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STOCHASTIC FRACTIONAL INTEGRODIFFERENTIAL EQUATIONS WITH CONTINUOUS CONDITIONS

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Abstract. In this paper, we use a smoothness method to obtain an existence of a strong solution for a general class of stochastic fractional integrodifferential equations. We established also the uniqueness of solution. We consider the case of continuous coefficients, which satisfy some so-called $\rho-$condition.

1. Introduction

An integrodifferential equation is an equation that includes both integrals and derivatives of a certain unknown function. In theory, but also in applications, these equations play an important role in the mathematical modelling of various fields taking into account the effects of real-world problems such as physics, biology, engineering sciences see among other ([2]). Furthermore, over the past twenty years, with the advent of fractional differential equations as a field of research, more and more researchers have become interested in the study of fractional calculus. Since then, different concepts of derivatives and fractional integrals have been introduced, such as the integrals and derivatives of Riemann-Liouville, Caputo, Grunwald Letnikov, Riesz (see [6], [11]). Fractional differential equations ([10]) can also effectively describe the dynamic behaviour of real-life phenomena with more accuracy than integer order equations. With the development of fractional calculus, fractional integrodifferential equations appear in many domains such as electromagnetic waves ([17]) and dynamic population system ([20], [12]). Some authors ([4], [1], [3]) show different existence results for fractional integrodifferential equations.

On the other hand, the present of random effects is noted in many branches of science, in particular economics and finance, physics, demography, biology and medicine. In 1942, Itô introduced the stochastic differential equations (SDE) which will know a real expansion ([13], [16]). Therefore stochastic integrodifferential equations appear as a natural extension of stochastic differential equations. These equations are involved in stochastic feedback systems ([15]), option pricing ([5]) and population growth model ([9]). Nowadays, more and more researchers

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are interested in fractional stochastic equations ([7], [8], [9]). Using the classical Picard-Lindelöf successive approximation scheme, Pedjeu and Ladde [14] established existence and uniqueness of solution of stochastic fractional differential equations. They used the Riemann-Liouville type fractional integral. The stochastic fractional equations are effective in modeling hereditary and hidden properties of certain noise systems in mathematical finance ([18]), in ecology and epidemiology ([14]). Also, note that several dynamic processes in science and engineering are under the influence of random perturbations of both internal and external environmental natures. So to bring more precision to the model, Pedjeu and Ladde [14] consider fractional stochastic differential equations by introducing the concept of dynamic processes operating under a set of linearly independent time scales.
In the literature we find a lot of result dealing with existence and uniqueness of fractional integrodifferential equations ([3], [4], [2]). In this spirit Umamaheswari et al. [19] established an existence and uniqueness of solution of stochastic fractional integrodifferential equations under the Lipschitz condition by using the Picard-Lindelöf approximation. These results generalise those of Pedjeu and Ladde [14] by going from a fractional stochastic differential equation to a stochastic fractional integrodifferential equation. A natural question is: under which other condition on the coefficients does the stochastic fractional integrodifferential equations have a solution?

In this paper, inspired by a method developed in Rong [16, Theorem 170], we solve a general class of stochastic fractional integrodifferential equations with continuous conditions. We prove an existence and uniqueness result which extend the result of Umamaheswari et al. [19] in the case of coefficients satisfying rather weaker conditions.

The rest of the paper is organized as follows. In section 2, we present some useful preliminaries notions. In section 3, we prove our main results.

2. Preliminaries

Definition 2.1. ($Riemann-Liouville\ fractional\ integral$).
The Riemann-Liouville fractional integral operator of order α of a function $f \in L^1(\mathbb{R}_+)$ is defined by

$$I_{0^+}^{(\alpha)} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \qquad \forall \alpha > 0,$$

where $\Gamma(.)$ is the Euler Gamma function defined by

$$\Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1} \exp(-t) dt.$$

Definition 2.2. ($Riemann-Liouville\ fractional\ derivative$).
The Riemann-Liouville fractional derivative of order $\alpha > 0$, $n-1 < \alpha < n$,

$n \in \mathbb{N}^*$, is defined as

$$D_{0^+}^{(\alpha)} f(t) = \left(\frac{d}{dt}\right)^n I_{0^+}^{(n-\alpha)} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} f(s) ds,$$

where the function $f(t)$ has absolutely continuous derivatives upto order $(n-1)$.

Definition 2.3. (Multi-time scale Integral [14])
For $p \in \mathbb{N}$, $p > 1$, let $\{T_1, \cdots, T_p\}$ be a set of linearly independent time scales. Let

$$f : [a, b[\times \mathbb{R}^{p-1} \to \mathbb{R}^n$$

be a continuous function defined by $f(t) = f(T_1(t), \cdots, T_p(t))$. The multi-time scale integral of the composite function f over an interval $[t_0, t] \subseteq]a, b[$ is defined as the sum of p integrals with respect to the time-scales $T_1, \cdots, T_p$. We denote it by If,

$$If(t) := \sum_{j=1}^{p} I_j f(t),$$

where the sense of the integral

$$I_j f(t) = \int_{t_0}^{t} f(s) dT_j(s)$$

depends on the time scale T_j, for each $j = 1, \cdots, p$.

Example 2.4. For $p = 3$, consider the linearly independent set consisting of time scales $T_1(t) := t$, $T_2(t) := B(t)$ where B is the standard Wiener process, and $T_3(t) := t^\alpha$, $0 < \alpha < 1$ as defined before. In this case,

$$f(t) \equiv f(T_1(t), T_2(t), T_3(t)) \quad \textit{and} \quad If(t) = I_1 f(t) + I_2 f(t) + I_3 f(t)$$

where the integrals

$$I_1 f(t) = \int_{t_0}^{t} f(s) ds, \quad I_2 f(t) = \int_{t_0}^{t} f(s) dB(s), \quad I_3 f(t) = \int_{t_0}^{t} f(s) (ds)^\alpha$$

are Riemann, Itô-Doob, and Riemann-Liouville type, respectively.

