Gulf Journal of Mathematics
Vol 3, Issue 2 (2015) 5-11

# ON POWER ARMENDARIZ RINGS

CHAHRAZADE BAKKARI

Abstract. The study of Armendariz rings was initiated by Rege and Chhawchharia, based on a result of Armendariz related to the structure of reduced rings. Recently, Armendariz rings were generalized to power Armendariz rings in [12, 18]. We investigate the transfer of the power Armendariz property to subring retracts, trivial ring extensions, finite direct products, localizations, and amalgamated algebras along an ideal. Our aim is to provide new classes of rings satisfying this property.

## 1. Introduction

All rings considered below are commutative with unit and all modules are unital. Let $R$ be a commutative ring and let $Q(R)$ denote the total ring of quotients of $R$. A ring $R$ is called a total ring of quotients if $R = Q(R)$, that is every element of $R$ is invertible or zero-divisor.

A ring $R$ is called an Armendariz ring if whenever polynomials $f = \sum_{i=0}^{m} a_i X^i$ and $g = \sum_{i=0}^{n} b_i X^i \in R[X]$ satisfy $fg = 0$, we have $a_i b_j = 0$ for every $i$ and $j$. It is easy to see that subrings of Armendariz rings are also Armendariz. Recently, the authors in [12, 18] called a ring $R$ a power Armendariz ring if whenever polynomials $f = \sum_{i=0}^{m} a_i X^i$ and $g = \sum_{i=0}^{n} b_i X^i \in R[X]$ satisfy $fg = 0$, we have $a_i^n b_j^m = 0$ for every $i$ and $j$ for some a positive integers $n$ and $m$. An Armendariz ring is a power Armendariz ring. See for instance [1, 2, 12, 17, 18, 19].

Let $A$ be a ring, $E$ be an $A$-module and $R := A \propto E$ be the set of pairs $(a, e)$ with pairwise addition and multiplication given by $(a, e)(b, f) = (ab, af + be)$. $R$ is called the trivial ring extension of $A$ by $E$. Considerable work has been concerned with trivial ring extensions. Part of it has been summarized in Glaz's

*Date*: Received: Feb 28, 2015.

* Corresponding author.

2010 *Mathematics Subject Classification.* 13F05, 13C10, 13C11, 13F30, 13D05, 16D40, 16E10, 16E60.

*Key words and phrases.* Power Armendariz ring, subring retract, trivial ring extension, finite direct product, localization, and amalgamated algebras along an ideal.

book [10] and Huckaba's book (where $R$ is called the idealization of $E$ by $A$) [15]. See for instance [3, 10, 15, 16].

The amalgamation algebras along an ideal, introduced and studied by D'Anna, Finocchiaro and Fontana in [4, 5] and defined as follows:

Let $A$ and $B$ be two rings with unity, let $J$ be an ideal of $B$ and let $f : A \to B$ be a ring homomorphism. In this setting, we can consider the following subring of $A \times B$:

$$A \bowtie^f J := \{(a, f(a) + j) \mid a \in A, j \in J\}$$

called the amalgamation of $A$ and $B$ along $J$ with respect to $f$. In particular, they have studied amalgmations in the frame of pullbacks which allowed them to establish numerous (prime) ideal and ring-theoretic basic properties for this new construction. This construction is a generalization of the amalgamated duplication of a ring along an ideal (introduced and studied by D'Anna and Fontana in [6, 7, 8]). See for instance [4, 5].

The goal of this work is to exhibit a class of power Armendariz rings. For this purpose, we study the transfer of the power Armendariz property to subring retracts, trivial ring extensions, finite direct products, localizations, and amalgamated algebras along an ideal.

## 2. Main Results

We begin this section by a result of the transfer of power Armendariz property to subring retract.

**Theorem 2.1.** *Let $R$ be a ring and $A$ a subring retract of $R$.*
**1)** *If $R$ is a power Armendariz ring, then so is $A$.*
**2)** *Assume that $(A, M)$ is a local ring and the module retraction map $\phi$ verifies $(Ker(\phi))^2 \subseteq Ker(\phi)$, $Ker(\phi) \subseteq Nil(R)$ and $Ker(\phi)$ is a simple $A$-module such that $MKer(\phi) = 0$. Then $R$ is a power Armendariz ring if and only if so is $A$.*

Before proving Theorem 2.1, we establish the following Lemma.

