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GLOBALLY GENERATED VECTOR BUNDLES ON A PROJECTIVE SPACE BLOWN UP ALONG A LINE

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Abstract. We classify globally generated vector bundles on the projective space blown up along a line with very low first Chern class c_1.

1. Introduction

Let $\mathbb{P}^n$ be the n-dimensional projective space over an algebraically closed field k of characteristic 0. Globally generated vector bundles on projective varieties are fundamental objects in algebraic geometry. However, even in the case of $\mathbb{P}^n$, their classification for small first Chern class has only been studied fairly recently [1, 9, 12, 13]. Subsequently, Ballico, Huh and Malaspina investigated globally generated vector bundles on projective varieties such as smooth quadric threefolds [4], complete intersection Calabi-Yau threefolds [5], Segre threefolds [6, 7]. Moreover, Ballico studied the case of projective space blown up at finitely many points [3].

In this article, we consider similar questions on the projective space blown up along a line (we follow the notation of [8]). Let X be the projective space blown up along a line, $\tilde{H}$ the pull-back of a hyperplane, E the exceptional divisor. Our main result is the classification of globally generated vector bundles on X with $c_1 = 2\tilde{H} - E (= \xi + f$, cf. the next section), up to trivial factor.

Main Theorem. *Let $\pi : X \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}^2) \to \mathbb{P}^1$ be the natural projection and let ξ and f be the classes of $\mathcal{O}_{\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1)\oplus\mathcal{O}_{\mathbb{P}^1}^2)}(1)$ and $\pi^*\mathcal{O}_{\mathbb{P}^1}(1)$ respectively. Let $\mathcal{E}$ be a globally generated vector bundle of rank r at least 2 on X with $c_1 = \xi + f$ and $c_2 = \alpha\xi^2 + \beta\xi f$. If $\mathcal{E}$ has no trivial factor, then the possible rank r and $(s; \alpha, \beta)$ are as follows (s is the number of connected components of associated curve to $\mathcal{E}$ via the Hartshorne-Serre correspondence modulo trivial factor):*

(1) $r = 2, (1; 1, 0)$;
(2) $r = 2, (1; 0, 1)$: *In this case* $\mathcal{E} \cong \mathcal{O}_X(\xi) \oplus \mathcal{O}_X(f)$;
(3) $r = 3, 4, (1; 1, 1)$;
(4) $r = 2, 3, (2; 0, 2)$;
(5) $3 \leq r \leq 6, (1; 1, 2)$.

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Furthermore, there exists a globally generated vector bundle in each of these cases.

In fact, we present a description of the possible c_2 as represented by the associated curves (cf. Section 3).

Remark 1.1. The classification of globally generated vector bundles on $\mathbb{P}^n$ with $c_1 = 2$ is the main result of [12], and the classification of globally generated vector bundles on $\mathbb{P}^3$ blown up at a point with $c_1 = 2\tilde{H} - E$ is one of the main results of [3] (cf. Theorem 3.2. in loc. cit).

2. Preliminaries

Let $\sigma : X \to \mathbb{P}^3$ be the projective space blown up along a line R, and let $\tilde{H}, E, \xi$, and f be as in Introduction. Thus $\omega_X \cong \mathcal{O}_X(-3\xi - f)$, and we have an isomorphism

$$A(X) \cong \mathbb{Z}[\xi, f]/(f^2, \xi^3 - \xi^2 f),$$

so that $\xi^3 = \xi^2 f$ are the classes of a point.

Trivially $\pi^*\mathcal{O}_{\mathbb{P}^1}(1)$ is globally generated. On the other hand, since $\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}^2_{\mathbb{P}^1}$ is globally generated, it follows that the same holds for $\mathcal{O}_{\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1)\oplus\mathcal{O}^2_{\mathbb{P}^1})}(1)$. In fact, by [8, Remark 4.7.], $\mathcal{O}_X(a\xi + bf)$ is globally generated if and only if $a, b \geq 0$. Moreover, $\mathcal{O}_X(\xi) \cong \sigma^*\mathcal{O}_{\mathbb{P}^3}(1) = \mathcal{O}_X(\tilde{H})$.