Under the set of time scales in Example 2.4, we have the following stochastic fractional differential equation

$$dx(t) = b(t, x(t))dt + \sigma_1(t, x(t))dW_t + \sigma_2(t, x(t))(dt)^\alpha, \quad t \in [0, T]. \tag{2.1}$$

Remark 2.5.
(i) If $\sigma_2 = 0$ in Example 2.4, then the SDE (2.1) reduced to known Itô-Doob type stochastic

$$dx(t) = b(t, x(t))dt + \sigma_1(t, x(t))dW_t, \qquad x(0) = x_0$$

whose fundamental properties and applications have been well studied for more than half-century.

(ii) If $\sigma_1 = 0$ in Example 2.4, we have the following generalized version of the classical deterministic fractional differential equations

$$dx(t) = b(t, x(t))dt + \sigma_2(t, x(t))(dt)^\alpha, \qquad x(0) = x_0.$$

(iii) If $\sigma_1 = 0$ and $\sigma_2 = 0$, then (2.1) is to the deterministic

$$dx(t) = b(t, x(t))dt, \qquad x(0) = x_0.$$

We give the following lemma known as the bihari inequality (Mao [13, Theorem 8.2]).

Lemma 2.6. (*Bihari Inequality*)
Let $T > 0$, $u_0 \geq 0$, $u(t), v(t)$ be continuous nonnegative functions on $[0, T]$, and $\rho : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be a concave continuous non-decreasing function such that $\rho(x) > 0$ for $x > 0$. If

$$u(t) \leq u_0 + \int_0^t v(s)\rho\left(u(s)\right) ds, \qquad \forall t \in [0, T],$$

then

$$u(t) \leq G^{-1}\left(G(u_0) + \int_0^t v(s)ds\right)$$

holds for all $t \in [0, T]$ such that

$$G(u_0) + \int_0^t v(s)ds \in Dom(G^{-1}),$$

where

$$G(x) = \int_1^x \frac{ds}{\rho(s)}, \qquad x \geq 0.$$

G^{-1} is the inverse function of G and $Dom(G^{-1})$ is the domain $G^{-1}(.)$. In particular, if $u_0 = 0$ and

$$\int_{0^+} \frac{ds}{\rho(s)} = +\infty,$$

then $u(t) = 0$ for all $t \in [0, T]$.

3. Main results

Let Ω be a non-empty set, $\mathcal{F}$ a σ-algebra of sets of Ω and $\mathbb{P}$ a probability measure defined on $\mathcal{F}$. Assuming $W = (W_t)_{t\geq 0}$ be an m-dimensional Brownian motion on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For a fix real $T > 0$, let $J = [0, T]$. Given a real $\alpha \in]1/2, 1[$ and X_0 a random variable defined on $(\Omega, \mathcal{F}, \mathbb{P})$ independent of the σ-algebra $(\mathcal{F}_t)_{t\in J}$ generated by $\{W_s, 0 \leq s \leq t\}$. Define the filtration $\mathcal{F}_t^0$ generated by X_0 and $\{W_s, 0 \leq s \leq t\}$. Let $b, \sigma_2 \in \mathcal{C}\left(J \times \mathbb{R}^p \times \mathbb{R}^{p\times p}, \mathbb{R}^p\right)$, $\sigma_1 \in \mathcal{C}\left(J \times \mathbb{R}^p \times \mathbb{R}^{p\times p}, \mathbb{R}^{p\times m}\right)$ and $f_i, g_i, h_i \in \mathcal{C}\left(J \times J \times \mathbb{R}^p, \mathbb{R}^p\right)$, $i = 1, 2, \cdots, p$. Consider the following stochastic fractional integrodifferential equation: for $t \in J$,

$$\begin{aligned} dX_t = \;& b(t, X_t, \int_0^t f_1(t, s, X_s)ds, \int_0^t f_2(t, s, X_s)ds, \cdots, \int_0^t f_p(t, s, X_s)ds)dt \\ +\;& \sigma_1(t, X_t, \int_0^t g_1(t, s, X_s)ds, \int_0^t g_2(t, s, X_s)ds, \cdots, \int_0^t g_p(t, s, X_s)ds)dW_t \\ &+ \sigma_2(t, X_t, \int_0^t h_1(t, s, X_s)ds, \int_0^t h_2(t, s, X_s)ds, \cdots, h_p(t, s, X_s)ds)(dt)^\alpha. \end{aligned} \tag{3.1}$$

We can rewrite the above equation in its equivalent integral form as follows:

$$\begin{aligned} X_t &= X_0 + \int_0^t b(s, X_s, \int_0^s f_1(s,r,X_r)dr, \cdots, \int_0^s f_p(s,r,X_r)dr)ds \\ &+ \int_0^t \sigma_1(s, X_s, \int_0^s g_1(s,r,X_r)dr, \cdots, \int_0^s g_p(s,r,X_r)dr)dW_s \\ &+ \alpha \int_0^t (t-s)^{\alpha-1}\sigma_2(s, X_s, \int_0^s h_1(s,r,X_r)dr, \cdots, \int_0^s h_p(s,r,X_r)dr)ds. \end{aligned}$$

In the following we assume that $b, \sigma_1, \sigma_2, f_j, g_j, h_j$, $j = 1, 2, \cdots, p$ satisfy assumptions (**H**):
Let $F_j = \int_0^t f_j(t,s,X_s)ds$, $G_j = \int_0^t g_j(t,s,X_s)ds$, $H_j = \int_0^t h_j(t,s,X_s)ds$, and $\widetilde{F}_j = \int_0^t f_j(t,s,\widetilde{X}_s)ds$, $\widetilde{G}_j = \int_0^t g_j(t,s,\widetilde{X}_s)ds$, $\widetilde{H}_j = \int_0^t h_j(t,s,\widetilde{X}_s)ds$,

(H.1) Linear growth condition

$$|b(t,X_t,F_1,\cdots,F_p)| + |\sigma_2(t,X_t,H_1,\cdots,H_p)| \le K(1+|X_t| + \sum_{j=1}^p |F_j| + \sum_{j=1}^p |H_j|)$$

$$|\sigma_1(t,X_t,G_1,G_2,\cdots,G_p)|^2 \le K^2(1+|X_t|^2 + \sum_{j=1}^p |G_j|^2)$$

$$|F_j| + |G_j| + |H_j| \le k_j(1+|X_t|),$$

for some constants $K > 0$ and $k_j > 0$, $j = 1, 2, ..., p$.

