

CSI- $\xi^\perp$ -RIEMANNIAN MAPS FROM KENMOTSU MANIFOLDS TO RIEMANNIAN MANIFOLDS

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ABSTRACT. In this article, we take into account a new class of semi-invariant $\xi^\perp$ -Riemannian maps, namely, CSI- $\xi^\perp$ (Clairaut semi-invariant- $\xi^\perp$)-Riemannian maps in contact geometry and discuss CSI- $\xi^\perp$ -Riemannian maps from Kenmotsu manifolds to Riemannian manifolds. We extend the notion of semi-invariant $\xi^\perp$ -Riemannian maps to CSI- $\xi^\perp$ -Riemannian maps and obtain a necessary and sufficient condition for this extension. We also investigate some geometric properties of these maps and exhibit a non-trivial example of discussed maps.

1. INTRODUCTION

The concept of Riemannian submersion was initiated by O' Neill [25] Gray [14] in 1966–67. The theory of almost Hermitian submersions was introduced by Watson [35] in 1976. The vital role and capability of Riemannian submersions in present science era can be seen in ([6], [10], [16], [11], [24], [15], [16]). In 1995, Riemannian submersion has been used as an application to robotics by Bedrossian and Spong [7]. Both of them showed it in the presence of a special category of robotic chains with zero Riemannian curvature when potential energy and friction phenomena are totally disregarded. Further, in 2004, an interesting application of Riemannian submersions for the theory of modeling and control of inessential robotic chain by Altafini [4]. He stated that there is a close contact between inverse kinematic in robotics and the pull back vectors. This bonding is provided by Riemannian submersions. The Riemannian submersions motivate the researchers to define the different new class of Riemannian submersions in the papers ([17], [18], [1], [26], [27], [28]). In 1992, the study of Riemannian maps between Riemannian manifold was introduced by Fischer [12] as a natural generalization of isometric immersions and Riemannian submersions. In [33], Sahin defined anti-invariant Riemannian map from an almost Hermitian manifold to a Riemannian manifold as a generalization of anti-invariant Riemannian submersions [29] from almost Hermitian manifolds to Riemannian manifolds. .

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In fundamental differential geometry, Clairaut's reaction [8] states that $r \sin \Theta$ is constant where Θ represents the angle between the velocity vector of a geodesic and a meridian, and r represents the distance to the axis of a surface of revolution. In [8], Bishop introduced Clairaut submersions and obtained a necessary and sufficient condition for Riemannian submersion to be Clairaut Riemannian submersions. Allison [3] considered Clairaut submersions in different way and showed applications of such submersions in static space-times. Lee et al. [21], took Kaehler manifold as total manifold and investigated new conditions in the article [29]. A similar study was done by several geometers in ([34], [13], [5], [31], [19], [22]).

In [20], Lee introduced the notion of anti invariant $\xi^\perp$ - Riemannian Submersions from almost contact Riemannian manifolds onto Riemannian manifolds in 2013. He also investigated geometric properties of such submersions. In 2017, Akyol, Sari and Aksoy generalized anti invariant $\xi^\perp$ -Riemannian Submersions and defined the semi-invariant $\xi^\perp$ -Riemannian Submersions [1] as well as semi-slant $\xi^\perp$ -Riemannian Submersions [2]. The authors obtained the geometry of the total space and the base space for the existence of discussed submersions.

In this paper, we are going to characterize the properties of CSI- $\xi^\perp$ -Riemannian maps from Kenmotsu manifolds to Riemannian manifolds which is an extension of semi-invariant $\xi^\perp$ -Riemannian maps. We exhibit our work as follows: In the section 2, we collect main notions, basic definitions and formulae for other sections. In the section 3, we give the definition of discussed maps from Kenmotsu manifolds to Riemannian manifolds and also investigate differential geometric properties of such maps. The last section 4 is concerned with a non-trivial example of the CSI- $\xi^\perp$ -Riemannian map from the Kenmotsu manifold to a Riemannian manifold.

2. PRELIMINARIES

A $(2n+1)$ -dimensional differentiable manifold N_1 admitting a $(1, 1)$ tensor field ϕ , a contravariant vector field ξ , a 1-form η such that

$$\phi^2 = -I + \eta \otimes \xi, \phi \circ \xi = 0, \eta \circ \xi = 0, \quad (2.1)$$

$$\eta(\xi) = 1, \quad (2.2)$$

where I is the identity tensor. The manifold N_1 with an almost contact structure (ϕ, ξ, η) is called an almost contact manifold.

An almost contact structure (ϕ, ξ, η, g_1) becomes an almost contact metric structure if there exists a Riemannian metric g_1 on an almost contact manifold N_1 satisfying the following conditions

$$g_1(\phi U_1, \phi W_2) = g_1(U_1, W_2) - \eta(U_1)\eta(W_2), g_1(\phi U_1, W_2) = -g_1(U_1, \phi W_2), \quad (2.3)$$

$$g_1(U_1, \xi) = \eta(U_1), \quad (2.4)$$

where U_1, W_2 are the vector fields on N_1 . Now, the manifold N_1 is called an almost contact metric manifold. The fundamental 2-form Φ is defined by $\Phi(U_1, W_2) = g_1(U_1, \phi W_2)$.

