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ANNIHILATOR GRAPHS DERIVED FROM GROUP RINGS

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Abstract. Let RG be the group ring of the group G over a ring R and let $\mathcal{Z}^*(RG)$ be the collection of all non-zero zero divisors in a finite group ring RG. And, for $x \in \mathcal{Z}^*(RG), ann(x) = \{r \in RG/rx = 0\}$. The annihilator graph of RG denoted as $\mathcal{AG}(RG)$, and is defined as the graph whose vertex set is the elements of non-zero zero divisors in RG and two distinct vertices v_x and v_y are adjacent if and only if $ann(x) \cap ann(y) \neq \{0\}$. In this paper we try to characterize the properties of annihilator graphs in group rings.

1. Introduction and preliminaries

In algebraic graph theory we have many graph theoretical methods to solve the problems in algebra. Also there are many graph theoretical problems arising from rings and groups [3]. In this paper, we consider the concept from group rings [4], [8]. The central role of group ring in group representation theory was established by E. Noether and R. Brauer during the period 1927 to 1929.

Let R be a ring and G be a group, then the group ring RG is the free R-module with basis G which consists of elements of the form $\sum_{g\epsilon G} a_g g$, where $a_g \epsilon R$ and $g \epsilon G$, with addition defined as

$$\sum_{g\epsilon G} a_g g + \sum_{g\epsilon G} b_g g = \sum_{g\epsilon G} (a_g + b_g) g$$

and the product defined as,

$$(\sum_{g\epsilon G} a_g g)(\sum_{h\epsilon G} b_h g) = (\sum_{g,h\epsilon G} a_g b_h gh)$$

RG is the set of all linear combinations of elements in G with coefficients in R. If R is commutative then RG is called group algebra. Let $\mathcal{U}(RG)$ denote the group of units in RG and let $Z^*(RG)$ denote the set of all non-zero zero divisors. And, for $x \in \mathcal{Z}^*(RG)$, define $ann(x) = \{r \in RG/rx = 0\}$. Also for a given group ring RG and a finite subset X of the group G, we shall denote $\hat{X}$ the following element of RG: $\hat{X} = \sum_{x\epsilon X} x$. In particular, if $X = G$, then $\hat{G} = \sum_{x\epsilon G} x \in Z(RG)$, since $\hat{G}(1 - g) = 0$ for any $g \in G$ with $g \neq 1$.

Ralph P Grimaldi defined a graph in Z_n, based on elements and units in Z_n.

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Also in 2017, Mojgan Afkhami defined the annihilator graph in group rings[9]. But here we give another definition of annihilator graphs in finite group rings and study their structures.

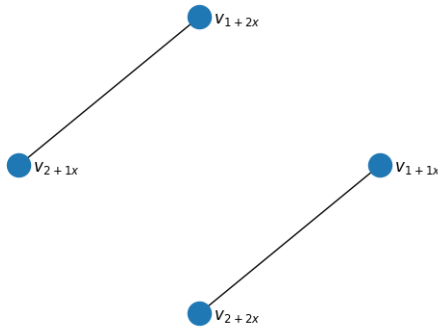
Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. If $\{u, v\}$ is a member of E then we say u and v are adjacent and the corresponding edge is denoted as uv. We denote the complete graph with n vertices by K_n [7]. If there exist a path joining two vertices of the graph then the graph is said to be connected. Otherwise it is disconnected. For a graph G, G^k represents a graph with the vertex set is same as in G and two vertices are adjacent in G^k when their distance in G is at most k.
In 2014, Farid Aliniaeifard and Y.Li[5] defined a zero divisor graph $\Gamma(RG)$ in a group ring RG, where the set of vertices of $\Gamma(RG)$ is $Z^*(RG)$, and two distinct vertices x and y are adjacent if and only if $xy = 0$. In this paper we define an annihilator graph, denoted by $\mathcal{AG}(RG)$, of a group ring RG, and study a relationship between the annihilator graphs and zero divisor graphs.