(H.2) The functions $b(t,.,\cdots,.)$, $\sigma_1(t,.,\cdots,.)$ and $\sigma_2(t,.,\cdots,.)$ are continuous in all variables.

(H.3) For each $N = 1, 2, \cdots$,

$$\begin{aligned} &2\left\langle (X_t - \widetilde{X}_t), (b(t,X_t,F_1,F_2,\cdots,F_p) - b(t,\widetilde{X}_t,\widetilde{F}_1,\widetilde{F}_2,\cdots,\widetilde{F}_p))\right\rangle \\ &+\left\langle (X_t - \widetilde{X}_t), (\sigma_2(t,X,G_1,G_2,\cdots,G_p) - \sigma_2(t,\widetilde{X}_t,\tilde{G}_1,\widetilde{G}_2,\cdots,\widetilde{G}_p))\right\rangle \\ &+|\sigma_1(t,X_t,H_1,H_2,\cdots,H_p) - \sigma_1\left(t,\widetilde{X}_t,\tilde{H}_1,\widetilde{H}_2,\cdots,\widetilde{H}_p\right)|^2 \\ &\le c(t)\rho\left(|X_t - \widetilde{X}_t|^2\right) \end{aligned}$$

as $|X_t| \vee |\widetilde{X}_t| \le N$, $t \in [0,T]$, where $\int_0^T c(t)dt < +\infty$ and $\rho(u) \ge 0$, as $u \ge 0$, is non-random, strictly increasing, continuous and concave such that $\int_{0^+} \frac{1}{\rho(u)}du = +\infty$.

We consider the following sets (where $\mathbb{E}$ denote the mathematical expectation with respect to the probability measure $\mathbb{P}$):

- $\mathcal{S}^2_{[0,T]}(\mathbb{R}^p)$ the space of $\mathcal{F}^0_t$-adapted càdlàg processes

$$\psi : [0,T]\times\Omega \to \mathbb{R}^p, \quad \| \psi \|^2_{\mathcal{S}^2} = \mathbb{E}\left(\sup_{0\leqslant t\leqslant T} | \psi_t |^2\right) < \infty,$$

- $\mathcal{M}^2_{[0,T]}(\mathbb{R}^d)$ the space of $\mathcal{F}^0_t$-progressively measurable processes

$$\psi : [0,T] \times \Omega \to \mathbb{R}^p, \quad \| \psi \|^2_{\mathcal{M}^2} = \mathbb{E} \int_0^T \mid \psi_t \mid^2 dt < \infty.$$

The main result in this section is given by:

Theorem 3.1. *Under assumption* $(\mathbf{H})$ *and* $\mathbb{E}\left(\mid X_0 \mid^2\right) < +\infty$*, the SDE*(3.1) *has a unique solution which is* $t-$*continuous such that* $X_t \in \mathcal{S}^2$*.*

We need the following two lemmas before proving this theorem.

Lemma 3.2. *Under assumptions* $(\mathbf{H.1})$ *and* $(\mathbf{H.2})$ *there exist functions* $b^n, \sigma_1^n, \sigma_2^n, n = 1, 2, \cdots$ *satisfying the following conditions:*

1) $|b^n(t, X_t, F_1, \cdots, F_p)| + |\sigma_2^n(t, X_t, H_1, \cdots, H_p)| \leq C\,(1 + |X_t|)$ *as* $n \geq N_0$,

$$|\sigma_1^n(t, X_t, G_1, G_2, \cdots, G_p)|^2 \leq C\left(1 + |X_t|^2\right),$$

where $N_0 > 0$ *and* $C > 0$ *are constants.*

2) $|b^n(t, X_t, F_1, \cdots, F_p) - b^n(t, \widetilde{X}_t, \widetilde{F}_1, \cdots, \widetilde{F}_p)| \leq C_n \mid X_t - \widetilde{X}_t \mid$,
$|\sigma_2^n(t, X_t, H_1, \cdots, H_p) - \sigma_2^n(t, \widetilde{X}_t, \widetilde{H}_1, \cdots, \widetilde{H}_p)| \leq C_n \mid X_t - \widetilde{X}_t \mid$,
$|\sigma_1^n(t, X_t, G_1, G_2, \cdots, G_p) - \sigma_1^n(t, \widetilde{X}_t, \tilde{G}_1, \tilde{G}_2, \cdots, \tilde{G}_p)|^2 \leq C_n \mid X_t - \widetilde{X}_t \mid^2$,
where $C_n \geq 0$ *is a constant depending on* n*.*

3) *For any* $N > 0$ *and for each* $t \geq 0$, $\omega \in \Omega$, *as* $n \longrightarrow +\infty$

$$\sup_{|X_t| \leq N} |b^n(t, X_t, F_1, F_2, \cdots, F_p) - b(t, X_t, F_1, F_2, \cdots, F_p)| \longrightarrow 0,$$

$$\sup_{|X_t| \leq N} |\sigma_1^n(t, X_t, G_1, G_2, \cdots, G_p) - \sigma_1(t, X_t, G_1, G_2, \cdots, G_p)| \longrightarrow 0,$$

$$\sup_{|X_t| \leq N} |\sigma_2^n(t, X_t, H_1, H_2, \cdots, H_p) - \sigma_2(t, X_t, H_1, H_2, \cdots, H_p)| \longrightarrow 0.$$

Proof. Let, for $j = 1, 2, \cdots, p$,
$\bar{F}_j^n = \int_0^t f_j(t, s, X_s^n - \frac{\bar{x}}{n})ds$, $\bar{G}_j^n = \int_0^t g_j(t, s, X_s^n - \frac{\bar{x}}{n})ds$, $\bar{H}_j^n = \int_0^t h_j(t, s, X_s^n - \frac{\bar{x}}{n})ds$.
Define

$$b^n(t, X_t, F_1, F_2, \cdots, F_p) = \int_{\mathbb{R}^d} b(t, X_t - \frac{\overline{x}}{n}, \overline{F}_1, \overline{F}_2, \cdots, \overline{F}_p) J(\overline{x}) d\overline{x}, \tag{3.2}$$

where for all $u \in \mathbb{R}^d$

$$J(u) = \begin{cases} c \exp\left[-(1 - \mid u \mid^2)^{-1}\right], & \text{for } \mid u \mid < 1, \\ 0, & \text{otherwise,} \end{cases}$$

and c is a constant satisfying $\int_{\mathbb{R}^d} J(u) du = 1$.