Now N_1 is called a Kenmotsu manifold [9] if

$$(\nabla_{Z_1}\phi)X_1 = g_1(\phi Z_1, X_1)\xi - \eta(X_1)\phi Z_1 \quad (2.5)$$

for any vector fields Z_1 and X_1 on N_1 , where ∇ is the Riemannian connection of the Riemannian metric g_1 . If $(N_1, \phi, \xi, \eta, g_1)$ be a Kenmotsu manifold, then the following equation holds:

$$\nabla_{Z_1}\xi = Z_1 - \eta(Z_1)\xi. \quad (2.6)$$

Let X_1 and Z_1 are vector fields on N_1 , define O'Neill's tensors [25] $\mathcal{T}$ and $\mathcal{A}$ by

$$\mathcal{A}_{X_1}Z_1 = \mathcal{H}\nabla_{\mathcal{H}X_1}\mathcal{V}Z_1 + \mathcal{V}\nabla_{\mathcal{H}X_1}\mathcal{H}Z_1, \quad (2.7)$$

$$\mathcal{T}_{X_1}Z_1 = \mathcal{H}\nabla_{\mathcal{V}X_1}\mathcal{V}Z_1 + \mathcal{V}\nabla_{\mathcal{V}X_1}\mathcal{H}Z_1 \quad (2.8)$$

It is simple to observe about the skew-symmetric operators $\mathcal{T}$ and $\mathcal{A}$ on the tangent bundle of N_1 reversing the vertical and the horizontal distributions.

From equations (2.7) and (2.8), we have

$$\nabla_{Y_1}Z_1 = \mathcal{T}_{Y_1}Z_1 + \mathcal{V}\nabla_{Y_1}Z_1, \quad (2.9)$$

$$\nabla_{Y_1}U_1 = \mathcal{T}_{Y_1}U_1 + \mathcal{H}\nabla_{Y_1}U_1, \quad (2.10)$$

$$\nabla_{U_1}Y_1 = \mathcal{A}_{U_1}Y_1 + \mathcal{V}\nabla_{U_1}Y_1, \quad (2.11)$$

$$\nabla_{U_1}W_1 = \mathcal{H}\nabla_{U_1}W_1 + \mathcal{A}_{U_1}W_1 \quad (2.12)$$

for all $Y_1, Z_1 \in \Gamma(\ker \pi_*)$ and $U_1, W_1 \in \Gamma(\ker \pi_*)^\perp$, where $\mathcal{H}\nabla_{Y_1}U_1 = \mathcal{A}_{U_1}Y_1$, if U_1 is basic. It is simple to take account that $\mathcal{T}$ acts as the second fundamental form on the fibers, while $\mathcal{A}$ works on the horizontal distribution.

A Riemannian map is a Riemannian map with totally umbilical fibers if [6]

$$\mathcal{T}_{Y_1}Z_1 = g_1(Y_1, Z_1)H \quad (2.13)$$

for all $Y_1, Z_1 \in \Gamma(\ker \pi_*)$, where H is the mean curvature vector field.

We also recall some basic definitions of different types of Riemannian maps related to CSI- $\xi^\perp$ -Riemannian maps from ([30], [32], [33])

3. EXTENSION OF SEMI-INVARIANT- $\xi^\perp$ -RIEMANNIAN MAP TO CSI- $\xi^\perp$ -RIEMANNIAN MAP

In this section, we study CSI- $\xi^\perp$ -Riemannian map as an extension of semi-invariant- $\xi^\perp$ -Riemannian map. We also discuss necessary and sufficient condition for such extension.

Definition 3.1. [19] Let $\pi : (N_1, \phi, \xi, \eta, g_1) \rightarrow (N_2, g_2)$ be a Riemannian map from an almost contact metric manifold to a Riemannian manifold. A Riemannian map π is called a semi-invariant $\xi^\perp$ -Riemannian map if there is a distribution $D_1 \subset (\ker \pi_*)$ such that

$$(\ker \pi_*) = D_1 \oplus D_2, \phi(D_1) = D_1, \phi(D_2) \subset (\ker \pi_*)^\perp,$$

where D_2 is orthogonal complementary to D_1 in $(\ker \pi_*)$.

In the theory of Riemannian map, Sahin [30] introduces the notion of Clairaut map. Now we extend the above definition (2.4) for CSI- $\xi^\perp$ -Riemannian map by using Sahin's work in following way:

Definition 3.2. A Riemannian map $\pi : (N_1, g_1) \rightarrow (N_2, g_2)$ is called a Clairaut map if there exists a positive function r on N_1 , such that, for any geodesic γ on N_1 , the function $(r \circ \gamma) \sin \Theta$ is constant, where, for any t , $\Theta(t)$ is the angle between $\dot{\gamma}$ and the horizontal space at $\gamma(t)$.

He also provide the following necessary and sufficient condition for a Riemannian map to be a Clairaut map:

Theorem 3.3. Let $\pi : (N_1, g_1) \rightarrow (N_2, g_2)$ be a Riemannian map with associated fibers. If each fiber in such a manner that it is totally umbilical and has the mean curvature vector field $H = -\nabla h$, where ∇h is the gradient of the function h with respect to g_1 . Then, π is a Clairaut Riemannian map with $r = e^h$.

From definition (3.2) we can give the following definition:

Definition 3.4. A semi-invariant $\xi^\perp$ -Riemannian map π from a Kenmotsu manifold $(N_1, \phi, \xi, \eta, g_1)$ to a Riemannian manifold (N_2, g_2) is called CSI- $\xi^\perp$ -Riemannian map if it satisfies the condition of Clairaut Riemannian map i.e. if each fiber is totally umbilical in nature and has the mean curvature vector field $H = -\nabla h$ is the gradient of the function h with respect to g_1 . Then, the π is a Clairaut Riemannian map with $r = e^h$.

Now, using definition (3.4), we have

$$(\ker \pi_*) = D_1 \oplus D_2, \phi(D_1) = D_1, \phi(D_2) \subseteq (\ker \pi_*)^\perp.$$

Thus for any $U_1 \in (\ker \pi_*)$, we put

$$U_1 = PU_1 + QU_1, \quad (3.1)$$

where $PU_1 \in \Gamma(D_1)$ and $QU_1 \in \Gamma(D_2)$.

In addition, for $X_1 \in (\ker \pi_*)$, we get

$$\phi X_1 = \psi X_1 + \omega X_1, \quad (3.2)$$

where $\psi X_1 \in \Gamma(D_1)$ and $\omega X_1 \in \Gamma(\phi D_2)$.