**Lemma 2.2.** *Let $A$ be a subring of $R$. If $R$ is a power Armendariz ring, then so is $A$.*

*Proof.* Straithforward. □

*Proof.* of Theorem 2.1.
**1)** If $R$ is a power Armendariz ring, then so is $A$ by Lemma 2.2.
**2)** Assume that $(A, M)$ is a local ring, the module retraction map $\phi$ verifies $(Ker(\phi))^2 \subseteq Ker(\phi)$, $Ker(\phi) \subseteq Nil(R)$ and $Ker(\phi)$ is a simple $A$-module such that $MKer(\phi) = 0$ and $A$ is a power Armendariz ring.

Set $V = Ker(\phi)$, and let $f(X) = \sum_{i=0}^{n}(a_i + v_i)X^i (\neq 0)$ and $g(X) = \sum_{j=0}^{m}(b_j + w_j)X^j (\neq 0)$ be two polynomials of $R[X]$, where $a_i, b_j \in A$ and $v_i, w_j \in V$ for each $i, j$ such that $fg = 0$. Without loss of generality, we may assume that $C_R(f) \neq R$ and $C_R(g) \neq R$. Hence, $a_i + v_i$ and $b_j + w_j$ are not invertibles in $R$ for each $i, j$ and so $a_i$ and $b_j$ are not invertibles in $A$ for each $i, j$ since $v_i, w_j \in V \subseteq Nil(R)$ (as the sum of a nilpotent element and unit is unit). Hence, $a_i, b_j \in M$ since $(A, M)$ is local and so $a_iV = b_jV = 0$ for each $i, j$. Then:

$$C_R(f) = \sum_{i=0}^{n}(a_i + v_i)A + \sum_{i=0}^{n} v_iV$$

$$C_R(g) = \sum_{j=0}^{m}(b_j + w_j)A + \sum_{j=0}^{m} w_jV.$$

Set the polynomials of $A[X]$ $f_A(X) = \sum_{i=0}^{n} a_iX^i$ and $g_A(X) = \sum_{j=0}^{m} b_jX^j$. We claim that $v_iV = 0$ for each $i$ or $w_jV = 0$ for each $j$.
Indeed, assume that there exists $i_0$ and $j_0$ such that $v_{i_0}V \neq 0$ and $w_{j_0}V \neq 0$. We may assume that $i_0$ and $j_0$ are the smallest such integer. Then, $v_{i_0}V = w_{j_0}V = V$ since $v_{i_0}V \neq 0$ and $w_{j_0}V \neq 0$ and $V$ is a simple $A$-module; thus $v_{i_0}w_{j_0}V = V$. Therefore, $C_R(fg) = \sum_{k=0}^{n+m}[(\sum_{i+j=k}(a_i + v_i)(b_j + w_j))(A + V)] = \sum_{k=0}^{n+m}[(\sum_{i+j=k}(a_ib_j + v_iw_j))(A + V)] = C_A(f_Ag_A) + V = V$ (since $fg = 0$ and for $k = i_0 + j_0$, we have $\sum_{i+j=k} v_iw_jV = v_{i_0}w_{j_0}V = V$), a contradiction since $fg = 0$. Hence, $v_iV = 0$ for each $i$ or $w_jV = 0$ for each $j$.
Without loss of generality, we may assume that $v_iV = 0$ for each $i$. Hence, $(a_i + v_i)^p = a_i^p$ for each positive integer $p$ (since $MV = 0$ and $v_iV = 0$) and $(b_j + w_j)^q = b_j^q + w_j^q$ for each positive integer $q$ (since $MV = 0$).
Let $n, m$ be a positive integers such that $a_i^n b_j^m = 0$ for each $i, j$ since $A$ is a power Armendariz ring and $f_Ag_A = 0$. Therefore, $(a_i + v_i)^n(b_j + w_j)^m = a_i^n(b_j^m + w_j^m) = a_i^n b_j^m = 0$ since $a_i^n w_j^m = 0$ (since $MV = 0$) and this completes the proof of Theorem 2.1. □

First, we examine the context of trivial ring extension of a local ring $(A, M)$ by an $A$-module $E$ such that $ME = 0$. Remark that this ring is a total ring by the proof of [10, Theorem 2.6 (1)].