1) By definition (3.2) of b^n we have

$$|b^n(t, X_t, F_1, F_2, \cdots, F_p)| \leqslant \int_{\mathbb{R}^d} |b(t, X_t - \frac{\overline{x}}{n}, \overline{F}_1, \overline{F}_2, \cdots, \overline{F}_p)| J(\overline{x}) d\overline{x},$$

and using the assumption (**H.1**) we have

$$\begin{aligned}|b^n(t,X_t,F_1,\cdots,F_p)| &\leqslant \int_{\mathbb{R}^d} K\left(1+|X_t-\frac{\overline{x}}{n}|+\sum_{i=1}^{p}k_i(1+|X_t-\frac{\overline{x}}{n}|)\right)J(\overline{x})d\overline{x}\\ &\leqslant K(1+\sum_{i=1}^{p}k_i)\left[\int_{\mathbb{R}^d}(1+|X_t|)J(\overline{x})d\overline{x}+\frac{1}{n}\int_{\mathbb{R}^d}\overline{x}J(\overline{x})d\overline{x}\right].\end{aligned}$$

Let $N_0=\displaystyle\int_{\mathbb{R}^d}\overline{x}J(\overline{x})d\overline{x}$, we obtain

$$\begin{aligned}|b^n(t,X_t,F_1,F_2,\cdots,F_p)| &\leqslant K(1+\sum_{i=1}^{p}k_i)\left[1+|X_t|+\frac{N_0}{n}\right]\\ &\leqslant 2K(1+\sum_{i=1}^{p}k_i)(1+|X_t|), \quad \text{for} \quad \frac{N_0}{n}\le 1.\end{aligned}$$

Then b^n satisfies the point 1).

2) we have

$$\left|b^n(t,X_t,F_1,\cdots,F_p)-b^n(t,\widetilde{X}_t,\widetilde{F}_1,\cdots,\widetilde{F}_p)\right|=$$

$$\left|\int_{\mathbb{R}^d} b(t,X_t-\frac{\overline{x}}{n},\int_0^t f_1(t,s,X_s^n-\frac{\bar{x}}{n})ds,\cdots,\int_0^t f_p(t,s,X_s^n-\frac{\bar{x}}{n})ds)J(\overline{x})d\overline{x}\right.$$
$$\left.-\int_{\mathbb{R}^d} b(t,\widetilde{X}_t-\frac{\overline{x}}{n},\int_0^t f_1(t,s,\widetilde{X}_s^n-\frac{\bar{x}}{n})ds,\cdots,\int_0^t f_p(t,s,\widetilde{X}_s^n-\frac{\bar{x}}{n})ds)J(\overline{x})d\overline{x}\right|,$$

then by a change of variable, we have

$$\begin{aligned}&\left|b^n(t,X_t,F_1,\cdots,F_p)-b^n(t,\tilde{X}_t,\tilde{F}_1,\cdots,\tilde{F}_p)\right|\\ &\qquad\leqslant n^d\int_{\mathbb{R}^d}\left|b(t,\overline{x},\overline{F}_1,\cdots,\overline{F}_p)\right|\left|J(n(X_t-\overline{x}))-J(n(\tilde{X}_t-\overline{x}))\right|d\overline{x}\\ &\qquad\leqslant n^d\int_{\mathbb{R}^d}K(1+\sum_{i=1}^{p}k_i)(1+\mid\overline{x}\mid)\left|J\left(n(X_t-\overline{x})\right)-J(n(\tilde{X}_t-\overline{x}))\right|d\overline{x}\\ &\qquad\leqslant C_n\mid X_t-\tilde{X}_t\mid\end{aligned}$$

because

$$n^dK(1+\sum_{i=1}^{p}k_i)\int_{\mathbb{R}^d}\int_0^1(1+\mid\overline{x}\mid)grad\left[J\left(n(X_t-\overline{x}+\theta(\tilde{X}_t-\overline{x}))\right)\right]d\theta d\overline{x}$$
$$\le C_n\mid X_t-\tilde{X}_t\mid.$$

3) By Heine-Borel's finite covering theorem for any $N>0$ and any given $\varepsilon>0$ one can find a $\delta>0$ may depend on t and ω such that as $\frac{1}{n}<\delta$, for all $\mid X_t\mid\le N$

$$\left|b(t,X_t-\frac{1}{n},\int_0^t f_1(t,s,X_s-\frac{1}{n})ds,..,\int_0^t f_p(t,s,X_s-\frac{1}{n})ds)-b(t,X_t,F_1,..,F_p)\right|<\varepsilon$$

because b is continuous. Hence, as $n > \frac{1}{\delta}$,

$$\begin{aligned}
\sup_{|X_t|\le N}\Big|b^n(t,X_t,F_1,\cdots,F_p)-b(t,X_t,F_1,\cdots,F_p)\Big| &= \\
\sup_{|X_t|\le N}\Big|\int_{\mathbb{R}^d} b(t,X_t-\frac{\overline{x}}{n},\overline{F}_1,\cdots,\overline{F}_p)J(\overline{x})d\overline{x}-b(t,X_t,F_1,\cdots,F_p)\Big| \\
&\le \int_{\mathbb{R}^d}\sup_{|X_t|\le N}\Big|b(t,X_t-\frac{\overline{x}}{n},\overline{F}_1,\cdots,\overline{F}_p)-b(t,X_t,F_1,\cdots,F_p)\Big|J(\overline{x})d\overline{x} \\
&\le \int_{|\overline{x}|<1}\sup_{|X_t|\le N}\Big|b(t,X_t-\frac{\overline{x}}{n},\overline{F}_1,\cdots,\overline{F}_p)-b(t,X_t,F_1,\cdots,F_p)\Big|J(\overline{x})d\overline{x} \\
&\le \varepsilon.
\end{aligned}$$