The horizontal distribution $\Gamma(\ker \pi_*)^\perp$ is decomposed as

$$\Gamma(\ker \pi_*)^\perp = \phi(D_2) \oplus \mu.$$

Here μ is an invariant distribution of ϕ and contains ξ .

Also for $Z_1 \in \Gamma(\ker \pi_*)^\perp$, we have

$$\phi Z_1 = BZ_1 + CZ_1, \quad (3.3)$$

where $BZ_1 \in \Gamma(D_2)$ and $CZ_1 \in \Gamma(\mu)$.

Now, we use $(N_1, \phi, \xi, \eta, g_1)$ for Kenmotsu manifold and (N_2, g_2) for Riemannian manifold in this section and further sections.

Lemma 3.5. Let $\pi : (N_1, \phi, \xi, \eta, g_1) \rightarrow (N_2, g_2)$ be a semi-invariant $\xi^\perp$ -Riemannian map. Then, we get

$$\mathcal{V}\nabla_{Y_1}\psi Z_1 + \mathcal{T}_{Y_1}\omega Z_1 = B\mathcal{T}_{Y_1}Z_1 + \psi\mathcal{V}\nabla_{Y_1}Z_1, \quad (3.4)$$

$$\mathcal{T}_{Y_1}\psi Z_1 + \mathcal{H}\nabla_{Y_1}\omega Z_1 = C\mathcal{T}_{Y_1}Z_1 + \omega\mathcal{V}\nabla_{Y_1}Z_1 + g_1(\psi Y_1, Z_1)\xi, \quad (3.5)$$

$$\mathcal{V}\nabla_{X_1}BV_1 + \mathcal{A}_{X_1}CV_1 + \eta(V_1)BX_1 = B\mathcal{H}\nabla_{X_1}V_1 + \psi\mathcal{A}_{X_1}V_1, \quad (3.6)$$

$$\mathcal{A}_{X_1}BV_1 + \mathcal{H}\nabla_{X_1}CV_1 + \eta(V_1)CX_1 = C\mathcal{H}\nabla_{X_1}V_1 + \omega\mathcal{A}_{X_1}V_1 + g_1(CX_1, V_1)\xi, \quad (3.7)$$

$$\mathcal{V}\nabla_{Y_1}BX_1 + \mathcal{T}_{Y_1}CX_1 + \eta(X_1)\psi Y_1 = \psi\mathcal{T}_{Y_1}X_1 + B\mathcal{H}\nabla_{Y_1}X_1, \quad (3.8)$$

$$\mathcal{T}_{Y_1}BX_1 + \mathcal{H}\nabla_{Y_1}CX_1 + \eta(X_1)\omega Y_1 = \omega\mathcal{T}_{Y_1}X_1 + C\mathcal{H}\nabla_{Y_1}X_1 + g_1(\omega Y_1, X_1)\xi, \quad (3.9)$$

$$\mathcal{V}\nabla_{X_1}\psi Y_1 + \mathcal{A}_{X_1}\omega Y_1 = B\mathcal{A}_{X_1}Y_1 + \psi\mathcal{V}\nabla_{X_1}Y_1, \quad (3.10)$$

$$\mathcal{A}_{X_1}\psi Y_1 + \mathcal{H}\nabla_{X_1}\omega Y_1 = C\mathcal{A}_{X_1}Y_1 + \omega\mathcal{V}\nabla_{X_1}Y_1 + g_1(BX_1, Y_1)\xi, \quad (3.11)$$

where $Y_1, Z_1 \in \Gamma(\ker \pi_*)$ and $X_1, V_1 \in \Gamma(\ker \pi_*)^\perp$.

Proof. Using (2.5), (2.9)-(2.12), (3.2) and (3.3), we get Lemma 3.5. $\square$

Lemma 3.6. *Let $\pi : (N_1, \phi, \xi, \eta, g_1) \rightarrow (N_2, g_2)$ be a semi-invariant $\xi^\perp$ -Riemannian map. If $\gamma : I_2 \subset R \rightarrow N_1$ is a regular curve, $Z_1(t)$ is the vertical and $U_1(t)$ is the horizontal components of the tangent vector field $\dot{\gamma} = E$ of $\gamma(t)$, then γ is a geodesic if and only if along γ the following equations hold:*

$$\mathcal{V}\nabla_{\dot{\gamma}}\psi Z_1 + \mathcal{V}\nabla_{\dot{\gamma}}BU_1 + (\mathcal{T}_{Z_1} + \mathcal{A}_{U_1})\omega Z_1 + (\mathcal{T}_{Z_1} + \mathcal{A}_{U_1})CU_1 - \eta(U_1)(\psi Z_1 + BU_1) = 0,$$

$$\mathcal{H}\nabla_{\dot{\gamma}}\omega Z_1 + \mathcal{H}\nabla_{\dot{\gamma}}CU_1 + (\mathcal{T}_{Z_1} + \mathcal{A}_{U_1})\psi Z_1 + (\mathcal{T}_{Z_1} + \mathcal{A}_{U_1})BU_1 + g_1(\phi\dot{\gamma}, \dot{\gamma})\xi - \eta(U_1)(\omega Z_1 + CU_1) = 0.$$