**Theorem 2.3.** *Let $(A, M)$ be a local ring, $E$ an $A$-module such that $ME = 0$, and let $R := A \propto E$ be the trivial ring extension of $A$ by $E$. Then, $R$ is a power Armendariz ring if and only if so is $A$.*

*Proof.* If $R$ is a power Armendariz ring, then so is $A$ since $A$ is a subring of $R$. Conversely, assume that $A$ is a power Armendariz ring. Let $f = \sum_{i=0}^{n}(a_i, e_i)X^i$ and $g = \sum_{i=0}^{m}(b_i, f_i)X^i$ be two polynomials in $R[X]$ such that $fg = 0$, where $n$ and $m$ are positive integers. Two cases are then possible.

**Case 1**. $a_i \notin M$ for some $i = 0, \ldots, n$. In this case, $a_i$ is invertible in $A$ and then $(a_i, e_i)$ is invertible in $R$. Hence, $C_R(f) = R$ and $f$ is a Gaussian polynomial (by [8, Remark 1.1]) and so $C_R(f)C_R(g) = C_R(fg) = 0$ as desired.

**Case 2**. $a_i \in M$ for each $i = 0, \ldots, n$. Two cases are then possible:

$*$ If there exists $b_j \notin M$ for some $j = 0, \ldots, m$, then by case 1, $g$ is A Gaussian polynomial and then $C_R(f)C_R(g) = C_R(fg) = 0$ as desired.

$*$ If $b_j \in M$ for each $j = 0, \ldots, m$, we set the two polynomials of $A[X]$: $f_A = \sum_{i=0}^{n} a_i X^i$ and $g_A = \sum_{i=0}^{m} b_i X^i$. We have $f_A g_A = 0$ since $fg = 0$. Hence, there exists a positive integers $n, m$ such that $a_i^n b_j^m = 0$ since $A$ is a power Armendariz ring. Therefore, $(a_i, e_i)^n (b_j, e_j)^m = 0$ (since $a_i, b_j \in M$ for each $i, j$ and $ME = 0$) and this completes the proof of Theorem 2.3. □

By Theorem 2.3 and since each domain is Armendariz and so power Armendariz, we have:

**Corollary 2.4.** *Let $(A, M)$ be a local domain, $E$ an $A$-module such that $ME = 0$. Then the trivial ring extension $R := A \propto E$ of $A$ by $E$ is a power Armendariz ring.*

Now, we will construct a wide class of rings satisfying the Armendariz property. For this, we study the transfer of this property to direct product.

**Theorem 2.5.** *Let $(R_i)_{i=1,\ldots,n}$ be a family of rings. Then $\prod_{i=1}^{n} R_i$ is a power Armendariz ring if and only if so is $R_i$ for each $i = 1, \ldots, n$.*

*Proof.* We will prove the result for $i = 1, 2$, and the Theorem will be established by induction on $n$.

If $R_1 \times R_2$ is a power Armendariz ring, then so $R_1$ and $R_2$ by Theorem 2.1.

Conversely, assume that $R_1$ and $R_2$ are power Armendariz rings. Let $f = \sum_{i=0}^{n}(a_i, e_i)X^i$ and $g = \sum_{i=0}^{m}(b_i, f_i)X^i$ be two polynomials in $(R_1 \times R_2)[X]$ such that $fg = 0$, where $n$ and $m$ are positive integers. Set $f_1 = \sum_{i=0}^{n} a_iX^i \in R_1[X]$, $f_2 = \sum_{i=0}^{n} e_iX^i \in R_2[X]$, $g_1 = \sum_{i=0}^{m} b_iX^i \in R_1[X]$ and $g_2 = \sum_{i=0}^{m} f_iX^i \in R_2[X]$. We have $0 = fg = (f_1g_1, f_2g_2)$ which implies that $f_1g_1 = 0$ and $f_2g_2 = 0$. Hence, there exists $n_1, n_2, m_1, m_2$ a positive integers such that $a_i^{n_1}b_j^{m_1} = 0$ and $e_i^{n_2}f_j^{m_2} = 0$ for each $i, j$ since $R_1$ and $R_2$ are power Armendariz rings. Therefore, $(a_i, e_i)^n(b_j, f_j)^m = 0$, where $n = sup(n_1, n_2)$ and $m = sup(m_1, m_2)$ and this completes the proof of Theorem 2.4. □

By Theorem 2.6 and since each domain is power Armendariz, we have:

**Corollary 2.6.** *Let $(R_i)_{i=1,\dots,n}$ be a family of domains. Then $\prod_{i=1}^{n} R_i$ is a power Armendariz ring.*

Now, we study the localization of power Armendariz ring.