Then, taking $\varepsilon \to 0$, we have

$$\sup_{|X_t|\le N}\Big|b^n(t,X_t,F_1,\cdots,F_p)-b(t,X_t,F_1,\cdots,F_p)\Big| \longrightarrow 0.$$

Thus point 3) is true for b^n.
By defining σ_1^n and σ_2^n in the same way as b^n in (3.2), we show that σ_1^n and σ_2^n satisfy this lemma .

□

Lemma 3.3. *Consider b^n, σ_1^n and σ_2^n obtained from the above Lemma 3.2. There exists an unique solution $(X_t^n)_{0\le t\le T}$ which is t-continuous and satisfying the following SDE:*

$$\begin{aligned}
X_t^n = \quad & X_0 + \int_0^t b^n(s,X_s^n,\int_0^s f_1(s,r,X_r^n)dr,\cdots,\int_0^s f_p(s,r,X_r^n)dr)ds \\
+ \quad & \int_0^t \sigma_1^n(s,X_s^n,\int_0^s g_1(s,r,X_r^n)dr,\cdots,\int_0^s g_p(s,r,X_r^n)dr)dW_s \\
+\,\alpha & \int_0^t (t-s)^{\alpha-1}\sigma_2^n(s,X_s^n,\int_0^s h_1(s,r,X_r^n)dr,\cdots,\int_0^s h_p(s,r,X_r^n)dr)ds \quad (3.3)
\end{aligned}$$

such that $\mathbb{E}(\sup_{0\le t\le T}|X_t^n|^2) < C(\alpha,T)$.

Proof. By Umamaheswari et al.[19, Theorem 2], this equation (3.3) has a pathwise unique strong solution $(X_t^n)_{0\le t\le T}$ t-continuous verifying $\sup_{0\le t\le T}\mathbb{E}\left(|X_t^n|^2\right) < +\infty$.
Now, show that there exist a constant $C(\alpha,T)$ such that $\mathbb{E}(\sup_{0\le t\le T}|X_t^n|^2) \le C(\alpha,T)$.
By Cauchy-Schwarz inequality and using $(a+b+c+d)^2 \le 4\left(a^2+b^2+c^2+d^2\right)$,

we have

$$\begin{aligned}
\mid X_t^n \mid^2 \leqslant \quad & 4 \mid X_0 \mid^2 + 4T\int_0^t \left| b^n(s, X_s^n, \int_0^s f_1(s,r,X_r^n)dr, \cdots, \int_0^s f_p(s,r,X_r^n)dr) \right|^2 ds \\
&+ 4\left| \int_0^t \sigma_1^n(s, X_s^n, \int_0^s g_1(s,r,X_r^n)dr, \cdots, \int_0^s g_p(s,r,X_r^n)dr)dW_s \right|^2 \\
&+ 4(\int_0^t \alpha^2(t-s)^{2\alpha-2}ds) \left(\int_0^t \left|\sigma_2^n(s, X_s^n, \int_0^s h_1(s,r,X_r^n)dr, .., \int_0^s h_p(s,r,X_r^n)dr)\right|^2 ds \right),
\end{aligned}$$

hence,

$$\begin{aligned}
\sup_{0\le t\le T} \mid X_t^n \mid^2 \quad \le \quad & 4 \mid X_0 \mid^2 + 4T\int_0^T \left| b^n(s, X_s^n, \int_0^s f_1(s,r,X_r^n)dr, \cdots, \int_0^s f_p(s,r,X_r^n)dr) \right|^2 ds \\
&+ \quad 4 \sup_{0\le t\le T} \left| \int_0^t \sigma_1^n(s, X_s^n, \int_0^s g_1(s,r,X_r^n)dr, \cdots, \int_0^s g_p(s,r,X_r^n)dr)dW_s \right|^2 \\
&+ \quad 4\alpha^2 \frac{T^{2\alpha-1}}{2\alpha-1} \int_0^T \left|\sigma_2^n(s, X_s^n, \int_0^s h_1(s,r,X_r^n)dr, \cdots, \int_0^s h_p(s,r,X_r^n)dr)\right|^2 ds,
\end{aligned}$$

by taking the mathematical expectation and using the Lemma 3.2, we obtain

$$\mathbb{E}(\sup_{0\le t\le T} \mid X_t^n \mid^2) \quad \le \quad 4\left(\mathbb{E}\left(\mid X_0 \mid^2\right) + (T+1+\alpha^2 \frac{T^{2\alpha-1}}{2\alpha-1})(1+\sum_{i=1}^p k_i) \int_0^T (1+\mathbb{E}\left(\mid X_s^n \mid^2\right) ds \right)$$

and since $\mathbb{E}\left(|X_t^n|^2\right) < +\infty$, there exist a constant positive M such that

$$\mathbb{E}\left(\sup_{0\le t\le T} \mid X_t^n \mid^2 \right) \le C(\alpha, T),$$

where

$$C(\alpha, T) = 4\left(\mathbb{E}\left(\mid X_0 \mid^2\right) + (T+1+\alpha^2 \frac{T^{2\alpha-1}}{2\alpha-1})(1+\sum_{i=1}^p k_i)(1+M)T \right).$$

□

Now we can give the proof of Theorem 3.1.