Since the normal bundle of the blown up line R inside $\mathbb{P}^3$ is $\mathcal{N}_{R|\mathbb{P}^3} \cong \mathcal{O}_{\mathbb{P}^3}(1)^2$, it follows that $E = \sigma^{-1}(R) \cong \mathbb{P}^1 \times \mathbb{P}^1$ and σ induces an isomorphism $X \setminus \sigma^{-1}(R) \cong \mathbb{P}^3 \setminus R$. Let $H \subseteq \mathbb{P}^3$ be a plane through R. On the one hand, $\sigma^{-1}(H)$ is in the class of ξ. On the other hand, $\sigma^{-1}(H)$ is the union of E with the strict transform of H. Such a strict transform is in the linear system $|f|$, hence E is the unique element in $|\xi - f|$.

Let $\mathcal{E}$ be a globally generated vector bundle of rank r on X with the first Chern class c_1. Then it fits into the exact sequence

$$0 \to \mathcal{O}_X^{r-1} \to \mathcal{E} \to \mathcal{I}_C(c_1) \to 0, \tag{2.1}$$

where C is a smooth curve on X by [9, Section 2. G.]. The associated curve C represents the second Chern class c_2 of $\mathcal{E}$.

For the construction of vector bundles as in (2.1), we use [2, Theorem 1.1.] for $L := \mathcal{O}_X(c_1)$. Note that for a smooth curve C, $\bigwedge_C := \bigwedge^2 N \otimes L^\vee|_C \cong \omega_C \otimes \omega_X^\vee \otimes L^\vee|_C$. By the discussion in the beginning of [2] (cf. also [5, Theorem 2.8.]), for the existence of a globally generated vector bundle of rank r, $\bigwedge_C$ must be globally generated (and trivial if $r = 2$), and an $(r-1)$-dimensional vector subspace of $h^0(\bigwedge_C)$ corresponds to a globally generated vector bundle of rank r without trivial factor, i.e. with no factor isomorphic to $\mathcal{O}_X$. Conversely if $\bigwedge_C$ is globally generated, we can construct a vector bundle $\mathcal{E}$ by [2, Theorem 1.1.] and $\mathcal{E}$ is globally generated if and only if $\mathcal{I}_C(c_1)$ is globally generated.

Proposition 2.1. ([11, Proposition 1]) *Let $\mathcal{E}$ be a globally generated vector bundle of rank r on a reduced irreducible projective variety V over k such that $h^0(\mathcal{E}(-c_1)) \neq 0$. Then we have*

$$\mathcal{E} \cong \mathcal{O}_V^{r-1} \oplus \mathcal{O}_V(c_1).$$

By this proposition, when $c_1 = 0$ the only globally generated vector bundle of rank r on X is $\mathcal{O}_X^r$.

Proposition 2.2. *Let $\mathcal{E}$ be a globally generated vector bundle of rank at least 2 on X with $c_1 = a\xi, a \geq 1$, $\sigma : X \to \mathbb{P}^3$ the blow up map. Then there exists a globally generated vector bundle $\mathcal{F}$ on $\mathbb{P}^3$ such that $\mathcal{E} = \sigma^*\mathcal{F}$.*

Proof. Since $E \in |\xi - f|$, we have $\xi \cdot E = \xi(\xi - f) = \xi^2 - \xi f$. By [8, Remark 3.1.], this is a class of a line, say $N \in |\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,0)|$, in $E \cong \mathbb{P}^1 \times \mathbb{P}^1$. Moreover, since a general plane $H \subseteq \mathbb{P}^3$ intersects R at a point, $\xi^2 - \xi f$ is the class of a fiber of $\sigma|_E : E \cong \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$.

