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ALMOST N-PERFECT RINGS

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Abstract. In this paper, we introduce and define the notion of almost n-perfect ring which is a generalization of almost perfect ring, and we investigate the transfer of this property to some ring extensions. Also, We give a characterization of global weak injective dimension property.

1. Introduction

Throughout this paper all rings are commutative with identity element and all modules are unital. For an R-module M, we denote by $pd_R(M)$ and $fd_R(M)$, respectively, the usual projective and flat dimension of M. $gldim(R)$ is the classical global dimension of R.

In 2005, Ding and Mao introduced in [5], the following definitions:

Definition 1.1. (1) An R-module M is called a cotorsion module if $Ext^1_R(F, M) = 0$ for any flat R-module F.
(2) The cotorsion dimension of an R-module M denoted by $cd_R(M)$ is the smallest integer n such that $Ext^{n+1}_R(F, M) = 0$ for any flat R-module F.
(3) The global cotorsion dimension of R, denoted by $C - gldim(R)$, is the quantity: $C - gldim(R) = sup\{cd_R(M)|M \quad R - module\}$.
(4) R is n-perfect ring if $pd_R(F) \leq n$ for any flat R-module F.

We recall the following Proposition proved in [5], which will be useful:

Proposition 1.2. *For any ring R and for any positive integer $n \geq 0$ the following conditions are equivalents:*

(1) $C - gldim(R) \leq n$.
(2) $cd_R(M) \leq n$ *for any* $R-module$ M.
(3) $Ext^{n+1}_R(F, M) = 0$ *for any R-module F and M where F is flat.*
(4) $Ext^i_R(F, M) = 0$ *for any $i > n$ and for any R-module F and M where F is flat.*
(5) $pd_R(F) \leq n$ *for any flat R module F.*
(6) *R is n-perfect.*

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For more details of these notions see [2, 5].
In 2006, Lee introduced in [10], the following definition: An $R-$module M is called a weak injective module if $Ext_R^1(F, M) = 0$ for any R-module F of flat dimension ≤ 1.
In 2009, Fuchs and Lee introduced in [9], the weak injective dimension of a module M , denoted $wid(M)$, to be the smallest integer n such that $Ext_R^{n+1}(F, M) = 0$ for all R-modules F of flat dimension ≤ 1.
In 2003, Bazzoni and Salce introduced in [1], the class of almost perfect rings as follow:

Definition 1.3. A ring R is almost perfect if R/I is perfect for any proper ideal I of R.

And they proved that an almost perfect ring which is not a domain is always perfect. Therefore, they focused their work on almost perfect domain.

In this paper, we introduce and define the notion of almost n-perfect ring which is a generalization of almost perfect ring, and we investigate the transfer of this property to some ring extensions. Also, We give a characterization of global weak injective dimension property.

In Section 2 we characterize the global weak injective dimension and we show that The usual global dimension, the global weak injective dimension, and the cotorsion global dimension of rings are linked via inequalities, as follows:

$$C - gldim(R) \leq Wi - gldim(R) \leq gldim(R).$$

After, we introduce the almost n-perfect ring which is a generalization of almost perfect rings(Definition 1.3). In Theorem 2.7 we characterize the almost n-perfect domain which is the main result of this paper.
In Section 3 we are interesting to studying the transfer of weak-injective global dimensions in polynomial rings Theorem 3.1, and $D + M$ constructions Theorem 3.2.

2. Almost n-perfect rings

In this section, we introduce almost n-perfect rings which are generalizations of almost n-perfect rings. Also we characterize the global weak-injective dimension of rings.

We start by giving the definition of global weak-injective dimension of rings.

Definition 2.1. The global weak-injective dimension of R is the supremum of weak-injective dimensions of all R-modules, denoted:

$$Wi - gldim(R) = \sup\{wid(M)/M \quad R - module\}$$

The following Proposition gives a characterization of global weak-injective dimension property

Proposition 2.2. *Let R be a ring and let n be a positive integer. The following statements are equivalent:*

(1) *$Wi - gldim(R) \leq n$;*

(2) *$wid(M) \leq n$ for R-module M;*
(3) *$Ext_R^{n+1}(F, M) = 0$ for any $i > n+1$ and for R-module F with $fd(F) \leq 1$ and M an R-module;*
(4) *$Ext_R^i(F, M) = 0$ for all R-module F with $fd(F) \leq 1$ and M an R-module;*
(5) *$pd(F) \leq n$ for all R-module F with $fd(F) \leq 1$.*

Proof. 1) $\Rightarrow$ 2) Straightforward.
2) $\Rightarrow$ 3) Follows from the definition of weak injective dimension.
3) $\Rightarrow$ 4) Straightforward.
4) $\Rightarrow$ 5) From [11, Theorem 9.5].
5) $\Rightarrow$ 1) Using the fact $pd_R(F) \leq n$ for any R-module with $fd(F) \leq 1$, then by [11, Theorem 9.5], we have $Ext_R^{n+1}(F, M)$ for any R-module M. Then $wid(M) \leq n$. Therefore, $Wi - gldim(R) = \sup\{wid(M)/M \quad R - module\} \leq n$. □

The usual global dimension, the global weak injective dimension, and the cotorsion global dimension of rings are linked via inequalities, as follows:

Proposition 2.3. *Let R be a ring then:*

$$C - gldim(R) \leq Wi - gldim(R) \leq gldim(R)$$

Proof. The first inequality since the class of flat modules is embedded in the class of modules of flat dimension ≤ 1.