Proof. Let $\gamma : I_2 \rightarrow N_1$ be a regular curve on N_1 . Since $Z_1(t)$ and $U_1(t)$ are the vertical and horizontal parts of the tangent vector field $\dot{\gamma}(t)$, i.e., $\dot{\gamma}(t) = Z_1(t) + U_1(t)$. Using (2.5), (2.9)-(2.12), (3.2) and (3.3), we get

$$\begin{aligned} & \phi\nabla_{\dot{\gamma}}\dot{\gamma} \\ &= \nabla_{\dot{\gamma}}\phi\dot{\gamma} - (\nabla_{\dot{\gamma}}\phi)\dot{\gamma}, \\ &= \nabla_{Z_1}\psi Z_1 + \nabla_{Z_1}\omega Z_1 + \nabla_{Z_1}BU_1 + \nabla_{Z_1}CU_1 + \nabla_{U_1}\psi Z_1 + \nabla_{U_1}\omega Z_1 + \\ & \quad \nabla_{U_1}BU_1 + \nabla_{U_1}CU_1 - g_1(\phi\dot{\gamma}, \dot{\gamma})\xi + \eta(U_1)(\psi Z_1 + BU_1) + \eta(U_1)(\omega Z_1 + CU_1), \\ &= \mathcal{T}_{Z_1}\psi Z_1 + \mathcal{V}\nabla_{Z_1}\psi Z_1 + \mathcal{T}_{Z_1}\omega Z_1 + \mathcal{H}\nabla_{Z_1}\omega Z_1 + \mathcal{T}_{Z_1}BU_1 + \mathcal{V}\nabla_{Z_1}BU_1 + \\ & \quad \mathcal{T}_{Z_1}CU_1 + \mathcal{H}\nabla_{Z_1}CU_1 + \mathcal{A}_{U_1}\psi Z_1 + \mathcal{V}\nabla_{U_1}\psi Z_1 + \mathcal{H}\nabla_{U_1}\omega Z_1 + \mathcal{A}_{U_1}\omega Z_1 + \\ & \quad \mathcal{A}_{U_1}BU_1 + \mathcal{V}\nabla_{U_1}BU_1 + \mathcal{H}\nabla_{U_1}CU_1 + \mathcal{A}_{U_1}CU_1 - g_1(\phi\dot{\gamma}, \dot{\gamma})\xi + \\ & \quad \eta(U_1)(\psi Z_1 + BU_1) + \eta(U_1)(\omega Z_1 + CU_1). \end{aligned}$$

Taking the vertical and horizontal components in above equation, we have

$$\mathcal{V}\phi\nabla_{\dot{\gamma}}\dot{\gamma} = \mathcal{V}\nabla_{\dot{\gamma}}\psi Z_1 + \mathcal{V}\nabla_{\dot{\gamma}}BU_1 + (\mathcal{T}_{Z_1} + \mathcal{A}_{U_1})\omega Z_1 + (\mathcal{T}_{Z_1} + \mathcal{A}_{U_1})CU_1 + \eta(U_1)(\psi Z_1 + BU_1),$$

$$\begin{aligned} & \mathcal{H}\phi\nabla_{\dot{\gamma}}\dot{\gamma} \\ &= \mathcal{H}\nabla_{\dot{\gamma}}\omega Z_1 + \mathcal{H}\nabla_{\dot{\gamma}}CU_1 + (\mathcal{T}_{Z_1} + \mathcal{A}_{U_1})\psi Z_1 + (\mathcal{T}_{Z_1} + \mathcal{A}_{U_1})BU_1 - \\ & \quad g_1(\phi\dot{\gamma}, \dot{\gamma})\xi + \eta(U_1)(\omega Z_1 + CU_1), \end{aligned}$$

Now, γ is a geodesic on N_1 if and only if $\mathcal{V}\phi\nabla_{\dot{\gamma}}\dot{\gamma} = 0$ and $\mathcal{H}\phi\nabla_{\dot{\gamma}}\dot{\gamma} = 0$, which completes the proof. $\square$

Theorem 3.7. *Let $\pi : (N_1, \phi, \xi, \eta, g_1) \rightarrow (N_2, g_2)$ be a semi-invariant $\xi^\perp$ -Riemannian map. Then π is a CSI- $\xi^\perp$ -Riemannian map with $r = e^h$ if and only if*

$$\begin{aligned} & (g_1(\nabla h, Z_1) + \eta(Z_1))\|Y_1\|^2 \\ &= -g_1(\mathcal{V}\nabla_{\dot{\gamma}}BZ_1, \psi Y_1) - g_1(\mathcal{H}\nabla_{\dot{\gamma}}CZ_1, \omega Y_1) - g_1((\mathcal{T}_{Y_1} + \mathcal{A}_{Z_1})CZ_1, \psi Y_1) - \\ & \quad g_1((\mathcal{T}_{Y_1} + \mathcal{A}_{Z_1})BZ_1, \omega Y_1) \end{aligned}$$

where $\gamma : I_2 \rightarrow N_1$ is a geodesic on N_1 and Y_1, Z_1 are vertical and horizontal components of $\dot{\gamma}(t)$.

Proof. Let $\gamma : I_2 \rightarrow N_1$ be a geodesic on N_1 with $Y_1(t) = \mathcal{V}\dot{\gamma}(t)$ and $Z_1(t) = \mathcal{H}\dot{\gamma}(t)$. Let $\Theta(t)$ denote the angle in $[0, \pi]$ between $\dot{\gamma}(t)$ and $Z_1(t)$. Assuming $\nu = \|\dot{\gamma}(t)\|^2$ then we get

$$g_1(Y_1(t), Y_1(t)) = \nu \sin^2 \Theta(t), \quad (3.12)$$

$$g_1(Z_1(t), Z_1(t)) = \nu \cos^2 \Theta(t). \quad (3.13)$$

Now, differentiating (3.12), we get

$$\frac{d}{dt}g_1(Y_1(t), Y_1(t)) = 2\nu \cos \Theta(t) \sin \Theta(t) \frac{d\Theta}{dt}.$$

Using (2.3), we get

$$g_1(\phi \nabla_{\dot{\gamma}} Y_1, \phi Y_1) = \nu \cos \Theta(t) \sin \Theta(t) \frac{d\Theta}{dt}. \quad (3.14)$$

Now, using (2.5), we get

$$\nabla_{\dot{\gamma}} \phi Y_1 = \phi \nabla_{\dot{\gamma}} Y_1 + g_1(\phi \dot{\gamma}, Y_1) \xi,$$

$$\begin{aligned} & g_1(\phi \nabla_{\dot{\gamma}} Y_1, \phi Y_1) \\ = & g_1(\nabla_{\dot{\gamma}} \phi Y_1, \phi Y_1), \\ = & g_1(\mathcal{V} \nabla_{\dot{\gamma}} \psi Y_1, \psi Y_1) + g_1(\mathcal{H} \nabla_{\dot{\gamma}} \omega Y_1, \omega Y_1) + g_1((\mathcal{A}_{Z_1} + \mathcal{T}_{Y_1}) \psi Y_1, \omega Y_1) + \\ & g_1((\mathcal{A}_{Z_1} + \mathcal{T}_{Y_1}) \omega Y_1, \psi Y_1). \end{aligned}$$