Let $\mathcal{E}$ be a globally generated vector bundle with $c_1 = a\xi$ and C be its associated curve. Assume that $C \cap E$ is non-empty. Since $\mathcal{I}_{C\cap E}(a\xi)|_E \cong \mathcal{I}_{C\cap E|\mathbb{P}^1\times\mathbb{P}^1}(a,0)$ is globally generated, $C \cap E$ is a disjoint union of lines. Hence C contains a line $N \in |\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,0)|$ as a connected component.

Now, consider the line bundle $\omega_X^\vee \otimes L^\vee|_N = \mathcal{O}_X((3-a)\xi + f)|_N$, which has degree 1 on N. Since N is rational, $\bigwedge_N$ is not globally generated. This contradicts the fact that C is the associated curve of $\mathcal{E}$.

We conclude that $C \cap E = \emptyset$. Then $C \cong \sigma(C)$. $\mathcal{I}_{C|X}(a\xi)$ is globally generated if and only if $\mathcal{I}_{\sigma(C)|\mathbb{P}^3}(a)$ is globally generated, and $\bigwedge_C \cong \omega_C \otimes \omega_X^\vee \otimes L^\vee|_C = \omega_C \otimes \mathcal{O}_X(4\tilde{H} - E) \otimes \mathcal{O}_X(-a\tilde{H})|_C \cong \omega_{\sigma(C)} \otimes \mathcal{O}_{\mathbb{P}^3}(4) \otimes \mathcal{O}_{\mathbb{P}^3}(-a)|_{\sigma(C)} \cong \bigwedge_{\sigma(C)}$.

Therefore, there exists a globally generated vector bundle $\mathcal{F}$ on $\mathbb{P}^3$ with $c_1 = \mathcal{O}_{\mathbb{P}^3}(a)$, such that $\mathcal{E} \cong \sigma^*\mathcal{F}$. □

Proposition 2.3. *Let $\mathcal{E}$ be a globally generated vector bundle of rank r at least 2 on X with $c_1 = bf, b \geq 1$, $\pi : X \to \mathbb{P}^1$ the projective bundle map. Then there exists a globally generated vector bundle $\mathcal{F}$ on $\mathbb{P}^1$ such that $\mathcal{E} = \pi^*\mathcal{F}$.*

Proof. Let $\mathcal{E}$ be a globally generated vector bundle with $c_1 = bf$, and $F := \pi^{-1}(p) \cong \mathbb{P}^2$ be a fiber. Then the restricted bundle $\mathcal{E}|_F$ is a globally generated vector bundle on $\mathbb{P}^2$ with $c_1 = 0$. By Proposition 2.1, $\mathcal{E}|_F \cong \mathcal{O}_{\mathbb{P}^2}^r$. The base-change theorem [10, page 11] implies $\pi_*\mathcal{E}$ is locally free of rank r.

The natural map $\pi^*\pi_*\mathcal{E} \to \mathcal{E}$ is surjective since $\mathcal{E}$ is globally generated and, being a surjective morphism between two vector bundles of the same rank, it is an isomorphism. By the theorem of Grothendieck [10, Theorem 2.1.1.], $\pi_*\mathcal{E}$ is a direct sum of line bundles and hence globally generated since its pull-back must be globally generated. □

Let $\mathcal{E}$ be a globally generated vector bundle with $c_1 = a\xi + bf$. If $c_1 = a\xi$, $\mathcal{E}$ is a pull-back of a globally generated vector bundle on $\mathbb{P}^3$ by σ (Proposition 2.2). If $c_1 = bf$, $\mathcal{E}$ is a pull-back of a globally generated vector bundle on $\mathbb{P}^1$ by π (Proposition 2.3). Thus, in these cases, the classification of globally generated vector bundles on X is reduced to the classification of globally generated vector bundles on $\mathbb{P}^n$.