2. Characterization of Annihilator Graphs in Group Rings

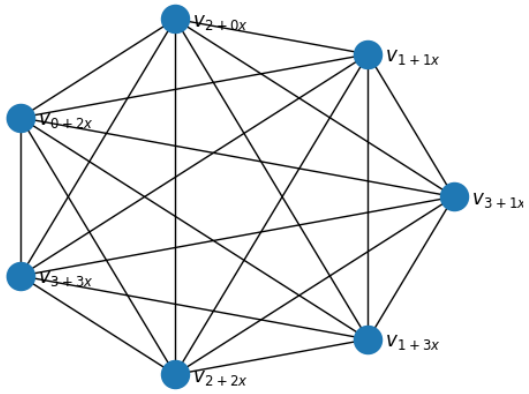
In this section we characterize the properties of annihilator graphs in Group Rings. Throughout this section we use Z_nC_m in terms of RG, where Z_n is the ring of integers modulo n and
$C_m = < x/x^m = 1 >$.

Definition 2.1. Let RG be a group ring. We construct a graph associated with the set of all non-zero, zero divisors of RG denoted by $\mathcal{AG}(RG)$, is called the annihilator graph of RG. The set of all non-zero zero divisors of RG are considered to be the vertices of the graph and two vertices v_x and v_y are adjacent if and only if $ann(x) \cap ann(y) \neq \{0\}$ for $x \neq y$; where v_x represents the vertices corresponding to the element $x \epsilon \mathcal{Z}^*(RG)$.

Example 2.2. Consider the group ring Z_3C_2 ; where $Z_3 = \{0, 1, 2\}$ and $C_2 = \{1, x\}$. Let $\mathcal{Z}^*(RG)$ be the set of all non-zero, zero divisors in RG. Then the following figure represents the corresponding annihilator graph of Z_3C_2.



Example 2.3. Consider the group ring Z_4C_2 ; where $Z_4 = \{0, 1, 2, 3\}$ and $C_2 = \{1, x\}$. Let $\mathcal{Z}^*(RG)$ be the set of all non-zero zero divisors in RG. Then the following figure represents the corresponding annihilator graph of Z_4C_2.



Theorem 2.4. *Let RG be a finite commutative group ring with non-zero identity, then the following conditions hold:*

a. *Let $n \geq 2$. Then $\mathcal{AG}(Z_nC_n)$ is complete.*
b. *If $R \cong Z_{p^k}$ and $G \cong C_p; k \geq 1$ and p is a prime number then $\mathcal{AG}(RG)$ is a complete graph.*

Proof. [a]. Let $RG = Z_nC_n$. From the definition of annihilator graph, we have $|V[\mathcal{AG}(Z_nC_n)]| = |Z^*(Z_nC_n)|$. We have to prove that $\mathcal{AG}(Z_nC_n)$ is a complete graph. That is to show for any two distinct vertices x and $y \in \mathcal{AG}(Z_nC_n)$, $ann(x) \cap ann(y) \neq \{0\}$. For the group ring Z_nC_n, we have $Char(R) = |G| = n$. Here we divide the proof into two cases.

(i) when n is prime.
(ii) when n is not a prime

Case-(i): when n is prime.
In this case Z_n is a field. Therefore $|Z(R)| = 1$.
Subcase-i: Let $|R| \geq 3$, then $|G| \geq 3$ also. We have $\hat{G} = \sum_{g \in G} g \in Z^*(RG)$ Let x and y be non-zero distinct element in R such that $x, y \neq 1$ and $\hat{G} \in Z(RG)$ then $x\hat{G}$ and $y\hat{G} \in Z^*(RG)$.
If $x\hat{G}.y\hat{G} = 0$, then $x\hat{G} \in ann(y\hat{G})$ and $y\hat{G} \in ann(x\hat{G})$.
Also we have for $|R| \geq 3, \hat{G}^2 = 0$.
Therefore, $x\hat{G}.\hat{G} = 0$ implies $\hat{G} \in ann(x\hat{G})$.
and $y\hat{G}.\hat{G} = 0$ implies $\hat{G} \in ann(y\hat{G})$.
Therefore, $\hat{G} \in ann(x\hat{G}) \cap ann(y\hat{G})$ for all $x, y \in R$.
Now take some $z \neq x\hat{G} \in Z^*(RG)$ then $z = \sum_{g \in G} a_g g$.
Consider $z\hat{G} = \sum_{g \in G} a_g g. \sum_{g \in G} g$
$= \sum_{g \in G} a_g. \sum_{g \in G} g$