Using Lemma 3.6, in above equation, we get

$$\begin{aligned} & g_1(\phi \nabla_{\dot{\gamma}} Y_1, \phi Y_1) \quad (3.15) \\ = & -g_1(\mathcal{V} \nabla_{\dot{\gamma}} B Z_1, \psi Y_1) - g_1(\mathcal{H} \nabla_{\dot{\gamma}} C Z_1, \omega Y_1) - g_1((\mathcal{T}_{Y_1} + \mathcal{A}_{Z_1}) C Z_1, \psi Y_1) - \\ & g_1((\mathcal{T}_{Y_1} + \mathcal{A}_{Z_1}) B Z_1, \omega Y_1) - \eta(Z_1) g_1(Y_1, Y_1). \end{aligned}$$

From equations (3.14) and (3.15), we have

$$\begin{aligned} & \nu \cos \Theta(t) \sin \Theta(t) \frac{d\Theta}{dt} \quad (3.16) \\ = & -g_1(\mathcal{V} \nabla_{\dot{\gamma}} B Z_1, \psi Y_1) - g_1(\mathcal{H} \nabla_{\dot{\gamma}} C Z_1, \omega Y_1) - g_1((\mathcal{T}_{Y_1} + \mathcal{A}_{Z_1}) C Z_1, \psi Y_1) - \\ & g_1((\mathcal{T}_{Y_1} + \mathcal{A}_{Z_1}) B Z_1, \omega Y_1) - \eta(Z_1) g_1(Y_1, Y_1). \end{aligned}$$

Moreover, π is a CSI- $\xi^\perp$ -Riemannian map with $r = e^h$ if and only if $\frac{d}{dt}(e^{h \circ \gamma} \sin \Theta) = 0$, i.e., $e^{h \circ \gamma}(\cos \Theta \frac{d\Theta}{dt} + \sin \Theta \frac{dh}{dt}) = 0$. By multiplying this with non-zero factor $v \sin \Theta$, we have

$$\begin{aligned} -v \cos \Theta \sin \Theta \frac{d\Theta}{dt} &= v \sin^2 \Theta \frac{dh}{dt}, \quad (3.17) \\ v \cos \Theta \sin \Theta \frac{d\Theta}{dt} &= -g_1(Y_1, Y_1) \frac{dh}{dt}, \\ v \cos \Theta \sin \Theta \frac{d\Theta}{dt} &= -g_1(\nabla h, \dot{\gamma}) \|Y_1\|^2, \\ v \cos \Theta \sin \Theta \frac{d\Theta}{dt} &= -g_1(\nabla h, Z_1) \|Y_1\|^2. \end{aligned}$$

Thus, from equations (3.16) and (3.17), we have

$$\begin{aligned} & (g_1(\nabla h, Z_1) - \eta(Z_1))\|Y_1\|^2 \\ = & g_1(\mathcal{V}\nabla_{\dot{\gamma}}BZ_1, \psi Y_1) + g_1(\mathcal{H}\nabla_{\dot{\gamma}}CZ_1, \omega Y_1) + g_1((\mathcal{T}_{Y_1} + \mathcal{A}_{Z_1})CZ_1, \psi Y_1) + \\ & g_1((\mathcal{T}_{Y_1} + \mathcal{A}_{Z_1})BZ_1, \omega Y_1). \end{aligned}$$

□

Corollary 3.8. *Let $\pi : (N_1, \phi, \xi, \eta, g_1) \rightarrow (N_2, g_2)$ be a semi-invariant $\xi^\perp$ -Riemannian map. Then,*

$$g_1(\nabla h, \xi) = 1.$$

4. CSI- $\xi^\perp$ -RIEMANNIAN MAPS FROM A KENMOTSU MANIFOLDS

In this section, we take account of CSI- $\xi^\perp$ -Riemannian maps from Kenmotsu manifolds to Riemannian manifolds. We also discuss geometric properties and obtain some fundamental results for discussed maps.

Theorem 4.1. *Let $\pi : (N_1, \phi, \xi, \eta, g_1) \rightarrow (N_2, g_2)$ be a CSI- $\xi^\perp$ -Riemannian map. Then, we get*

$$\mathcal{A}_{\phi W_1}\phi Y_2 = Y_2(h)W_1 \quad (4.1)$$

for $Y_2 \in \Gamma(\mu)$ and $W_1 \in \Gamma(D_2)$, such that ϕY_2 is basic.

Proof. For $X_1, Y_1 \in \Gamma(D_2)$, using equation (2.13) and Theorem 3.3, we get

$$\mathcal{T}_{X_1}Y_1 = -g_1(X_1, Y_1)\text{grad}h. \quad (4.2)$$

Taking inner product in (4.2) with ϕW_1 , we have

$$g_1(\mathcal{T}_{X_1}Y_1, \phi W_1) = -g_1(X_1, Y_1)g_1(\text{grad}h, \phi W_1), \quad (4.3)$$

for all $W_1 \in \Gamma(D_2)$.

