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TRIVIAL RING EXTENSIONS AND AMALGAMATIONS OF PERIODIC RINGS

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Abstract. In this paper, we investigate the transfer of the notion of periodic ring to trivial ring extensions of commutative ring by modules and amalgamation of rings. Namely, we prove that the trivial ring extension $A \propto E$ is a periodic ring if and only if so is A. It is also shown that the amalgamation $A \bowtie^f J$ is a periodic ring if and only if so is A and J satisfies the periodic-like property.

1. Introduction

All rings considered in this paper are assumed to be commutative with identity element $\neq 0$; and all modules are unital. A ring R is said to be a periodic ring if for each $x \in R$, $\{x^n \ ; \ n \in \mathbb{N}\}$ is a finite set, equivalently, for every $x \in R$ there exist distinct positive integers n and m such that $x^n = x^m$. This definition is a particular case of the class of rings which are neither commutative nor unitary. The structure and commutativity of periodic rings are well discussed more detail in the papers [1, 2, 6] and the reference therein.
Given a ring A and an A-module E, the set $R = A \propto E$ of pairs (a, x) with pairwise addition and multiplication is giving by $(a, x)(b, y) = (ab, ay + bx)$ is called the trivial ring extension of A by E (also called the idealization of E over A). Considerable work, part of it summarized in Glaz's book [5] and Huckaba's book [7], has been concerned with trivial ring extensions.

Let $f : A \longrightarrow B$ be a ring homomorphism and let J be an ideal of B. The following sub-ring of $A \times B$:

$$A \bowtie^f J = \{(a, f(a) + j) \ ; \ a \in A, j \in J\}$$

is said to be amalgamation of A with B along J with respect to f. This construction is generalization case of the amalgamated duplication of a ring along an ideal, introduced and studied in [4], see for instance [3].

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The main purpose of this paper is to investigate the transfer of the notion "periodic ring" to trivial ring extensions and amalgamation of rings. Then we generate new and original families of examples of periodic rings.

2. Main results

We begin this section with a result which studies the transfer of the periodic-like properties to trivial ring extension.

Theorem 2.1. *Let A be a ring, E an A-module and let $A \propto E$ be the trivial ring extension of A by E. Then $A \propto E$ is a periodic ring if and only if so is A.*

We need the following lemma before proving Theorem 2.1.

Lemma 2.2. *Every periodic ring has a nonzero characteristic.*

Proof. Let A be a periodic ring. Suppose that 0 is the characteristic of A, there exist distinct positive integers n and m such that $(2.1_A)^n = (2.1_A)^m$. Hence $(2^n - 2^m)\,1_A = 0$, therefore $2^n = 2^m$ and so $n = m$. We have the desired contradiction. □

Proof. of Teorem 2.1 Assume that $A \propto E$ is a periodic ring. Let $a \in A$, there exist distinct positive integers p and q such that $(a,0)^p = (a,0)^q$. Then $a^p = a^q$, therefore A is a periodic ring.

Conversely, suppose that A is a periodic ring and let (a,x) be an element of $A \propto E$. By induction we claim that for each $k \geq 1$, $(a,x)^k = (a^k, ka^{k-1}x)$. Indeed, it is certainly true for $k = 1$. Suppose the equality is true for k, then

$$(a,x)^{k+1} = \left(a^k, ka^{k-1}x\right)(a,x) = \left(a^{k+1}, (k+1)a^k x\right)$$

We denote by r the characteristic of A. There are distinct positive integers k and l such that $a^k = a^l$. Then $a^{kr} = a^{lr}$, therefore $a^{kr+1} = a^{lr+1}$. On the other hand, $rb = 0$ for each element b of A. It follows that

$$\begin{aligned}(a,x)^{rk+1} - (a,x)^{rl+1} &= \left(a^{rk+1}, (rk+1)a^{rk}x\right) - \left(a^{rl+1}, (rl+1)a^{rl}x\right)\\ &= \left(0, r(k-l)a^{rk}x\right)\\ &= 0.\end{aligned}$$

We deduce that $(a,x)^{rk+1} = (a,x)^{rl+1}$. By using the above lemma we have $rk+1 \neq rl+1$. Thus $A \propto E$ is a periodic ring. This completes the proof of Theorem 2.1. □

Theorem 2.1 enriches the literature with new examples, as shown below.