**Theorem 2.7.** *Let $R$ be a power Armendariz ring and $S$ is a multiplicative subset of $R$. Then $S^{-1}R$ is a power Armendariz ring.*

*Proof.* Without loss of generality, we may consider the polynomials of the form $S^{-1}f$ and $S^{-1}g$ where $f = \sum_{i=0}^{n} a_iX^i$ and $g = \sum_{i=0}^{m} b_iX^i \in R[X]$, such that $S^{-1}(f)S^{-1}(g) = 0$. Hence, there exists $t \in S$ such that $tfg = 0$ and so $(ta_i)^n b_j^m = 0$ for some positive integers $n, m$ and for each $i, j$. Therefore, $(a_i/1)^n(b_j/1)^m = 0$ in $S^{-1}R$, as desired. □

We end this paper by studying the amalgamated of power Armendariz ring.

**Theorem 2.8.** *Let $(A, B)$ be a pair of rings, $f : A \to B$ be a ring homomorphism and $J$ be a proper ideal of $B$, then*

1. *If $A \bowtie^f J$ is a power Armendariz ring then so is $A$.*
2. *If $A$ and $f(A) + J$ are power Armendariz rings then so is $A \bowtie^f J$.*
3. *Assume that $J$ contains a regular element. Then, $A \bowtie^f J$ is a power Armendariz ring if and only if $f(A) + J$ and $A$ are power Armendariz rings.*

*Proof.* **1)** If $A \bowtie^f J$ is power Armendariz, then so is $A$ by Theorem 2.1.

**2)** Assume that $A$ and $f(A)+J$ are power Armendariz and we must show that $A \bowtie^f J$ is power Armendariz.
let $F(x) = \sum_{i=0}^{\infty}(a_i, f(a_i)+j_i)x^i$ and $G(x) = \sum_{j=0}^{\infty}(b_j, f(b_j)+k_j)x^j$
in $A \bowtie^f J[[x]]$ such that $F(x)G(x) = 0$.
Set $f_B(x) = \sum_{i=0}^{\infty}(f(a_i)+j_i)x^i$ and $g_B(x) = \sum_{j=0}^{\infty}(f(b_j)+k_j)x^j$
$f_A(x) = \sum_{i=0}^{\infty} a_i x^i$ and $g_A(x) = \sum_{j=0}^{\infty} b_j x^j$
$F(x)G(x) = 0$ implies that $f_A(x)g_A(x) = 0$ and $f_B(x)g_B(x) = 0$
which implies that $(f(a_i)+j_i)^{n_1}(f(b_j)+k_j)^{m_1} = 0$ and $(a_i^n)^2(b_j^m)^2 = 0$ for each $i, j$ since $f(A)+J$ and $A$ are power Armendariz rings. Therefore, $(a_i, f(a_i)+j_i)^n(b_j, f(b_j)+k_j)^m = 0$ for each $i, j$ and for $n := max(n_1, n_2)$ and $m := max(m_1, m_2)$ and so $A \bowtie^f J$ is a power Armendariz ring.

**3)** Assume that $J$ contains a regular element $e$.
Assume that $A \bowtie^f J$ is a power Armendariz, we show that $f(A)+J$ is power Armendariz.
Let $f_A(x) = \sum_{i=0}^{\infty}(f(a_i)+j_i)x^i$ and $g_A(x) = \sum_{j=0}^{\infty}(f(b_j)+k_j)x^j$
such that $f_A(x)g_A(x) = 0$
we have

$$f_A(x)g_A(x) = \sum_{k=0}^{\infty}(\sum_{i+j=k}(f(a_i)+j_i)(f(b_j)+k_j))x^k = 0$$

Set

$$F(x) = \sum_{i=0}^{\infty}(0, e(f(a_i)+j_i))x^i \; and \; G(x) = \sum_{j=0}^{\infty}(0, e(f(b_j)+k_j))x^j$$

we have

$$\begin{aligned} F(x)G(x) &= \sum_{k=0}^{\infty}(\sum_{i+j=k}(0, e^2(f(a_i)+j_i)(f(b_j)+k_j))x^k \\ &= \sum_{k=0}^{\infty}(0, e^2\sum_{i+j=k}(f(a_i)+j_i)(f(b_j)+k_j))x^k = 0 \end{aligned}$$

