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ON THE PYTHAGOREAN ORTHOGONALITY OF HILBERT BOUNDED LINEAR OPERATORS

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Abstract. In this work, we characterize the Pythagorean orthogonality of bounded linear operators using the concept of norm attainment set and disjoint support in Hilbert spaces.

1. Introduction and preliminaries

The notation of orthogonality relation in a general normed linear space was introduced by Robert [8]. Later, in 1935, Birkhoff introduced Birkhoff orthogonality [2], which is one of the most important orthogonalities defined in normed linear space. James [5] provided a comprehensive study of the properties of Birkhoff orthogonality. Due to this, Birkhoff orthogonality is also referred to as Birkhoff-James (BJ) orthogonality. Later, James [4] introduced the Isosceles orthogonality. Furthermore, James [4] introduced the Pythagorean orthogonality in normed linear space, which generalizes the result in an Euclidean space stating that two vectors are perpendicular if and only if there is a right triangle having two vectors as legs. As a generalization of orthogonality from an inner product space to normed linear space, a new concept of orthogonality in terms of bounded linear operators has been used by many different researchers. Bhatia and Semrl [1] introduced a concept of orthogonality in terms of matrices (operators) by taking the concept of BJ-orthogonality.

Recently, Sain et al. [7] studied the problem of characterizing BJ-orthogonality for bounded linear operators. Botazzi et al. [3] studied BJ-orthogonality and isosceles orthogonality of bounded linear operator between Hilbert space and Banach space. They have characterized isosceles orthogonality of bounded linear operator between Hilbert space with support functional. They also explore a relation between BJ-orthogonality and isosceles orthogonality in general Banach space.

Now, we present some notations and definitions to be used in the sequel.

Suppose that H is a real Hilbert space. Let $B_H = \{x \in X : \|x\| \leq 1\}$ be the unit ball of H and $S_H = \{x \in X : \|x\| = 1\}$ be the unit sphere of H. Let $B(H)$ denote the Banach space of all bounded linear operators from H to H.

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Let M_E denote the collection of all unit vectors in H at which $E \in B(H)$ attains norm, i.e.

$$M_E = \{x \in S_H : \|Ex\| = \|E\|\}.$$

As observed, in [9], [10], the structure of a bounded linear operator's norm attainment set is crucial for understanding orthogonality. The notion of a bounded linear operator's norm attainment set, on the other hand, was demonstrated, and it was shown that this notion is closely related to the geometry of the space of bounded linear operators between Banach spaces [6].

Similarly, the norm attainment set of $E + \lambda F$ and $E - \lambda F$ are $M_{E+\lambda F}$ and $M_{E-\lambda F}$ respectively where $E, F \in B(H)$. For $E \in B(H)$, E^* denotes the adjoint of E.

Definition 1.1. For any two elements $x, y \in X$, x is said to be orthogonal to y in the sense of Pythagorean orthogonal [4],

$$x \perp_P y \text{ if and only if } \|x + y\|^2 = \|x\|^2 + \|y\|^2.$$

Definition 1.2. [3] Let $E, F \in B(H)$. Two operators E, F have disjoint support if and only if $EF^* = F^*E = 0$.

Motivated by the fascinating geometric interpretation of the norms of vectors within the Hilbert space, we characterize the Pythagorean orthogonality of bounded linear operators using the concept of norm attainment set and disjoint support.

2. Main results

In this section, we focus on the characterization of Pythagorean orthogonality of bounded linear operator in Hilbert spaces.

Theorem 2.1. *Suppose $E, F \in B(H)$, where H is a real Hilbert space. If E and F have disjoint support, then $E \perp_p F$.*

Proof. For $h \in S_H$, $E, F \in B(H)$, we have

$$\begin{aligned} \|(E+F)h\|^2 &= \|Eh\|^2 + \|Fh\|^2 + 2Re\langle Eh, Fh\rangle \\ &= \|Eh\|^2 + \|Fh\|^2 + 2Re\langle h, F^*Eh\rangle \\ &= \|Eh\|^2 + \|Fh\|^2 + 2Re\langle h, 0h\rangle \\ &= \|Eh\|^2 + \|Fh\|^2. \end{aligned}$$

Since $h \in S_H$, we have $\|(E+F)\|^2 = \|E\|^2 + \|F\|^2$. Hence $E \perp_p F$. □

In the following example, we illustrate that if all pairs of operators $E, F \in B(H)$ with $E \perp_P F$, then the operators E and F may not have disjoint support.