From (2.3), (2.5) and (4.3), we obtain

$$g_1(\nabla_{X_1}\phi Y_1, W_1) = g_1(X_1, Y_1)g_1(\text{grad}h, \phi W_1).$$

Since ∇ is metric connection, using (2.13) and (2.10) in above equation, we get

$$g_1(X_1, W_1)g_1(\text{grad}h, \phi Y_1) = g_1(X_1, Y_1)g_1(\text{grad}h, \phi W_1). \quad (4.4)$$

Taking $W_1 = Y_1$ and interchanging the role of X_1 and Y_1 , we obtain

$$g_1(Y_1, Y_1)g_1(\text{grad}h, \phi X_1) = g_1(X_1, Y_1)g_1(\text{grad}h, \phi Y_1). \quad (4.5)$$

Using (4.4) with $W_1 = X_1$ in (4.5), we have

$$g_1(\text{grad}h, JX_1) = \frac{(g_1(X_1, Y_1))^2}{\|X_1\|^2\|Y_1\|^2}g_1(\text{grad}h, \phi X_1). \quad (4.6)$$

If $\text{grad}h \in \Gamma(\phi(D_2))$, then (4.6) and the condition of equality in the Schwarz inequality implies that either h is constant on $\phi(D_2)$ or the fibers are one dimensional.

On the other hand, using (2.3) and (2.5), we get

$$g_1(\phi\nabla_{X_1}W_1, \phi Y_2) = g_1(\nabla_{X_1}\phi W_1, \phi Y_2)$$

for $Y_2 \in \Gamma(\mu)$ and $Y_2 \neq \xi$. Now, using (2.3), we obtain

$$g_1(\nabla_{X_1} \phi W_1, \phi Y_2) = g_1(\nabla_{X_1} W_1, Y_2).$$

Using (2.9) and (2.13) in above equation, we get

$$g_1(\nabla_{X_1} \phi W_1, \phi Y_2) = -g_1(X_1, W_1)g_1(\text{grad}h, Y_2).$$

Since ϕW_1 is basic and using the fact that $\mathcal{H}\nabla_{X_1} \phi W_1 = \mathcal{A}_{\phi W_1} X_1$, we get

$$\begin{aligned} g_1(\nabla_{X_1} \phi W_1, \phi Y_2) &= -g_1(X_1, W_1)g_1(\text{grad}h, Y_2), \\ g_1(\mathcal{A}_{\phi W_1} X_1, \phi Y_2) &= -g_1(X_1, W_1)g_1(\text{grad}h, Y_2), \\ g_1(\mathcal{A}_{\phi W_1} \phi Y_2, X_1) &= g_1(X_1, W_1)g_1(\text{grad}h, Y_2) \\ g_1(\mathcal{A}_{\phi W_1} \phi Y_2, X_1) &= g_1(X_1, W_1)g_1(\nabla h, Y_2). \end{aligned} \quad (4.7)$$

Since $\mathcal{A}_{\phi W_1} \phi Y_2$ and W_1 are vertical and ∇h is horizontal, we obtain equation (4.1). $\square$

Theorem 4.2. *Let $\pi : (N_1, \phi, \xi, \eta, g_1) \rightarrow (N_2, g_2)$ be a CSI- $\xi^\perp$ -Riemannian map with $r = e^h$ and $\dim(D_2) > 1$. Then, $\bar{\nabla}_{X_1}^\pi \pi_*(\phi Z_1) = X_1(h)\pi_*(\phi Z_1)$, for all $Z_1 \in \Gamma(D_2)$ and $X_1 \in \Gamma(\ker \pi_*)^\perp$.*

Proof. From Theorem 3.3, fibers are totally umbilical with mean curvature vector field $H = -\text{grad}h$, we have

$$\begin{aligned} -g_1(\nabla_{Z_1} X_1, Z_2) &= g_1(\nabla_{Z_1} Z_2, X_1), \\ -g_1(\nabla_{Z_1} X_1, Z_2) &= -g_1(Z_1, Z_2)g_1(\text{grad}h, X_1), \end{aligned}$$

for all $Z_1, Z_2 \in \Gamma(D_2)$ and $X_1 \in \Gamma(\ker \pi_*)^\perp$.

Using (2.3) and (2.5) in above equation, we get

$$g_1(\nabla_{X_1} \phi Z_1, \phi Z_2) = g_1(\phi Z_1, \phi Z_2)g_1(\text{grad}h, X_1). \quad (4.8)$$

Since π is semi-invariant Riemannian map, we have

$$g_2(\pi_*(\nabla_{X_1} \phi Z_1), \pi_*(\phi Z_2)) = g_2(\pi_*(\phi Z_1), \pi_*(\phi Z_2))g_1(\text{grad}h, X_1). \quad (4.9)$$

Using (2.13) in (4.9), we get

$$g_2(\bar{\nabla}_{X_1}^\pi \pi_*(\phi Z_1), \pi_*(\phi Z_2)) = g_2(\pi_*(\phi Z_1), \pi_*(\phi Z_2))g_1(\text{grad}h, X_1), \quad (4.10)$$

which implies $\bar{\nabla}_{X_1}^\pi \pi_*(\phi Z_1) = X_1(h)\pi_*(\phi Z_1)$, for all $Z_1 \in \Gamma(D_2)$ and $X_1 \in \Gamma(\ker \pi_*)^\perp$, hence the proof. $\square$

Proposition 4.3. *Let $\pi : (N_1, \phi, \xi, \eta, g_1) \rightarrow (N_2, g_2)$ be a CSI- $\xi^\perp$ -Riemannian map with $r = e^h$. If $\mathcal{T}$ is not equal to zero identically, then the invariant distribution D_1 does not define a totally geodesic foliation on N_1 .*

Proof. For $Z_1, Z_2 \in \Gamma(D_1)$ and $X_1 \in \Gamma(D_2)$, using equations (2.3), (2.5) and (2.13), we get

$$\begin{aligned} g_1(\nabla_{Z_1} Z_2, X_1) &= g_1(\nabla_{Z_1} \phi Z_2, \phi X_1), \\ &= g_1(\mathcal{T}_{Z_1} \phi Z_2, \phi X_1), \\ &= -g_1(Z_1, \phi Z_2)g_1(\text{grad}h, \phi X_1). \end{aligned}$$

Thus, the assertion can be seen from above equation and the fact that $gradh \in \phi(D_2)$. $\square$

Proposition 4.4. *Let $\pi : (N_1, \phi, \xi, \eta, g_1) \rightarrow (N_2, g_2)$ be a CSI- $\xi^\perp$ -Riemannian map with $r = e^h$. Then, D_2 is not totally geodesic foliation on N_1 .*

Proof. For $X_1, X_2 \in \Gamma(D_2)$ and $\xi \in \Gamma(\ker \pi_*)^\perp$, using (2.6), we get

$$g_1(\nabla_{X_1} X_2, \xi) = -g_1(\nabla_{X_1} \xi, X_2) = -(X_1, X_2) \neq 0.$$

Hence D_2 is not totally geodesic foliation on N_1 . $\square$

Using Propositions (4.3) and (4.4), one can give the following Theorem.