$= \sum_{g\in G} a_g.\hat{G}$
$= 0.\hat{G}$
Therefore, $\hat{G} \in ann(z)$ for all $z \neq x\hat{G} \in Z^*(RG)$.
Similarlly $z \in ann(\hat{G})$. Also we have $\hat{G} \in ann(x\hat{G})$. Therefore, $ann(z) \cap ann(x\hat{G}) \neq \{0\}$ for all $z, x\hat{G} \in Z^*(RG)$. Again if we take any two distinct $z, s \neq x\hat{G} \in Z^*(RG)$, we have $z\hat{G} = s\hat{G} = 0$. That implies $\hat{G} \in ann(z) \cap ann(s)$. Therefore in all cases we get a non empty intersection. Hence $\mathcal{AG}(Z_nC_n)$ is a complete graph.
Subcase-(ii): $|R| < 3$
In this case $|G| < 3$. Therefore $RG = Z_2C_2$. Here $\hat{G} = 1 + g \in Z^*(Z_2C_2)$ This is the only one element in $Z^*(Z_2C_2)$. Therefore $\mathcal{AG}(Z_2C_2) \cong K_1$.
Case-(ii): When n is not a prime.
Subcase-(i): When $|Z(R)| = 2$. Since $|R| \leq |Z(R)|^2$, and R is not a field, $|R| = 4$. Therefore, $|G|$ is also equal to 4. Let $Z(R) = \{0, x\}$, then $x^2 = 0$. Since $|G| = 4$, there exist non-identity distinct elements g_1, g_2, g_3 such that $G = \{1, g_1, g_2, g_3\}$. Here $\hat{G} = \sum_{g\in G} g = 1 + g_1 + g_2 + g_3$
$\hat{G}.\hat{G} = (1 + g_1 + g_2 + g_3).(1 + g_1 + g_2 + g_3)$
$= 0$
Therefore, $\hat{G} \in ann(\hat{G})$.
Consider $x\hat{G}.\hat{G} = x.0 = 0$. That implies $\hat{G} \in ann(x\hat{G})$. Therefore, $ann(x\hat{G}) \cap ann(\hat{G}) \neq \{0\}$
Again take some $y = \sum_{g\epsilon G} a_g g$ and $z = \sum_{g\epsilon G} b_g g \in Z^*(RG)$; we get $y\hat{G} = 0$ and $z\hat{G} = 0$. Therefore $\hat{G} \in ann(y) \cap ann(z)$ for all $y, z \in Z^*(RG)$.
Hence if we take any two distinct vertices v_x and $v_y$in $\mathcal{AG}(Z_4C_4)$ we get $ann(x) \cap ann(y) \neq \{0\}$. Therefore, $\mathcal{AG}(Z_4C_4)$ is complete.
Subcase-(ii): When $|Z(R)| > 2$
That implies $|R| > 5$ and $|G| > 5$. Since $Z(R) \subseteq Z(RG)$, we have for $x, y \in Z(R); x, y \in Z^*(RG)$.
Also for all $x, y \in Z(R)$ we have $x\hat{G} = 0$ and $y\hat{G} = 0$. That implies $\hat{G} \in ann(x)$ and $ann(y)$. That is, $\hat{G} \in ann(x) \cap ann(y)$. Therefore, for all $x, y \in Z^*(R) \subset Z^*(RG)$, we have v_x and v_y are adjacent. Now take some $r = \sum_{g\epsilon G} a_g g$ and $s = \sum_{g\epsilon G} b_g g$ in $Z^*(RG)$. Then from the above cases we can prove that $\hat{G} \in ann(r)$ and $ann(s)$. That is, $\hat{G} \in ann(r) \cap ann(s)$. Therefore, if we take any two distinct pair of elements x and y in $Z^*(RG)$ there exist an edge between v_x and v_y in $\mathcal{AG}(Z_nC_n)$.
Hence in all cases the result is true.