3. Proof of the Main Theorem

Lemma 3.1. *Let $\mathcal{E}$ be a globally generated vector bundle on X with $c_1 = \xi + bf$, $b \geq 0$ and $\pi : X \to \mathbb{P}^1$ be the projective bundle map. Let the associated curve of*

$\mathcal{E}$ be C . If C has a connected component which is not in a fiber of π, then C is connected and rational.

Proof. Let $F := \pi^{-1}(p) \cong \mathbb{P}^2$ be a fiber. Since $\mathcal{I}_C(\xi + bf)$ is globally generated, $\mathcal{I}_C(\xi + bf)|_F \cong \mathcal{I}_{C\cap F|\mathbb{P}^2}(1)$ is globally generated.

Let C_i be a connected component of C which is not in a fiber of π. Since $\mathcal{I}_{C\cap F|\mathbb{P}^2}(1)$ is globally generated, $\deg(C_i \cap F) = 1$. Hence $\pi|_{C_i} : C_i \to \mathbb{P}^1$ is of degree 1 and C_i is rational. If there exists another component C_j not contained in a fiber, then there exists a fiber F such that $C \cap F$ is 0-dimensional and $\deg(C \cap F) \geq 2$. Then, a line through two points of $C \cap F$ is in the base locus of $\mathcal{I}_{C\cap F|\mathbb{P}^2}(1)$. This contradicts the fact that $\mathcal{I}_{C\cap F|\mathbb{P}^2}(1)$ is globally generated. Hence at most one component is not in a fiber of π. If there exists another component C_j contained in a fiber F, then $C \cap F$ contains C_j and a point of $C_i \cap F$ not on C_j. Again, this contradicts the fact that $\mathcal{I}_{C\cap F|\mathbb{P}^2}(1)$ is globally generated, and so $C = C_i$ is connected. □

Now, let $\mathcal{E}$ be a globally generated vector bundle of rank r at least 2 on X with $c_1 = \xi + f$, $C = \bigcup_{i=1}^{s} C_i$ its associated curve.

Remark 3.2. (cf. [6, Remark 2.4, 2.7.], [7, Remark 2.7.]) By assumption $\mathcal{I}_C(\xi+f)$ is globally generated. Let us take two general divisors $M_1, M_2 \in |\mathcal{I}_C(\xi + f)|$. Set $Y := M_1 \cap M_2$. By the Bertini theorem we have $Y = C \cup D$ with either $D = \emptyset$ or D a reduced curve containing no component of C and smooth outside $C \cap D$. Each connected component C_i of C appears with multiplicity one in Y, because affixing points $p_i \in C_i$ for every i, we may find a divisor $T \in |\mathcal{I}_C(\xi + f)|$ not containing the tangent line of C_i at p_i. Y is also connected since we have $h^0(\mathcal{O}_Y) = 1$ by vanishing of cohomologies (by [8, Proposition 4.1.]) and a Koszul complex standard exact sequence. The adjunction formula gives $\omega_Y \cong \mathcal{O}_Y(-\xi + f)$ and so we have

$$\begin{aligned} 2p_a(Y) - 2 &= \deg(\xi + f)(\xi + f)(-\xi + f) \\ &= \deg(-\xi^3 - \xi^2 f) \\ &= -2 \end{aligned}$$

Hence $p_a(Y) = 0$ and Y is rational.

Lemma 3.3. *If C has a connected component in E, then C is contained in E and its class is ξ^2 or ξf.*

Proof. By assumption, $\mathcal{I}_{C\cap E}(\xi+f)|_E$ is globally generated. By the adjunction theorem on E, $\omega_E = \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-2,-2) = \mathcal{O}_X(-4\tilde{H}+2E)|_E = \mathcal{O}_X(-2\xi-2f)|_E$. Hence $\mathcal{I}_{C\cap E}(\xi+f)|_E \cong \mathcal{I}_{C\cap E}(2\tilde{H} - E)|_E \cong \mathcal{I}_{C\cap E,\mathbb{P}^1\times\mathbb{P}^1}(1,1)$ and a curve in $|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1)|$ has the class $(\xi + f)(\xi - f) = \xi^2$.