The second inequality, hold since if for some positive integer n we have $gldim(R) = sup\{pd_R(M)/M \quad R - module\} = n$, then $pd_R(M) \leq n$ for any R-module M. In particular $pd_R(F) \leq n$ for any R-module F of flat dimension ≤ 1 and by Proposition 2.3, we have $Wi - gldim(R) \leq n$. □

Proposition 2.4. *Let R be a ring then:*

(1) *$Wi - gldim(R) = 0$, then R is perfect.*
(2) *If R is an integral domain and $Wi - gldim(R) \leq 1$, then R is almost perfect domain.*

Proof. (1) Suppose that $W_i - gldim(R) = 0$ and let F be a flat module. By Proposition 2.3, $pd_R(F) = 0$. Hence, by application to Proposition 1.2, it follows that R is perfect.
(2) Immediately from [9, Theorem 6.1].

□

Now, we define the notion of almost n-perfect rings as follow:

Definition 2.5. A ring R is called almost n-perfect ring if the factor ring R/I is n-perfect for any nonzero ideal I of R.

Remark 2.6. (1) Almost 1-perfect ring is the well known almost perfect ring.
(2) If R is an almost n-perfect ring, then R is almost m-perfect ring for any integer $m \geq n$.
(3) Every perfect ring is almost n-perfect for any integer $n \geq 1$.

The next result gives a characterization of almost n-perfect domain by using global weak injective dimension of ring.

Theorem 2.7. *Let R be an integral domain and Q its field of fractions. If R is almost n-perfect, then $Wigldim(R) \leq n+1$*

we have equivalence if $pd_R(Q) \leq n$.

Proof. To prove this Theorem, we need this Lemma.

Lemma 2.8. [9, Theorem 5.3] *Let R be an integral domain and M an R-module. If M is of cotorsion dimension n. Then it has weak-injective dimension n or n +1.*

Proof of Theorem2.7, Suppose that R is almost n-perfect. Then for any ideal I of R we have R/I is n-perfect. Let M be any R module for any flat R-module F, we have $Tor_p^R(R/I, F) = 0$ for all $p > 0$. Thus, we may apply [4, Proposition 4.1.3] which gives: $Ext_R^{n+1}(F, M) \cong Ext_{R/I}^{n+1}(F \otimes_R R/I, M) = 0$. Therefore, $cd_R(M) \leq n$ and from Lemma 2.8 $wid_R(M) \leq n+1$ and so $Wigldim(R) \leq n+1$.

Now, suppose that $pd_R(Q) \leqslant n$ and $Wigldim(R) \leq n+1$. Let F be a flat R-module, from [9, Proposition 3.2], there exists an exact sequence of R-modules of the form:

$$0 \longrightarrow F \longrightarrow \bigoplus Q \longrightarrow A \longrightarrow 0,$$

since F is flat then so is A. For any R-module M, we have the exact sequence:

$$Ext^{n+1}(\bigoplus Q, M) \longrightarrow Ext^{n+1}(F, M) \longrightarrow Ext^{n+2}(A, M).$$

Since $pd_R(Q) \leq n$, $Ext^{n+1}(\bigoplus Q, M) = 0$ and $Ext^{n+2}(A, M) = 0$ since $Wigldim(R) \leqslant n+1$. So, $Ext^{n+1}(F; M) = 0$ and $cd_R(M) \leq n$. Hence, R is almost n-perfect. $\square$

3. Transfer of Almost n-perfect property under certain ring extensions construction.

In this section, we examine the transfer of almost n-perfect property to polynomial rings and D+M constructions.

The following Theorem study the almost n-perfect property in polynomial rings.

Theorem 3.1. *Let R be an integral domain and Q its field of fractions. Consider $R[X_1, X_2, \cdots, X_n]$ be the polynomial ring in n indeterminates over a ring R. If $pd_R(Q) \leq m$, then R is almost m-perfect if and only if $R[X_1, X_2, \cdots, X_n]$ is almost $n+m$-perfect.*

Proof. The proof is done by induction on n and it suffices to check it for n=1. And so we only need to prove that R is almost m-perfect if and only if $R[X]$ is almost (m+1)perfect. One can easily check that $Q(X)$ is the field of fraction of the integral domain $R[X]$. Using the fact R[X] is free R-module, we can prove easily that if $pd_R(Q) \leq m$ than also $pd_{R[X]}(Q(X)) \leq m$.

With similar argument as [8, Example (iv)], it follows that $Wi-gldim(R[X]) \leq Wi-gldim(R)+1$. Then from Theorem 2.7 if R is almost m-perfect, then $Wi-gldim(R) \leq m+1$ and so $Wi-gldim(R[X]) \leq m+2$ and $R[X]$ is almost (m+1)-perfect.