[b]. From part-[a] of theorem 2.4, we get $\mathcal{AG}(Z_pC_p)$ is a complete graph, for $k = 1$. Therefore, for proving the teorem it is enough to show that for $k > 1$, $\mathcal{AG}(RG)$ is a complete graph
For $k > 1$, we have R is not an integral domain. Then there exist non-zero elements $x, y \in R$ such that $xy = 0$. It follows that for $x, y \in Z^*(R)$ and since $Z^*(R) \subset Z^*(RG)$, $x, y \in Z^*(RG)$ also.

Now consider the case for $p = 2$. That is, $RG = Z_{2^k}C_2$; $k > 1$. We have the zero divisors of Z_{2^k} are
$\{2, 4, 6, ..., 2^k-2\} = \{2.1, 2.2, 2.3, ..., 2.(2^{k-1}-1)\}$, which is a subset of $Z^*(Z_{2^k}C_2)$.
Claim: $2^{k-1}.\hat{G} \in ann(x)$ for all $x = \sum_{g\epsilon G} a_g g \in Z_{2^k}C_2$.
If $x \in Z^*(Z_{2^k})$, then x is of the form some $2r$ for $r = 1, 2, ..., (2^{k-1} - 1)$. Hence
$$\begin{aligned} x.2^{k-1}.\hat{G} &= 2r.2^{k-1}.\hat{G} \\ &= r2^k.\hat{G} \\ &\equiv 0(mod\ 2^k) \end{aligned}$$
That implies that $2^{k-1}.\hat{G} \in ann(x)$ for all $x \in Z*(R)$. That means $2^{k-1}.\hat{G} \in ann(x)$ for all $x \in Z*(RG)$.
Now suppose that x is not of the form $2r$. That is $x = \sum_{g\epsilon G} a_g g$.
$$\begin{aligned} \text{Then } 2^{k-1}.\hat{G}.x &= 2^{k-1}.\hat{G}.\textstyle\sum_{g\epsilon G} a_g g \\ &= (2^{k-1}.\textstyle\sum_{g\epsilon G} a_g)\hat{G}. \\ &= 0.\hat{G} \\ &= 0. \end{aligned}$$
That implies $2^{k-1}.\hat{G} \in ann(x)$ for all $x \in Z_{2^k}C_2$. Hence $\mathcal{AG}(Z_{2^k}C_2)$ is a complete graph.
Again consider the case for $p \neq 2$. Here we have $Z^*(Z_{p^k}) = \{p, 2p, 3p, ..., (p^{k-1} - 1).p\}$ which is a subset of $Z^*(Z_{p^k}C_p)$. That is, for $1 \leq r \leq p^{k-1} - 1, rp \in Z^*(Z_{p^k}) \subset Z^*(Z_{p^k}C_p)$. To prove $p^{k-1}(p-1).\hat{G} \in ann(x)$ for all $x \in Z^*(Z_{p^k}C_p)$.
For proving this, we divide the proof into 2 cases