So, there exists a positive integers $n, m$ such that $(0, e(f(a_i)+j_i))^n(0, e(f(b_j)+k_j))^m = 0$ since $A \bowtie^f J$ is a power Armendariz, which implies that

$$e^{n+m}(f(a_i)+j_i)^n(f(b_j)+k_j)^m = 0$$

for each $i, j$ and hence $(f(a_i)+j_i)^n(f(b_j)+k_j)^m = 0$ since $e$ is a regular in $J$. and this shows that $f(A)+J$ is a power Armendariz.
On the other hand if $A \bowtie^f J$ is power Armendariz then so is $A$ by (1).
Conversely, if $f(A)+J$ and $A$ are power Armendariz rings then so is $A \bowtie^f J$ by (2), as desired. □

## References


1. D.D. Anderson and V. Camillo, *Armendariz rings and Gaussian rings*, Comm.Algebra **2**6 (1998), 2265-2272.
2. E. Armendariz, *A note on extensions of Baer and P.P. rings*, J. Austral. Math. Soc., **1**8 (1974), 470-473. MR51, 3224.
3. C. Bakkari, S. Kabbaj, and N. Mahdou, *Trivial extensions defined by Prüfer conditions*, J. Pure Appl. Algebra **2**14 (2010), 53–60.
4. M. D'Anna, C. A. Finocchiaro and M. Fontana, *Amalgamated algebras along an ideal*, Commutative algebra and its applications, Walter de Gruyter, Berlin, (2009) 241-252.
5. M. D'Anna, C. A. Finocchiaro and M. Fontana, *Properties of chains of prime ideals in amalgamated algebras along an ideal*, J. Pure Appl. Algebra 214 (2010) 1633-1641.
6. M. D'Anna, *A construction of Gorenstein rings*, J. Algebra 306 (2006) 507-519.
7. M. D'Anna and M. Fontana, *The amalgamated duplication of a ring along a multiplicative-canonical ideal*, Ark. Mat. 45 (2007) 241-252.
8. M. D'Anna and M. Fontana, *An amalgamated duplication of a ring along an ideal : the basic properties*, J. Algebra Appl. 6 (2007) 443-459.
9. A. Forsythe, *Divisors of zero in polynomial rings*, Amer. Math. Monthly, **5**0 (1943), 7-8. MR4, 129.
10. S. Glaz, *Commutative coherent rings*, Lecture Notes in Mathematics, 1371. Springer-Verlag, Berlin, 1989.
11. S. Glaz, *The weak dimension of Gaussian rings*, Proc. Amer. Math. Soc. **1**33 (9) (2005), 2507–2513.
12. J.C. Han, T.K. Kwak, M.J. Lee, Y. Lee and Y.S. Seo, *On powers of coefficients of zero-dividing polynomials*, Submitted.
13. Y. Hirano and H. Tominaga, *Regular rings, V rings and their generalizations*, Hiroshima Math. J., **9** (1979), 137-149.
14. W. Heinzer and C. Huneke, *Gaussian polynomials and content ideals*, Proc. Amer. Math. Soc. **1**25 (1997), 739–745.
15. J. A. Huckaba, *Commutative rings with zero divisors*, Marcel Dekker, New York, 1988.
16. S. Kabbaj and N. Mahdou, *Trivial extensions defined by coherent-like conditions*, Comm. Algebra **3**2 (10) (2004), 3937–3953.
17. N.K. Kim and Y. Lee, *Armendariz rings and reduced rings*, J. Algebra **2**23 (2000) 477–488.
18. T.K. Kwak, M.J. Lee, and Y. Lee, *Quasi-Armendariz property on powers of coefficients*, International Electronic Journal of Algebra, **1**5 (2014), 208–217.
19. T.K. Lee and Y. Zhou, *Armendariz and reduced rings*, Comm. Algebra **3**2 (6) (2004), 2287–2299.
20. N. Mahdou and H. Mouanis, *Some homological properties of subring retract and applications to fixed rings*, Comm. Algebra **3**2 (5) (2004), 1823–1834.
21. T- K. Lee and T-L. Wong, *On Armendariz rings*, Houston J. Math. **2**9 (3) (2003) 583-593.
22. M. B. Rege and S. Chhawchharia, *Armendariz rings*, Proc. Japan Acad. Ser. A Math. Sci. **7**3 (1997) 14-17.
23. W. R. Scott, *Divisors of zero in polynomials rings*, Amer. Math. Monthly, **6**1 (1954), 336. MR, 672.



Department of Mathematics and Computer Sciences, Faculty of Science, University Moulay Ismail, Meknes, Morocco.

*E-mail address*: cbakkari@hotmail.com