Conversely, assume that $R[X]$ is almost (m+1)-perfect this equivalent to $Wi-gldim(R[X]) \leq m+2 < \infty$(from Theorem 2.7). Let F be a R-module such that

$fd_R(F) \leq 1$. Since $R[X]$ is a free R-module, it is easy to see that $fd_{R[X]}(F[X]) \leq 1$. And, $pd_R(F) = pd_R(F[X]) = pd_{R[X]}(F[X]) = n+2$. This means that $Wi - gldim(R) \leq m+2$. Assume that $wi - gldim(R) = m+2$. Then, by Theorem 2.3, there exists an R-module F of $fd_R(F) \leq 1$ such that $pd_R(F) = m+2$. Thus, there exists an R-module E such that $fd_R(E) \leq 1$, such that $Ext^{n+2}(E,F) \neq 0$. From [3, Example (7), page 9], the endomorphism $\mu : F \longrightarrow F$, defined by $\mu(f) = Xf$, is injective. Then, we may apply the Reess theorem [11, Theorem 9.37], which gives:

$$Ext^{m+3}_{R[X]}(E, F[X]) \cong Ext^{m+2}_{R}(E,F) \neq 0$$

Then, From [11, Exercice 9.20, page 258]

$$Ext^{m+3}_{R[X]}(E[X], F[X]) \cong Hom_R(R[X], Ext^{m+3}_{R[X]}(E, F[X])) \cong (Ext^{m+3}_{R[X]}(E, F[X]))^{\mathbb{N}} \neq 0.$$

Then, $wid_{R[X]}(F[X]) = m+3$, which contradicts with $Wi - gldim(R[X]) = m+2$, so $Wi - gldim(R) = m+1$ and R is almost m-perfect by Theorem 2.7. □

Let $T = K + M$ be an integral domain where K is a field and M is a maximal ideal of T. Let D a subring of K, and consider the ring $R = D + M$. Now we study the transfer transfer of almost n-perfect rings in $D + M$ constructions. This construction have proven to be useful in solving many open problems and conjectures for various contexts in ring theory (see for example [6, 7]).

Theorem 3.2. *Let T be a ring of the form $T = K + M$ where K is a field an M a maximal ideal of T. Let D be a subring of K where $frac(D) = K$. Consider $R = D + M$ and $frac(R) = Q$ such that $pd_R(Q) \leq n$ for some positive integer n, then:*

R is almost n-perfect if and only if D and T are almost n-perfect.

The proof of the theorem concludes from the following proposition.

Proposition 3.3. *Let T be a ring of the form $T = K + M$ where K is a field an M a maximal ideal of T. Let D be a subring of K where $frac(D) = K$. Consider $R = D + M$, then:*

$$Wi - gldim(R) = sup\{Wi - gldim(T), Wi - gldim(D)\}$$

The proof of the proposition concludes from the following lemma.

Lemma 3.4. *Let T be a ring of the form $T = K + M$ where K is a field an M a maximal ideal of T. Let D be a subring of K where $frac(D) = K$. Consider $R = D + M$, then for any R-module F such that $fd_R(F) \leq 1$ we have:*

$$pd_R(F) = sup\{pd_T(F \otimes_R T), pd_D(F/MF)\}$$

Proof. Suppose that $pd_R(F) = n$ and consider the following exact sequence of R-modules:
$0 \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \ \longrightarrow P_0 \longrightarrow F \longrightarrow 0$, where P_i are projective modules. Since T is flat, we obtain the following exact sequence of T-modules :

$$0 \longrightarrow P_n \otimes_R T \longrightarrow P_{n-1} \otimes_R T \longrightarrow \cdots \longrightarrow P_0 \otimes_R T \longrightarrow F \otimes_R T \longrightarrow 0$$

Then $pd_T(F \otimes_R T) \leq n$. On the other hand from [12, Proof of Theorem1.1], $Tor_1^R(D, F) = 0$, and since $fd_R(F) \leq 1$, then $Tor_p^R(D, F) = 0$ for any $p > 0$ and by [4, Prpopsition 4.1.3], for any D-module C and for any integer $n > 1$ we have $Ext_R^{n+1}(F, C) \cong Ext_D^{n+1}(F \otimes_R D, C)$, so $pd_D(F/MF) \leq n$ and then $sup\{pd_T(F \otimes_R T), pd_D(F/MF)\} \leq pd_R(F)$

Conversely, suppose that $sup\{fd_T(F \otimes_R T), fd_D(F/MF)\} = n$, for some positive integer n. And let the exact sequence of R-modules $0 \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow F \longrightarrow 0$ such that $P_0 \cdots P_{n-1}$ are projective. then, $P_n \otimes_R T$ and P_n/MP_n are projective T-module and D-module, respectively. Thus, from [12, Theorem 1.1] P_n is a projective R-module. Then $pd_R(F) \leq n$, so $pd_R(F) = sup\{pd_T(F \otimes_R T), pd_D(F/MF)\}$ as desired. $\square$

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