Theorem 4.5. *Let $\pi : (N_1, \phi, \xi, \eta, g_1) \rightarrow (N_2, g_2)$ be a CSI- $\xi^\perp$ -Riemannian map with $r = e^h$. Then, $(\ker \pi_*)$ is not totally geodesic foliation on N_1 .*

Theorem 4.6. *Let $\pi : (N_1, \phi, \xi, \eta, g_1) \rightarrow (N_2, g_2)$ be a CSI- $\xi^\perp$ -Riemannian map with $r = e^h$. Then, the anti-invariant distribution D_2 is one-dimensional.*

Proof. Since π is a Clairaut proper semi-invariant map, then either $\dim(D_2) = 1$ or $\dim(D_2) > 1$. If $\dim(D_2) > 1$, then we can choose $V_1, V_2 \in \Gamma(D_2)$ such that $\{V_1, V_2\}$ is orthonormal. Using (2.5), (2.9), (2.10), (3.2) and (3.3), we get

$$\begin{aligned} \mathcal{T}_{V_1} \phi V_2 + \mathcal{H} \nabla_{V_1} \phi V_2 &= \nabla_{V_1} \phi V_2, \\ \mathcal{T}_{V_1} \phi V_2 + \mathcal{H} \nabla_{V_1} \phi V_2 &= B \mathcal{T}_{V_1} V_2 + C \mathcal{T}_{V_1} V_2 + \psi \mathcal{V} \nabla_{V_1} V_2 + \omega \mathcal{V} \nabla_{V_1} V_2. \end{aligned}$$

Taking inner product above equation with V_1 , we have

$$g_1(\mathcal{T}_{V_1} \phi V_2, V_1) = g_1(B \mathcal{T}_{V_1} V_2, V_1) + g_1(\psi \mathcal{V} \nabla_{V_1} V_2, V_1). \quad (4.11)$$

Using (2.3), (2.5), (2.13) and (4.11), we get

$$g_1(\mathcal{T}_{V_1} V_1, \phi V_2) = -g_1(\mathcal{T}_{V_1} \phi V_2, V_1) = -g_1(gradh, \phi V_2) = g_1(\mathcal{T}_{V_1} V_2, \phi V_1). \quad (4.12)$$

Using above equation, we obtain

$$\begin{aligned} g_1(gradh, \phi V_2) &= g_1(\mathcal{T}_{V_1} V_2, \phi V_1), \\ g_1(gradh, \phi V_2) &= g_1(V_1, V_2) g_1(gradh, \phi V_1), \\ g_1(gradh, \phi V_2) &= 0. \end{aligned}$$

So, we get $gradh \perp \phi(D_2)$.

Therefore, the dimension of D_2 must be one. $\square$

5. EXAMPLE

Example 5.1. Let N_1 be a 5-dimensional space described by the following:

$$\mathbb{R}^5 = \{(x_1, x_2, y_1, y_2, z) \in \mathbb{R}^5 \mid (x_1, x_2, y_1, y_2) \neq (0, 0, 0, 0) \text{ and } z \neq 0\}.$$

Let $\eta = dz$ and $\xi = \frac{\partial}{\partial z}$ where η be a 1-form and ξ be a vector field. and its Riemannian metric g_1 and tensor field ϕ are defined as

$$g_1 = e^{2z}(dx_1)^2 + e^{2z}(dx_2)^2 + e^{2z}(dy_1)^2 + e^{2z}(dy_2)^2 + (dz)^2,$$

$$\phi = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

which gives a Kenmotsu structure (ϕ, ξ, η, g_1) on N_1 .

A ϕ -basis for this Kenmotsu structure can be defined by $\{e_1 = e^{-z} \frac{\partial}{\partial y_1}, e_2 = e^{-z} \frac{\partial}{\partial y_2}, e_3 = e^{-z} \frac{\partial}{\partial x_1}, e_4 = e^{-z} \frac{\partial}{\partial x_2}, e_5 = \xi = \frac{\partial}{\partial z}\}$.

Let N_2 be $\{(u_1, u_2) \in \mathbb{R}^2 | u_2 = z \neq 0\}$. We take the Riemannian metric $g_2 = e^{2z}(du_1)^2 + (du_2)^2$ on N_2 .

Now, we define the map $\pi : (N_1, \phi, \xi, \eta, g_1) \rightarrow (N_2, g_2)$ by:

$$\pi(x_1, x_2, y_1, y_2, z) = \left(\frac{x_1 + y_1}{\sqrt{2}}, 2022, z \right).$$

By direct computations, we have

$$\begin{aligned} \ker \pi_* &= \text{span}\{X_1 = e_2, X_2 = (e_1 - e_3), X_3 = e_4\}, \\ D_1 &= \text{span}\{X_1 = e_2, X_3 = e_4\}, D_2 = \text{span}\{X_2 = (e_1 - e_3)\}, \\ (\ker \pi_*)^\perp &= \text{span}\{U_1 = (e_1 + e_3), U_2 = \xi = e_2\}. \end{aligned}$$

After some computations, one can see that $\pi_*(U_1) = \sqrt{2}e^{-z} \frac{\partial}{\partial u_1}$, $\pi_*(U_2) = \frac{\partial}{\partial u_2}$ and $g_1(U_i, U_j) = g_2(\pi_*U_i, \pi_*U_j)$ for all $U_i, U_j \in \Gamma(\ker \pi_*)^\perp, i, j = 1, 2$. Thus π is semi-invariant $\xi^\perp$ -Riemannian map. Moreover it is easy to see that $\phi X_1 = -X_3, \phi X_2 = -U_1, \phi X_3 = X_1$.