Example 2.3. Let (p, q) be a pair of distinct prime integers and let $A = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z}$. We put $m = (p-1)(q-1)+1$. Since $x^{p-1} = 1$ for each nonzero element x of $\mathbb{Z}/p\mathbb{Z}$, then $x^m = x$ for any element $x \in \mathbb{Z}/p\mathbb{Z}$. It follows that $(x, y)^m = (x, y)$ for each element $(x, y) \in A$. Let $R = A^{\mathbb{N}}$ be the set of all sequences of elements of A and let E be an A-module. Since $u^m = u$ for each $u \in R$, then R is a periodic ring. We conclude that $R \propto E$ is a periodic ring.

Recall that a ring A is said to be a p-ring, where p is a prime integer, if $x^p = x$ and $px = 0$ for each element x of A. The following corollary is an immediate consequence of the above theorem.

Corollary 2.4. *Let A be a ring, E an A-module and let $R = A \propto E$ be the trivial ring extension of A by E.*

(1) *If A is a finite ring then R is a periodic ring.*
(2) *If A is a p-ring then R is a periodic ring.*

Proof. Since every finite ring (also p-ring) is a periodic ring, then the corollary follows from Theorem 2.1. □

Examples 2.5. Let p be a prime integer, n a positive integer and let $\mathbb{F}_q$ be the finite field with $q = p^n$ elements.

(1) Let E be any $\mathbb{F}_q$-vector space, then $\mathbb{F}_q \propto E$ is a periodic ring.
(2) Let M be any $\mathbb{F}_p^{\mathbb{N}}$-module, where $\mathbb{F}_p^{\mathbb{N}}$ is the set of all sequences of elements of $\mathbb{F}_p$. Then $\mathbb{F}_p^{\mathbb{N}} \propto M$ is a periodic ring.

In the next theorem we establish the transfer of periodic-like property to amalgamation of rings.

Theorem 2.6. *Let $f : A \longrightarrow B$ be a ring homomorphism, J an ideal of B an let $A \bowtie^f J$ be the amalgamation of A with B along J with respect to f. Then the following statements are equivalent:*

(1) *$A \bowtie^f J$ is a periodic ring.*
(2) *A and $f(A) + J$ are periodic rings.*
(3) *A is a periodic ring and J satisfied the periodic-like property i.e for each $x \in J$ there exists distinct positive integers n and m such that $x^m = x^n$.*

Proof. (1) $\Longrightarrow$ (2): Assume that $A \bowtie^f J$ is a periodic ring and let a be an element of A. There are distinct positive integers p and q such that $(a, f(a))^p = (a, f(a))^q$. Then $a^p = a^q$, therefore A is a periodic ring. Let (a, x) be an element of $A \times J$. Since $(a, f(a) + x) \in A \bowtie^f J$ there exist distinct positive integers p_1 and q_1 such that $(a, f(a) + x)^{p_1} = (a, f(a) + x)^{q_1}$. Hence $(f(a) + x)^{p_1} = (f(a) + x)^{q_1}$, and so $f(A) + J$ is a periodic ring.

(2) $\Longrightarrow$ (3): Since $J \subseteq f(A) + J$ and $f(A) + J$ is a periodic ring, we have the desired implication.

(3) $\Longrightarrow$ (1): Let n be a positive integer and let $a \in A$. Then $f(a)^n = f\left(a^n\right)$. Hence $f(A)$ is a periodic ring. we denote by l the characteristic of $f(A)$. By using Lemma 2.2, $l \neq 0$. Let (a, x) be an element of $A \times J$. From the assumption there are some positive integers $n_0 < n_1$ such that $x^{n_0} = x^{n_1}$. Then $\left(x^{n+n_0}\right)_{n\geq 0}$ is a periodic sequence since

$$x^{n+(n_1-n_0)+n_0} = x^{n+n_1} = x^{n+n_0}.$$

Therefore $\{x^n \ ; \ n \in \mathbb{N}\}$ and $\{f\left(a^n\right) \ ; \ n \in \mathbb{N}\}$ are finite sets. We deduce that there exists a positive integer m such that $\{x^n \ ; \ n \in \mathbb{N}\} = \{1, x, ..., x^m\}$ and that $\{f\left(a^n\right) \ ; \ n \in \mathbb{N}\} = \{1, f(a), ..., f(a^m)\}$. Now, consider a positive integer n, by the binomial theorem (which is valid in any commutative ring) we get the following equality