(i) for $x \in Z^*(Z_{p^k})$
(ii) for $x \notin Z^*(Z_{p^k})$

If $x \in Z^*(Z_{p^k})$, then x is of the form rp; where $r = 1, 2, ..., (p^{k-1} - 1)$. Consider
$$\begin{aligned} ann(rp) &= \{y \in Z_{p^k} : rp.y \equiv 0(mod\ p^k)\} \\ &= \{y \in Z_{p^k} : p^{k-1} | ry\} \end{aligned}$$
That implies, $p^{k-1} \in ann(rp)$. That is, $p^{k-1} \in ann(x)$ for all $x \in Z^*(Z_{p^k}) \subset Z^*(Z_{p^k}C_p)$
$$\begin{aligned} \text{Therefore, } p^{k-1}(p-1).\hat{G}.x &= p^k.\hat{G}.x - p^{k-1}.\hat{G}.x \\ &= 0 - 0\ \ (mod\ p^k) \\ &= 0 \end{aligned}$$
Therefore $p^{k-1}(p-1).\hat{G} \in ann(x)$ for all $x \in Z^*(Z_{p^k}C_p)$. Therefore, $ann(x) \cap ann(y) \neq \{0\}$ for all $x, y \in Z^*(Z_{p^k}) \subset Z^*(Z_{p^k}C_p)$
Now consider the case for $x \notin Z^*(Z_{p^k})$, then $x = \sum_{g\epsilon G} a_g g$
Claim : $p^{k-1}(p-1).\hat{G} \in ann(x)$ for all $x \in Z^*(Z_{p^k}C_p)$
$$\begin{aligned} p^{k-1}(p-1).\hat{G}.x &= p^{k-1}(p-1).\hat{G}.\textstyle\sum_{g\epsilon G} a_g g \\ &= p^{k-1}(p-1)\textstyle\sum_{g\epsilon G} a_g \hat{G} \\ &= [p^k \textstyle\sum_{g\epsilon G} a_g - p^{k-1} \sum_{g\epsilon G} a_g]\hat{G} \\ &= 0 - p^{k-1} \textstyle\sum_{g\epsilon G} a_g \hat{G} (mod\ p^k) \\ &= -p^k.\hat{G} (mod\ p^k) \\ &\equiv 0(mod\ p^k) \end{aligned}$$

Hence $p^{k-1}(p-1).\hat{G} \in ann(x)$ for all $x \in Z^*(Z_{p^k}C_p)$
That is, $ann(x) \cap ann(y) \neq \{0\}$ for all $x \neq y; x, y \in Z^*(Z_{p^k}C_p)$
Hence, $\mathcal{AG}(Z_{p^k}C_p)$ is the complete graph.

□

Theorem 2.5. *If p and q are two distinct prime integers, then $\mathcal{AG}(Z_pC_q)$ is always disconnected and has two components, each of which is a complete graph.*

Proof. Consider the group ring Z_pC_q, where p and q are distinct prime numbers. Using the above remark, we have to partitioned the vertex set of $\mathcal{AG}(Z_pC_q)$ into two sets; $V_1[\mathcal{AG}(Z_pC_q)]$ and $V_2[\mathcal{AG}(Z_pC_q)]$.
Where $V_1[\mathcal{AG}(Z_pC_q)]$ contains those vertices which are the vertex corresponds to the elements like $r\hat{G}, r \in Z_p$. And the vertices corresponding to the remaining elements belongs to $V_2[\mathcal{AG}(Z_pC_q)]$.
when $q = 2$, we have $|Z^*(Z_pC_2)| = 2(p-1)$. Also we have $\hat{G} = 1 + g$.
Let $V[\mathcal{AG}(Z_pC_2)] = V(G_1) \cup V(G_2)$, where $G_1 \cup G_2 = Z^*(Z_pC_2)$ and
$G_1 \cap G_2 = \phi$
Let us take $G_1 = \{\hat{G}, 2\hat{G}, 3\hat{G}, ..., (p-1)\hat{G}\}$
and $G_2 = \{\sum_{g\epsilon G} a_g g / \sum a_g \equiv 0(modp)\}; |G_2| = p-1$.
For proving the lemma, we have to show that the graph corresponds to G_1 and G_2 is equal to K_{p-1}. And there is no edge connecting G_1 and G_2.
Claim: $ann(x) \cap ann(y) \neq \{0\}$ for all $x, y \in G_1$.
Since $x, y \in G_1$, we have x and y is of the form $r\hat{G}$; where
$r = 1, 2, 3, ..., (p-1)$. Now if we take any $z \in Z_pC_2$, then
$z = \sum_{g\epsilon G} a_g g$; $\sum_{g\epsilon G} a_g \equiv 0(mod\ p)$
Therefore, clearly $z \in G_2$. Again $r\hat{G}.z = r\hat{G}.\sum_{g\epsilon G} a_g g$

$$\begin{aligned} &= r.\textstyle\sum_{g\epsilon G} a_g.\hat{G} \\ &= \textstyle\sum_{g\epsilon G} a_g.r.\hat{G} \\ &\equiv 0(mod\ p) \end{aligned}$$

