{1}{\prod\limits_{k=1}^{n}|c_k|}\right)^2}$$

$$= \prod_{k=1}^{n}|c_k|\frac{\left|2b_3(r^2+rs+s^2)-b_2^2(r+s)^2+2b_2(r+s)\sum\limits_{k=1}^{n}\frac{1-|c_k|^2}{c_k}\right|}{\left(\left(2\prod\limits_{k=1}^{n}|c_k|\right)^2-(|r+s||b_2|)^2\right)}$$

Also,we have

$$|B'(1)| = 1+\sum_{k=1}^{n}\frac{1-|c_k|^2}{|1-c_k|^2},\ 1\in\partial U.$$

Therefore, we obtain

$$\frac{2}{1+\prod\limits_{k=1}^{n}|c_k|\frac{\left|2b_3(r^2+rs+s^2)-b_2^2(r+s)^2+2b_2(r+s)\sum\limits_{k=1}^{n}\frac{1-|c_k|^2}{c_k}\right|}{\left(\left(2\prod\limits_{k=1}^{n}|c_k|\right)^2-(|r+s||b_2|)^2\right)}}$$

$$\leq \frac{2\prod\limits_{k=1}^{n}|c_k|+|r+s||b_2|}{2\prod\limits_{k=1}^{n}|c_k|-|r+s||b_2|}\left(\frac{2|rp'(r)-sp'(s)|}{|r-s|}-1-\sum_{k=1}^{n}\frac{1-|c_k|^2}{|1-c_k|^2}\right),$$

$$\frac{2\left(2\prod\limits_{k=1}^{n}|c_k|^2-|r+s||b_2|\right)^2}{\left(\left(2\prod\limits_{k=1}^{n}|c_k|\right)^2-(|r+s||b_2|)^2\right)+\prod\limits_{k=1}^{n}|c_k|\left|2b_3(r^2+rs+s^2)-b_2^2(r+s)^2+2b_2(r+s)\sum\limits_{k=1}^{n}\frac{1-|c_k|^2}{c_k}\right|}$$

$$\leq \frac{2|rp'(r)-sp'(s)|}{|r-s|}-1-\sum_{k=1}^{n}\frac{1-|c_k|^2}{|1-c_k|^2}$$

and so, we get inequality (2.5).

Now, we shall show that the inequality (2.5) is sharp. Let

$$\frac{p(rz)-p(sz)}{(r-s)\,z} = \frac{1-z^2\prod\limits_{k=1}^{n}\frac{z-c_k}{1-\overline{c_k}z}}{1+z^2\prod\limits_{k=1}^{n}\frac{z-c_k}{1-\overline{c_k}z}} = -1+\frac{2}{1+z^2\prod\limits_{k=1}^{n}\frac{z-c_k}{1-\overline{c_k}z}}.$$

If we take the derivative of both sides and substitute $z=1$, we obtain

$$|rp'(r)-sp'(s)|=\frac{|r-s|}{2}\left(2+_{k=1}^{n}\frac{1+c_k}{1-c_k}\right),$$

where $c_1, c_2, ..., c_n$ are positive real numbers.

On the other hand, we have

$$1+b_2(r+s)z+b_3\left(r^2+rs+s^2\right)z^2+...=1-\frac{2z^2\prod\limits_{k=1}^{n}\frac{z-c_k}{1-\overline{c_k}z}}{1+z^2\prod\limits_{k=1}^{n}\frac{z-c_k}{1-\overline{c_k}z}}.$$

Therefore, we take

$$b_2(r+s)+b_3\left(r^2+rs+s^2\right)z+...=-\frac{2z\prod\limits_{k=1}^{n}\frac{z-c_k}{1-\overline{c_k}z}}{1+z^2\prod\limits_{k=1}^{n}\frac{z-c_k}{1-\overline{c_k}z}}$$

and

$$(a+b)c_2+\left(a^2+ab+b^2\right)c_3z+...=\frac{(1-2\gamma)z\prod\limits_{i=1}^{n}\frac{z-\alpha_i}{1-\overline{\alpha_i}z}}{\gamma-(1-\gamma)z^2\prod\limits_{i=1}^{n}\frac{z-\alpha_i}{1-\overline{\alpha_i}z}}.$$