Proof. **Existence**: Consider the equation (3.3). For $j = 1, 2, \cdots, p$, let $F_j^n = \int_0^t f_j(t,s,X_s^n)ds$, $G_j^n = \int_0^t g_j(t,s,X_s^n)ds$, $H_j^n = \int_0^t h_j(t,s,X_s^n)ds$, and $\bar{F}_j^n = \int_0^t f_j(t,s,X_s^n - \frac{\bar{x}}{n})ds$, $\bar{G}_j^n = \int_0^t g_j(t,s,X_s^n - \frac{\bar{x}}{n})ds$, $\bar{H}_j^n = \int_0^t h_j(t,s,X_s^n - \frac{\bar{x}}{n})ds$. By Itô's formula, we have

$$\begin{aligned}
\mathbb{E}(|X_t^m - X_t^n|^2) = 2\mathbb{E}\int_0^t (X_s^m - X_s^n)(b^m(s, X_s^m, F_1^m, .., F_p^m) - b^n(s, X_s^n, F_1^n, .., F_p^n))ds \\
+\mathbb{E}\int_0^t |\sigma_1^m(s, X_s^m, G_1^m, \cdots, G_p^m) - \sigma_1^n(s, X_s^n, G_1^n, \cdots, G_p^n)|^2 ds \\
+2\alpha\mathbb{E}\int_0^t (t-s)^{\alpha-1}(X_s^m - X_s^n)(\sigma_2^m(s, X_s^m, H_1^m, .., H_p^m) - \sigma_2^n(s, X_s^n, H_1^n, .., H_p^n))ds.
\end{aligned}$$

Using the definition of the sequences b^n, σ_1^n and σ_2^n and by the notations

$$\Delta b^{m,n}(s) = b(s, X_s^m - \frac{\bar{x}}{m}, \bar{F}_1^m, \cdots, \bar{F}_p^m) - b(s, X_s^n - \frac{\bar{x}}{n}, \bar{F}_1^n, \cdots, \bar{F}_p^n),$$

$$\Delta\sigma_1^{m,n}(s)=\sigma_1(s,X_s^m-\frac{\bar{x}}{m},\bar{G}_1^m,\cdots,\bar{G}_p^m)-\sigma_1(s,X_s^n-\frac{\bar{x}}{n},\bar{G}_1^n,\cdots,\bar{G}_p^n)$$

and

$$\Delta\sigma_2^{m,n}(s)=\sigma_2(s,X_s^m-\frac{\bar{x}}{m},\bar{H}_1^m,\cdots,\bar{H}_p^m)-\sigma_2(s,X_s^n-\frac{\bar{x}}{n},\bar{H}_1^n,\cdots,\bar{H}_p^n),$$

we obtain

$$\begin{aligned}\mathbb{E}\left(|X_t^m-X_t^n|^2\right) &= \mathbb{E}\int_0^t\int_{\mathbb{R}^d}2(X_s^m-X_s^n)\Delta b^{m,n}(s)J(\bar{x})d\bar{x}ds\\ &+ \mathbb{E}\int_0^t\int_{\mathbb{R}^d}|\Delta\sigma_1^{m,n}(s)|^2J(\bar{x})d\bar{x}ds\\ &+ 2\alpha\mathbb{E}\int_0^t(t-s)^{\alpha-1}\int_{\mathbb{R}^d}(X_s^m-X_s^n)\Delta\sigma_2^{m,n}(s)J(\bar{x})d\bar{x}ds\end{aligned}$$

and by asumption **(H.3)**, we have

$$\begin{aligned}\mathbb{E}\left(|X_t^m-X_t^n|^2\right)\leq\ & 2\mathbb{E}\int_0^t\int_{\mathbb{R}^d}c(s)\rho(|X_s^m-X_s^n-(m^{-1}-n^{-1})\bar{x}|^2)J(\bar{x})d\bar{x}ds\\ &+2\mathbb{E}\int_0^t\int_{\mathbb{R}^d}|m^{-1}-n^{-1})\bar{x}|J(\bar{x})d\bar{x}ds\\ &+2\alpha\mathbb{E}\int_0^t(t-s)^{\alpha-1}\int_{\mathbb{R}^d}c(s)\rho(|X_s^m-X_s^n-(m^{-1}-n^{-1})\bar{x}|^2)J(\bar{x})d\bar{x}ds\\ &+2\alpha\mathbb{E}\int_0^t(t-s)^{\alpha-1}|m^{-1}-n^{-1}|\int_{\mathbb{R}^d}|\bar{x}|J(\bar{x})d\bar{x}ds,\end{aligned}$$

hence as $t\in[0,T]$,

$$\begin{aligned}\mathbb{E}\left(|X_t^m-X_t^n|^2\right) &\leq K(T,\alpha)\int_0^t c(s)\int_{\mathbb{R}^d}\rho(\mathbb{E}|X_s^m-X_s^n-(m^{-1}-n^{-1})\bar{x}|^2)J(\bar{x})d\bar{x}ds\\ &+ K(T,\alpha,N)(m^{-1}+n^{-1}).\end{aligned}\tag{3.4}$$

Since by Lemma 3.3, we have for all n

$$\mathbb{E}(\sup_{0\leq t\leq T}|X_t^n|^2)\leq C(T,\alpha).$$

So by Fatou's lemma, we obtain

$$\limsup_{m,n\to+\infty}\mathbb{E}\left(|X_t^m-X_t^n|^2\right)\leq K(T,\alpha)\int_0^t c(s)\rho_1(\limsup_{m,n\to+\infty}\mathbb{E}|X_s^m-X_s^n|^2)ds$$

where $\rho_1(u)=\rho(u)+u$. Therefore

$$\limsup_{m,n\to+\infty}\mathbb{E}\left(|X_t^m-X_t^n|^2\right)=0.$$

From inequality (3.4) one also finds that

$$\limsup_{m,n\to+\infty}\mathbb{E}\int_0^T|X_t^m-X_t^n|^2dt=0.$$

So there exist $(X_t)_t \in \mathcal{M}^2_{[0,T]}(\mathbb{R}^d)$ such that:

$$\lim_{n\to+\infty} \mathbb{E}\int_0^T |X_t^n - X_t|^2 dt = 0$$

and then for each $t \geq 0$, $\lim_{n\to+\infty} \mathbb{E}|X_t^n - X_t|^2 = 0$.
So $X_t^n \longrightarrow X_t$, in probability for each t, and one can choose a subsequence (n_k) denoted again (n) such that $\mathbb{P}-$ a.s as $n \longrightarrow +\infty$, $X_t^n \longrightarrow X_t^0$, $\forall t = r_k$, $\quad k = 1, 2, \cdots$; where $(r_k)_{k\geq 1} \subset [0,T]$ is the totality of rational numbers in $[0,T]$. Hence by Fatou's Lemma, we have

$$\mathbb{E}\left(\sup_{t\leq T} |X_t^0|^2\right) \leq \mathbb{E}\left(\sup_k \lim_{n\longrightarrow+\infty} |X_{r_k}^n|^2\right) \leq \liminf_{n\longrightarrow+\infty} \mathbb{E}\left(\sup_{t\leq T} |X_t^n|^2\right) \leq K(T,\alpha).$$

Since X_t is the uniform limit of a sequence of continuous functions, it is also continuous.

Now show that for $t \in J$, when $n \to +\infty$, $\mathbb{P} - a.s$:

$$\begin{aligned}
\int_0^t b^n(s, X_s^n, F_1^n, \cdots, F_p^n)ds &\to \int_0^t b(s, X_s, F_1, \cdots, F_p)ds, \\
\int_0^t \sigma_1^n(s, X_s^n, G_1^n, \cdots, G_p^n)dW_s &\to \int_0^t \sigma_1(s, X_s, G_1, \cdots, G_p)dW_s, \\
\int_0^t (t-s)^{\alpha-1}\sigma_2^n(s, X_s^n, H_1^n, \cdots, H_p^n)ds &\to \int_0^t (t-s)^{\alpha-1}\sigma_2(s, X_s, H_1, \cdots, H_p)ds.
\end{aligned}$$

One may assume that $\sup_{t\leq T} \mid X_t^n \mid \leq k_0$, for all $n \in \mathbb{N}$ and $\sup_{t\leq T} \mid X_t \mid \leq k_0$. However, as $t \in J$, for any $\varepsilon > 0$, by noting

$$p_n = \mathbb{P}\left(\left|\int_0^t b^n(s, X_s^n, F_1^n, \cdots, F_p^n)ds - \int_0^t b(s, X_s, F_1, \cdots, F_p)ds\right| > \varepsilon\right),$$

and

$$F_j^X = \int_0^s f_j(s, r, X)dr, \qquad j = 1, 2, \cdots, p,$$

we have by Markov's inequality

$$\begin{aligned}
p_n &\leq \frac{1}{\varepsilon}\mathbb{E}\int_0^T \left|b^n(s, X_s^n, F_1^n, \cdots, F_p^n) - b(s, X_s^n, F_1^n, \cdots, F_p^n)\right| ds \\
&+ \frac{1}{\varepsilon}\mathbb{E}\int_0^T \left|b(s, X_s^n, F_1^n, \cdots, F_p^n) - b(s, X_s, F_1, \cdots, F_p)\right| ds
\end{aligned}$$

and taking the supremum for all $\mid X \mid \leq k_0$, we obtain

$$\begin{aligned}
p_n &\leq \frac{1}{\varepsilon}\int_0^T \sup_{|X|\leq k_0} \left|b^n(s, X, F_1^X, \cdots, F_p^X) - b(s, X, F_1^X, \cdots, F_p^X)\right| ds \\
&+ \frac{1}{\varepsilon}\int_0^T \lim_{h\to 0} \sup_{|X|\leq k_0} \left|b(s, X+h, F_1^{X+h}, \cdots, F_1^{X+h}) - b(s, X, F_1^X, \cdots, F_p^X)\right| ds, \\
&:= I_1^n + I_2^h.
\end{aligned}$$

From Lemma 3.2, we have

$$\sup_{|X|\le k_0}\left|b^n(s,X,F_1^X,\cdots,F_p^X)-b(s,X,\int_0^s F_1^X,\cdots,F_p^X)\right|\to 0$$

and using the linear growth condition of the functions b and b^n, we can apply Lebesgue's dominated convergence theorem to get

$$\lim_{n\to+\infty} I_1^n=0.$$

Now since

$$\lim_{h\to 0}\sup_{|X|\le k_0}\left|b(s,X+h,F_1^{X+h},\cdots,F_p^{X+h},)-b(s,X,F_1^X,\cdots,F_p^X)\right|=0,$$

we can take small enough $\delta>0$ such that

$$\sup_{|X|\le k_0,|h|\le\delta}\left|b(s,X+h,F_1^{X+h},\cdots,F_p^{X+h})-b(s,X,F_1^X,\cdots,F_p^X)\right|<\varepsilon,$$

then for this δ

$$I_2^h=0.$$

So we have $p_n\to 0$, hence

$$\int_0^t b^n(s,X_s^n,F_1^n,\cdots,F_p^n)ds\longrightarrow\int_0^t b(s,X_s,F_1,\cdots,F_p)ds,\qquad \mathbb{P}-a.s.$$

Now we define q_n by

$$q_n=\mathbb{P}\left(\left|\int_0^t\sigma_1^n(s,X_s^n,G_1^n,\cdots,G_p^n)dW_s-\int_0^t\sigma_1(s,X_s,G_1,\cdots,G_p)dW_s\right|>\varepsilon\right),$$

and

$$G_j^X=\int_0^s g_j(s,r,X)dr,\qquad j=1,2,\cdots,p,$$

we deduce from Markov's inequality and Doob's inequality that

$$\begin{aligned}q_n\le{}&\frac{4}{\varepsilon^2}\mathbb{E}\int_0^T\left|\sigma_1^n(s,X_s^n,G_1^n,\cdots,G_p^n)-\sigma_1(s,X_s^n,G_1^n,\cdots,G_p^n)\right|^2ds\\&+\frac{4}{\varepsilon^2}\mathbb{E}\int_0^T\left|\sigma_1(s,X_s^n,G_1^n,\cdots,G_p^n)-\sigma_1(s,X_s,G_1,\cdots,G_p)\right|^2ds\end{aligned}$$

and taking the supremum we have

$$\begin{aligned}q_n\ \le\ &\frac{4}{\varepsilon^2}\int_0^T\sup_{|X|\le k_0}\left|\sigma_1^n(s,X,G_1^X,\cdots,G_p^X)-\sigma_1(s,X,G_1^X,\cdots,G_p^X)\right|^2ds\\&+\ \frac{4}{\varepsilon^2}\int_0^T\lim_{h\to 0}\sup_{|X|\le k_0}\left|\sigma_1(s,X+h,G_1^{X+h},\cdots,G_p^{X+h})-\sigma_1(s,X,G_1^X,\cdots,G_p^X)\right|^2ds\\:=\ &I_3^n+I_4^h.\end{aligned}$$