$$(f(a) + x)^n = \sum_{k=0}^{n} \binom{n}{k} f\left(a^k\right) x^{n-k}.$$

Thus $(f(a) + x)^n = \sum_{0\leq i,j\leq m} p_{i,j} f\left(a^i\right) x^j$, where $p_{i,j}$ is a positive integer for each $0 \leq i, j \leq m$. On the other hand, there are nonnegative integers $q_{i,j}$ and $r_{i,j}$ such that

$$p_{i,j} = q_{i,j}l + r_{i,j} \quad \text{and} \quad 0 \leq r_{i,j} \leq l - 1.$$

Hence $p_{i,j} f\left(a^i\right) x^j = r_{i,j} f\left(a^i\right) x^j$, since $l f\left(a^i\right) = 0$. It follows that

$$(f(a) + x)^n = \sum_{0\leq i,j\leq m} r_{i,j} f\left(a^i\right) x^j.$$

We conclude that $\{(f(a)+x)^n \ ; \ n \in \mathbb{N}\}$ is a finite set, and so $\{(a, f(a)+x)^n \ ; \ n \in \mathbb{N}\}$ is also a finite set. Finally, $A \bowtie^f J$ is a periodic ring. This completes the proof of Theorem 2.6. □

From the above theorem we deduce that if (A, B) is a pair of periodic rings, $f : A \longrightarrow B$ a ring homomorphism and J an ideal of B then $A \bowtie^f J$ is a periodic ring. The following example illustrates the previous result.

Example 2.7. Let $\mathbb{F}_p[x]$ be the polynomial ring over $\mathbb{F}_p$, where p is a prime integer, and let $(n, m) \in \mathbb{N}^2$ such that $n < m$. Let $P(x)$ be an element of $\mathbb{F}_p[x]$ divides $x^{p^m} - x^{p^n}$. Set $A = \dfrac{\mathbb{F}_p[x]}{\left(x^{p^m} - x^{p^n}\right)}$ and $J = P(x)A$. Consider the ring homomorphism $f : A \longrightarrow A$, defined by $f(a) = a^p$ for each $a \in A$. Then $A \bowtie^f J$ is a periodic ring.

Proof. By using the above result it suffices to prove that A is a periodic ring. Let $0 \neq Q(x) \in \mathbb{F}_p[x]$. By induction on $n = \deg Q$, the degree of the polynomial $Q(x)$, we claim that $(Q(x))^p = Q\left(x^p\right)$. Indeed, it is certainly true for $n = 0$, since $\mathbb{F}_p^*$ is a cyclic group with $p - 1$ elements. Assume that the statement is true for each $k \leq n$ and that $\deg Q = n+1$. Set $Q(x) = a_{n+1}x^{n+1} + Q_1(x)$, where $0 \neq a_{n+1} \in \mathbb{F}_p$

and $Q_1(x) \in \mathbb{F}_p[x]$ such that $\deg Q_1 \leq n$. By the binomial theorem, we obtain that

$$(Q(x))^p = \left(a_{n+1}x^{n+1}\right)^p + (Q_1(x))^p = a_{n+1}^p x^{p(n+1)} + Q_1\left(x^p\right).$$

Thus $(Q(x))^p = Q\left(x^p\right)$, as desired.
Now consider an element $Q(x)$ in $\mathbb{F}_p[x]$. From the previous part of the proof we can write $Q(x)^{p^k} = Q\left(x^{p^k}\right)$, where k is a non-negative integer. We denote by $\overline{Q(x)} = Q(x) + \left(x^{p^m} - x^{p^n}\right)$. Since $\overline{x}^{p^n} = \overline{x}^{p^m}$ we get the following equalities

$$\overline{Q(x)}^{p^m} = \overline{Q(x)^{p^m}} = \overline{Q\left(x^{p^m}\right)} = Q\left(\overline{x}^{p^m}\right) = Q\left(\overline{x}^{p^n}\right) = \overline{Q(x)}^{p^n}.$$

We conclude that A is a periodic ring, completing the proof. □

The following corollary is an immediate consequence of Theorem 2.6

Corollary 2.8. *Let A be a ring, I an ideal of A and let $A \bowtie I$ be the amalgamated duplication of A by I. Then $A \bowtie I$ is a periodic ring if and only if so is A.*

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