This implies $z \in ann(r\hat{G})$ for all $r = 1, 2, 3, ..., (p-1)$. Therefore, $ann(x) \cap ann(y) \neq \{0\}$ for all $x, y = r\hat{G}$. Since $|G_1| = p-1$, we have the annihilator graph corresponds to G_1 is K_{p-1}.
Similarly in G_2, we get for all $z_1 = \sum_{g\epsilon G} a_g g$, $r\hat{G}.z_1 = 0$. That implies $r.\hat{G} \in ann(z_1)$. Hence in G_2, $ann(z_1) \cap ann(z_2) \neq \{0\}$
Therefore, the annihilator graph corresponds to G_2 is K_{p-1}.
Now to show that there is no edge connecting $V(G_1)$ to $V(G_2)$.
From the above proof, it is clear that for all $r.\hat{G} \in G_1$,

$$\begin{aligned} ann(r\hat{G}) &= \{z/z = \textstyle\sum_{g\epsilon G} a_g g \in G_2 \ and\ \ \sum_{g\epsilon G} a_g \equiv 0(mod\ p)\} \\ &= G_2 \end{aligned}$$

And for all $z \in G_2$, $ann(z) = \{r\hat{G}/r = 1, 2, 3, ..., (p-1)\}$
$= G_1$.
But we have, $G_1 \cap G_2 = \phi$. Therefore, $ann(r\hat{G}) \cap ann(z) = \{0\}$. Therefore, $v_{r\hat{G}}$ is not adjacent to v_z for all $v_{r\hat{G}} \in V(G_1)$ and $v_z \in V(G_2)$. $G_1 \cup G_2$ is disconnected. Therefore $\mathcal{AG}(Z_pC_2)$ contains 2 copies of K_{p-1}. So now we consider the case for $q =$ odd prime. Let us take $V_1[\mathcal{AG}(Z_pC_q)] = \{v_{\hat{G}}, v_{2\hat{G}}, v_{3\hat{G}}, ..., v_{(p-1)\hat{G}}\}$.

and $V_2[\mathcal{AG}(Z_pC_q)] = \{\sum_{g\epsilon G} a_g g / \sum a_g \equiv 0(modp)\}$.
For proving the theorem, it is enough to show that for all $v_x, v_y \in V_1[\mathcal{AG}(Z_pC_q)$, $ann(x) \cap ann(y) \neq \{0\}$. Simillarly, in $V_2[\mathcal{AG}(Z_pC_q)]$.
And for all $v_x \in V_1[\mathcal{AG}(Z_pC_q)$,and $v_z \in V_2[\mathcal{AG}(Z_pC_q)], ann(x) \cap ann(y) = \{0\}$.
Let us take $v_x, v_y \in V_1[\mathcal{AG}(Z_pC_q)]$.
Then x and y is of the form $r\hat{G}$; where $r = 1, 2, 3, ..., (p-1)$.
For all $v_x, v_y \in V_1[\mathcal{AG}(Z_pC_q)]$, take some $z \in Z_pC_q$ with $z = \sum_{g\epsilon G} a_g g$ and $\sum a_g = 0$.
Then clearly $v_z \in V_2[\mathcal{AG}(Z_pC_q)]$.
Consider
$$\begin{aligned} x.z &= r\hat{G}.\sum_{g\epsilon G} a_g g \\ &= r.\sum_{g\epsilon G} a_g.\hat{G} \\ &= \sum_{g\epsilon G} a_g.r\hat{G} \\ &\equiv 0(mod\ p) \end{aligned}$$
Therefore, $z \in ann(x)$, for all $v_x \in V_1[\mathcal{AG}(Z_pC_q)]$.
Therefore $ann(x) \cap ann(y) \neq \{0\}$ for all $v_x, v_y \in V_1[\mathcal{AG}(Z_pC_q)]$.
Simillarly in $V_2[\mathcal{AG}(Z_pC_q)]$, $r\hat{G} \in ann(z)$ for all $v_z \in V_2[\mathcal{AG}(Z_pC_q)]$.
So v_z is adjacent to $v_{z'}$ for all $v_z, v_{z'} \in V_2[\mathcal{AG}(Z_pC_q)]$.
Now to show that $ann(x) \cap ann(y) = \{0\}$ for $v_x \in V_1[\mathcal{AG}(Z_pC_q)]$ and $v_z \in V_2[\mathcal{AG}(Z_pC_q)]$.
Since $v_x \in V_1[\mathcal{AG}(Z_pC_q)]$, x is of the form $r\hat{G}$. And annihilator of $r\hat{G}$ contains elements of the form $z = \sum_{g\epsilon G} a_g g$, where $\sum_{g\epsilon G} a_g = 0$.
That is, $ann(r\hat{G}) = \{z / v_z \in V_2[\mathcal{AG}(Z_pC_q)], z = \sum_{g\epsilon G} a_g g, \sum_{g\epsilon G} a_g = 0\}$.
Simillarly, for all $v_z \in V_2[\mathcal{AG}(Z_pC_q)], ann(z) = \{r\hat{G} / r = 1, 2, ..., (p-1)\}$
Therefore, clearly $ann(x) \cap ann(z) = \{0\}$ for all $v_x \in V_1[\mathcal{AG}(Z_pC_q)]$ and $v_z \in V_2[\mathcal{AG}(Z_pC_q)]$. Hence from the definition of annihilator graph, we get there is no edge connecting $V_1[\mathcal{AG}(Z_pC_q)]$ and $V_2[\mathcal{AG}(Z_pC_q)]$. Hence $\mathcal{AG}(Z_pC_q)$, contains two disconnected component with each component is complete. □