Now, we will obtain smooth function h on $\mathbb{R}^5$ satisfying $\mathcal{T}_X X = g_1(X, X)\nabla h$, for all $X \in \Gamma(\ker \pi_*)$.

We obtain following equation using the discussed Kenmotsu structure,

$$\begin{aligned} [e_1, e_1] &= [e_2, e_2] = [e_3, e_3] = [e_4, e_4] = [e_5, e_5] = 0, \\ [e_1, e_2] &= 0, [e_1, e_3] = 0, [e_1, e_4] = 0, [e_1, e_5] = e_1, \\ [e_2, e_3] &= 0, [e_2, e_4] = 0, [e_2, e_5] = e_2, [e_3, e_4] = 0, \\ [e_3, e_5] &= e_3, [e_4, e_5] = e_4, \end{aligned} \tag{5.1}$$

The Levi-Civita connection ∇ of the metric g_1 is defined by the Koszul's formula,

$$\begin{aligned} &2g_1(\nabla_E V, W) \\ &= U g_1(U, W) + V g_1(W, E) - W g_1(E, U) + g_1([E, U], W) - \\ &g_1([U, W], U) + g_1([W, E], U). \end{aligned} \tag{5.2}$$

Using (5.1) and (5.2), we get

$$\begin{aligned} \nabla_{e_1} e_1 &= \nabla_{e_2} e_2 = \nabla_{e_3} e_3 = \nabla_{e_4} e_4 = -\frac{\partial}{\partial z}, \\ \nabla_{e_1} e_2 &= \nabla_{e_1} e_3 = \nabla_{e_1} e_4 = \nabla_{e_2} e_1 = \nabla_{e_2} e_3 = \nabla_{e_2} e_4 = 0, \\ \nabla_{e_3} e_1 &= \nabla_{e_3} e_2 = \nabla_{e_3} e_4 = \nabla_{e_4} e_1 = \nabla_{e_4} e_2 = \nabla_{e_4} e_3 = 0. \end{aligned} \tag{5.3}$$

Therefore

$$\begin{aligned}\nabla_{X_1}X_1 &= \nabla_{e_2}e_2 = -\frac{\partial}{\partial z}, \nabla_{X_2}X_2 = \nabla_{e_1-e_3}e_1 - e_3 = -2\frac{\partial}{\partial z} \\ \nabla_{X_3}X_3 &= \nabla_{e_4}e_4 - \frac{\partial}{\partial z}, \nabla_{X_1}X_2 = 0, \nabla_{X_1}X_3 = 0, \\ \nabla_{X_2}X_3 &= 0, \nabla_{X_2}X_1 = 0, \nabla_{X_3}X_1 = 0, \nabla_{X_3}X_2 = 0.\end{aligned}\quad (5.4)$$

Now, we have

$$\begin{aligned}\mathcal{T}_X X &= \mathcal{T}_{\lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3} \lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3, \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}. \\ \mathcal{T}_X X &= \lambda_1^2 \mathcal{T}_{X_1} X_1 + \lambda_2^2 \mathcal{T}_{X_2} X_2 + \lambda_3^2 \mathcal{T}_{X_3} X_3 + \\ &\quad \lambda_1 \lambda_2 \mathcal{T}_{X_1} X_2 + \lambda_1 \lambda_3 \mathcal{T}_{X_1} X_3 + \lambda_2 \lambda_3 \mathcal{T}_{X_2} X_3 + \\ &\quad \lambda_1 \lambda_2 \mathcal{T}_{X_2} X_1 + \lambda_1 \lambda_3 \mathcal{T}_{X_3} X_1 + \lambda_2 \lambda_3 \mathcal{T}_{X_3} X_2.\end{aligned}\quad (5.5)$$

Using (5.3), we obtain

$$\begin{aligned}\mathcal{T}_{X_1} X_1 &= -\frac{\partial}{\partial z}, \mathcal{T}_{X_2} X_2 = -2\frac{\partial}{\partial z}, \mathcal{T}_{X_3} X_3 = -\frac{\partial}{\partial z}, \\ \mathcal{T}_{X_1} X_2 &= 0, \mathcal{T}_{X_1} X_3 = 0, \mathcal{T}_{X_2} X_3 = 0, \mathcal{T}_{X_2} X_1 = 0, \\ \mathcal{T}_{X_3} X_1 &= 0, \mathcal{T}_{X_3} X_2 = 0.\end{aligned}\quad (5.6)$$

Next, using (5.5) and (5.6), we get

$$\mathcal{T}_X X = -(\lambda_1^2 + 2\lambda_2^2 + \lambda_3^2 + \lambda_3^2) \frac{\partial}{\partial z}.$$

Since $X = \lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3$, so $g_1(\lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3, \lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3) = \lambda_1^2 + 2\lambda_2^2 + \lambda_3^2$. Now, $\nabla h = e^{-2Z} \frac{\partial h}{\partial y_1} \frac{\partial}{\partial y_1} + e^{-2Z} \frac{\partial h}{\partial y_2} \frac{\partial}{\partial y_2} + e^{-2Z} \frac{\partial h}{\partial x_1} \frac{\partial}{\partial x_1} + e^{-2Z} \frac{\partial h}{\partial x_2} \frac{\partial}{\partial x_2} + \frac{\partial h}{\partial z} \frac{\partial}{\partial z}$. Hence $\nabla h = \frac{\partial}{\partial z}$ for the function $h = z$. Therefore, $\mathcal{T}_X X = -g_1(X, X)\nabla h$, thus by Theorem (3.3), π is a CSI- $\xi^\perp$ -Riemannian map.

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