Notice from Lemma 3.2

$$\sup_{|X|\leq k_0}\left|\sigma_1^n(s,X,G_1^X,\cdots,G_p^X)-\sigma_1(s,X,G_1^X,\cdots,G_p^X)\right|^2\to 0,$$

so the Lebesgue's dominated convergence theorem give

$$\lim_{n\to+\infty} I_3^n=0.$$

Hence since,

$$\lim_{h\to 0}\sup_{|X|\leq k_0}\left|\sigma_1(s,X+h,G_1^{X+h}\cdots,G_p^{X+h})-\sigma_1(s,X,G_1^X,\cdots,G_p^X)\right|^2=0$$

we can take small enough $\delta>0$ such that

$$\sup_{|X|\leq k_0,|h|\leq\delta}\left|\sigma_1(s,X+h,G_1^{X+h},\cdots,G_p^{X+h})-\sigma_1(s,X,G_1^X,\cdots,G_p^X)\right|^2<\varepsilon.$$

Then for this $\delta>0$

$$I_4^h=0.$$

So,

$$\int_0^t\sigma_1^n(s,X_s^n,G_1^n,\cdots,G_p^n)dW_s\longrightarrow\int_0^t\sigma_1(s,X_s,G_1,\cdots,G_p)dW_s,\qquad\mathbb{P}-a.s.$$

In the same way one shows that as $n\to\infty$,

$$\int_0^t\sigma_2^n(s,X_s^n,H_1^n,\cdots,H_p^n)ds\longrightarrow\int_0^t\sigma_2(s,X_s,H_1,\cdots,H_p)ds,\qquad\mathbb{P}-a.s.$$

Then $(X_t)_{t\geq 0}$ is a solution of equation (3.1).

Uniqueness: Let $(X_t)_{t\geq0}$ and $(Y_t)_{t\geq0}$ be solutions processes through the initial value $X_0=Y_0$. We have

$$\begin{aligned}
X_t-Y_t &= \int_0^t b(s,X_s,\int_0^s f_1(s,r,X_r)dr,\cdots,\int_0^s f_p(s,r,X_r)dr)ds\\
&- \int_0^t b(s,Y_s,\int_0^s f_1(s,r,Y_r)dr,\cdots,\int_0^s f_p(s,r,Y_r)dr)ds\\
&+ \int_0^t \sigma_1(s,X_s,\int_0^s g_1(s,r,X_r)dr,\cdots,\int_0^s g_p(s,r,X_r)dr)dW_s\\
&- \int_0^t \sigma_1(s,Y_s,\int_0^s g_1(s,r,Y_r)dr,\cdots,\int_0^s g_p(s,r,Y_r)dr)dW_s\\
&+ \alpha\int_0^t (t-s)^{\alpha-1}\sigma_2(s,X_s,\int_0^s h_1(s,r,X_r)dr,\cdots,\int_0^s h_p(s,r,X_r)dr)ds\\
&- \alpha\int_0^t (t-s)^{\alpha-1}\sigma_2(s,Y_s,\int_0^s h_1(s,r,Y_r)dr,\cdots,\int_0^s h_p(s,r,Y_r)dr)ds\\
&:= \int_0^t\Delta^{X,Y}b(s)ds+\int_0^t\Delta^{X,Y}\sigma_1(s)dW_s+\alpha\int_0^t(t-s)^{\alpha-1}\Delta^{X,Y}\sigma_2(s)ds.
\end{aligned}$$

By Itô formula we have

$$\mathbb{E}\left(|X_t - Y_t|^2\right) = 2\mathbb{E}\int_0^t (X_s - Y_s)|\Delta^{X,Y} b(s)|ds + \mathbb{E}\int_0^t |\Delta^{X,Y}\sigma_1(s)|^2 ds$$
$$+2\alpha\mathbb{E}\int_0^t (t-s)^{\alpha-1}(X_s - Y_s)|\Delta^{X,Y}\sigma_2(s)|ds.$$

The assumption **(H.3)** implies

$$\begin{aligned}\mathbb{E}\left(|X_t - Y_t|^2\right) &\leq \int_0^t \rho\left(\mathbb{E}|X_s - Y_s|^2\right) + \int_0^t \rho\left(\mathbb{E}|X_s - Y_s|^2\right) + 2\alpha\int_0^t (t-s)^{\alpha-1}\rho\left(\mathbb{E}|X_s - Y_s|^2\right) \\ &\leq \ 2\int_0^t \rho\left(\mathbb{E}|X_s - Y_s|^2\right) ds + 2\alpha\int_0^t (t-s)^{\alpha-1}\rho\left(\mathbb{E}|X_s - Y_s|^2\right) ds \\ &\leq \ \int_0^t \left(3 + 2\alpha(t-s)^{\alpha-1}\right)\rho\left(\mathbb{E}|X_s - Y_s|^2\right) ds.\end{aligned}$$

Bihary's inequality gives

$$\mathbb{E}\left(|X_t - Y_t|^2\right) = 0.$$

Then

$$\mathbb{P}\left\{|X_t - Y_t| = 0, \quad \forall t \in [0, T]\right\} = 1,$$

hence, the solution of equation (3.1) is unique.

□

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