Corollary 2.6. *$\mathcal{AG}(Z_pC_2) = 2K_{p-1}$ where p is a prime number.*

Corollary 2.7. *If $RG = Z_pC_q$, for $p \neq q$ and p and q are prime number then $\omega(\mathcal{AG}(Z_pC_q)) = 2$.*

Theorem 2.8. *Let Z_mC_n be a finite group ring. Then*

(i) *$\mathcal{AG}(Z_pC_q) = [\Gamma(Z_pC_q)]^c$, where p and q are prime numbers.*
(ii) *$\Gamma(Z_{p^k}C_p)$ is the spanning subgraph of $\mathcal{AG}(Z_{p^k}C_p)$, for a prime number p.*

Where $\Gamma(Z_mC_n)$ is the zero-divisor graph corresponds to the group ring Z_mC_n.

Lemma 2.9. *The graphs $[\mathcal{AG}(RG)] \cup [\Gamma(RG)]$ and $[\Gamma(RG)]^2$ are identical graphs.*

Proof. Since the vertex set of $\Gamma(RG)$ and $\mathcal{AG}(RG)$ are same, it is enough to show that the edge set of $[\mathcal{AG}(RG)] \cup [\Gamma(RG)]$ and $[\Gamma(RG)]^2$ are same.
Let us assume that v_x is adjacent to v_y in $E\{[\mathcal{AG}(RG)] \cup [\Gamma(RG)]\}$. Then either $xy = 0$ or $ann(x) \cup ann(y) \neq \{0\}$. That means either $xy = 0$ or there exist some $z \in Z^*(RG)$ such that $z \in ann(x) \cup ann(y)$. That is, $xz = 0$ and $yz = 0$. In both cases, we get $d(v_x, v_y) \leq 2$. And hence $v_xv_y \in E\{[\Gamma(RG)]^2\}$.

Conversely, consider $v_xv_y \in E\{[\Gamma(RG)]^2\}$. Then $d(v_x, v_y) \leq 2$ in $[\Gamma(RG)]^2$. Let $d(v_x, v_y) = 1$. This implies that v_x and v_y are adjacent vertices. That is, $xy = 0$ Which implies $v_xv_y \in E[\Gamma(RG)]$. Now consider $d(v_x, v_y) = 2$. Then there exist another vertex v_w such that v_x is adjacent to v_w which is adjacent to v_y for some $w \in Z^*(RG)$. Which means $w \in ann(x)$ and $w \in ann(y)$.
That implies that $w \in ann(x) \cup ann(y)$. Therefore $v_xv_y \in E[\mathcal{AG}(RG)]$.
From both cases we get $v_xv_y \in E[\mathcal{AG}(RG)] \cup [\Gamma(RG)]$.

□

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