Passing to limit in the last equality yields $b_2=0$. Similarly, using straightforward calculations, we take $|b_3(r^2+rs+s^2)|=2\prod\limits_{k=1}^{n}|c_k|$. Thus, for $c_1, c_2, ..., c_n\in\mathbb{R}$, we obtain

$$\frac{|r-s|}{2}\left(1+\sum_{k=1}^{n}\frac{1-|c_k|^2}{|1-c_k|^2}+\frac{2\alpha^2}{\beta+\gamma}\right)=\frac{|r-s|}{2}\left(1+\sum_{k=1}^{n}\frac{1-|c_k|^2}{|1-c_k|^2}\right.$$
$$\left.+\frac{2\left(2\prod\limits_{k=1}^{n}|c_k|^2\right)^2}{\left(2\prod\limits_{k=1}^{n}|c_k|\right)^2+\prod\limits_{k=1}^{n}|c_k||2b_3(r^2+rs+s^2)|}\right)$$
$$=\frac{|r-s|}{2}\left(1+\sum_{k=1}^{n}\frac{1-|c_k|^2}{|1-c_k|^2}+\frac{2\left(2\prod\limits_{k=1}^{n}|c_k|^2\right)^2}{\left(2\prod\limits_{k=1}^{n}|c_k|\right)^2+4\left(\prod\limits_{k=1}^{n}|c_k|\right)^2}\right)=\frac{|r-s|}{2}\left(2+_{k=1}^{n}\frac{1+c_k}{1-c_k}\right).\qquad\square$$

The theorem giving the relationship between $b_2$ and $b_3$ is given below.

**Theorem 2.5.** *Let $p(z)=z+b_2z^2+b_3z^3+...$ be an analytic function in $U$, $\Re\left(\frac{p(rz)-p(sz)}{(r-s)z}\right)\geq 0$ for $z\in U$, where $r,s\in\mathbb{C}$, $r\neq s$, $|r|\leq 1$, $|s|\leq 1$, $1-\frac{p(rz)-p(sz)}{(r-s)z}$ has no zeros in $U$ except $z=0$ and $b_2>0$. Then, we have the inequality*

$$\left|2b_3\left(r^2+rs+s^2\right)-b_2^2(r+s)^2\right|\leq 4\left|b_2|r+s|\ln\left(\frac{b_2|r+s|}{2}\right)\right|. \qquad (2.6)$$

*Proof.* Let $b_2>0$ in the expression of the function $p(z)$. Having in mind the inequality (2.4) and the function $1-\frac{p(rz)-p(sz)}{(r-s)z}$ has no zeros in $U$ except $z=0$, we denote by $\ln q(z)$ the analytic branch of the logarithm normed by the condition

$$\ln q(0) = \ln\left(\frac{b_2\,|r+s|}{2}\right) < 0.$$

The auxiliary function

$$R(z) = \frac{\ln q(z) - \ln q(0)}{\ln q(z) + \ln q(0)}$$

is analytic in the unit disc $U$, $|R(z)| < 1$ for $z \in U$, $R(0) = 0$.

By Schwarz lemma, we obtain

$$\begin{aligned} 1 \;\geq\; & |R'(0)| = \frac{|2\ln q(0)|}{|\ln q(0) + \ln q(0)|^2}\left|\frac{q'(0)}{q(0)}\right| \\ =\; & \frac{-1}{2\ln q(0)}\left|\frac{q'(0)}{q(0)}\right|. \\ =\; & -\frac{\frac{\left|2b_3\left(r^2+rs+s^2\right)-b_2^2(r+s)^2\right|}{4}}{2\ln\left(\frac{b_2|r+s|}{2}\right)\frac{b_2|r+s|}{2}} \end{aligned}$$

and

$$\left|2b_3\left(r^2+rs+s^2\right) - b_2^2\,(r+s)^2\right| \leq 4\left|b_2\,(r+s)\ln\left(\frac{b_2\,|r+s|}{2}\right)\right|.$$

□

## 3. Conclusion

In this brief, assuming that $p(r) = p(s)$, $r, s \in \mathbb{C}$, we presented four theorems with their proofs for boundary analysis of derivative of analytic functions particularly at the point $z = 1$. In the theorems, inequalities are generally strengthened by taking into account the second and third terms of the Taylor expansion coefficients of the $p(z)$ function. In future studies, considering the Taylor expansion of the $p(z)$ function around two points, the module of the derivative of the function will be compiled below.

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