Example 2.2. Let H represent the two-dimensional real Hilbert space. We consider any $|||.|||$. Let $E = \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix}$ and $F = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$. Then,

(i) $\|E + F\|^2 = 5^2 = 4^2 + 3^2 = \|E\|^2 + \|F\|^2$,

(ii) $E^*F = \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix}$

which implies that E, F do not have disjoint support.

Theorem 2.3. *Suppose that $E, F \in B(H)$ and there exists $h_0, k_0 \in H$ such that $h_0 \in M_{E+F}$ and $k_0 \in M_E \cap M_F$. Then $\langle Eh_0, Fh_0\rangle \leq 0$ and $\langle Ek_0, Fk_0\rangle \geq 0$ if and only if $E \perp_p F$.*

Proof. Suppose that $m, n : H \to \mathbb{R}$ be given by

$$\begin{aligned} m(h) &= \|(E+F)h\|^2 = \|Eh\|^2 + \|Fh\|^2 + 2\langle Eh, Fh\rangle \text{ and} \\ n(h) &= \|Eh\|^2 + \|Fh\|^2. \end{aligned}$$

So, $m(h) - n(h) = 2\langle Eh, Fh\rangle$. Suppose that there exists $h_0, k_0 \in H$ such that $h_0 \in M_{E+F}$ and $k_0 \in M_E \cap M_F$ and $\langle Eh_0, Fh_0\rangle \leq 0$.
If possible let $n(k_0) = \|E\|^2 + \|F\|^2 < \|E+F\|^2 = m(h_0)$.
Then $n(k_0) < m(h_0)$. It is also clear that $n(h_0) \leq n(k_0)$. Then $n(h_0) \leq n(k_0) < m(h_0)$. This implies that $m(h_0) - n(h_0) > 0$ that is $2\langle Eh, Fh\rangle > 0$, which is a contradiction.

Hence $n(k_0) \geq m(h_0)$.
On the similar lines, we can prove that $n(k_0) \leq m(h_0)$.
Hence from both the cases we have, $\|(E+F)\|^2 = m(h_0) = n(k_0) = \|E\|^2 + \|F\|^2$.
Hence $E \perp_p F$.
Conversely, let $E \perp_p F$. Then $\|(E+F)\|^2 = m(h_0) = n(k_0) = \|E\|^2 + \|F\|^2$.
Now,

$$\begin{aligned} \langle Eh_0, Fh_0\rangle &= \frac{1}{2}[\|(E+F)h_0\|^2 - \|Eh_0\|^2 + \|Fh_0\|^2] \\ &\geq \frac{1}{2}[\|(E+F)\|^2 - \|E\|^2 + \|F\|^2] \\ &= 0. \end{aligned}$$

Therefore $\langle Eh_0, Fh_0\rangle \geq 0$.
Similarly, we can prove that $\langle Ek_0, Fk_0\rangle \leq 0$.
Hence proved. □

Corollary 2.4. *Let $E, F \in B(H)$ and suppose that there exits $h_1 \in M_{E+F} \cap M_E \cap M_F$ such that $\langle Eh_1, Fh_1\rangle = 0$. Then $E \perp_p F$.*

The next result adds a further requirement and provides another way to characterize Pythagorean orthogonality of bounded linear operators in a real Hilbert space with finite dimension.

Theorem 2.5. *Let H be a real Hilbert space with $dim(H) = n$ and $E, F \in B(H)$. Suppose that there exist sub-spaces H_1 and H_2 of H such that $M_{E+F} = S_{H_1}, M_E \cap M_F = S_{H_2}$ and $dim(H_1) + dim(H_2) > n$. Then, the following statements are equivalent:*
(i) $E \perp_p F$.
(ii) There exist $x_0 \in M_{E+F} \cap M_E \cap M_F$ such that $\langle Ex_0, Fx_0\rangle = 0$.

Proof. Let (i) holds. Since $dim(H_1) + dim(H_2) > n$, there exist $0 \neq x_1 \in H_1 \cap H_2$. Consider $x_0 = \frac{x_1}{\|x_1\|}$, then $x_0 \in M_{E+F} \cap M_E + M_F$.

Now,

$$\begin{aligned}\langle Ex_0, Fx_0\rangle &= \frac{1}{2}[\|(E+F)h_0\|^2 - \|Eh_0\|^2 + \|Fh_0\|^2] \\ &= \frac{1}{2}[\|(E+F)\|^2 - \|E\|^2 + \|F\|^2] \\ &= 0.\end{aligned}$$

Therefore (i) implies (ii).
By the Corollary 2.4, it is easy to see that (ii) implies (i). Hence proved. □

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