Next, let us review the explicit description of X, the projective space $\mathbb{P}^3$ blown up along a line R. Let x_0, x_1, x_2, x_3 be the coordinates of $\mathbb{P}^3$ and let the line R be defined by the vanishing of x_0, x_1. Let y_0, y_1 be the coordinates of $\mathbb{P}^1$. Then, X is the closed subscheme in $\mathbb{P}^3 \times \mathbb{P}^1$ defined by the equation $x_0y_1 = x_1y_0$.

Let $H \subseteq \mathbb{P}^3$ be a plane. If H does not contain R, it intersects R at a point and its strict transform (pull-back) in X intersects E in a line of class $\xi^2 - \xi f$, which is the fiber of $\sigma|_E : E \cong \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ at the point $H \cap R$.

If H contains R, we can write H as $a_0x_0 + a_1x_1 = 0$, where $a_0, a_1 \in k$, and the class of its strict transform in X is f. Since X is defined by the equation $x_0y_1 = x_1y_0$, the strict transform of H in X intersects E at the point $(x_0, x_1, x_2, x_3; y_0, y_1) = (0, 0, x_2, x_3; -a_1, a_0)$, which is a point on a line in the section of $\sigma|_E : E \cong \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$, with class ξf. Each strict transform of a hyperplane containing R intersects E in each line in the linear system $|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(0,1)|$.

Let $Q \subseteq \mathbb{P}^3$ be an irreducible quadric cone containing R. Then the strict transform of Q in X intersects E in the union of the line in $|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,0)|$ corresponding to the fiber of $\sigma|_E : E \cong \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ at the vertex of Q and the line in $|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(0,1)|$ corresponding to the tangent plane of Q along R.

Let $Q \subseteq \mathbb{P}^3$ be an irreducible smooth quadric surface containing R. We can write Q as $a_{00}x_0^2 + a_{01}x_0x_1 + a_{11}x_1^2 + f(x_2, x_3)x_0 + g(x_2, x_3)x_1 = 0$, where $f(x_2, x_3)$ and $g(x_2, x_3)$ are linear polynomials in x_2, x_3. Hence the strict transform of Q in X intersects E in the coordinates $(x_0, x_1, x_2, x_3; y_0, y_1) = (0, 0, x_2, x_3; -g(x_2, x_3), f(x_2, x_3))$. Since Q is smooth, f and g are nonzero, so this intersection is a rational curve in the linear system $|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1)|$.

Suppose C contains a line $N \in |\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,0)|$ with class $\xi^2 - \xi f$ in E. The degree of the line bundle $\mathcal{O}_X(2\xi)$ restricted to N is zero. Then $\bigwedge_N \cong \omega_N \otimes \omega_X^\vee \otimes L^\vee|_N \cong \omega_N \otimes \mathcal{O}_X(2\xi)|_N$ is not globally generated, a contradiction.

Assume $N := C \cap E \in |\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(0,1)|$ is a line with class ξf in E. The element of $|\mathcal{I}_C(\xi + f)|$ is the strict transform of a union of two planes, one of them containing R, or a quadric cone containing R. If $|\mathcal{I}_C(\xi + f)|$ consists only of the strict transforms of two planes, then the strict transform of the plane defined by N is in the base locus of $\mathcal{I}_C(\xi + f)$. We conclude that there is a quadric cone containing $\sigma(C) \cup R$. Assume that $C \neq N$. There is a complete intersection Y containing C, whose class is $\xi^2 + 2\xi f$ (we may assume $Y \cap E = N$). The residual curve to N in Y is $D = \xi^2 + \xi f$. $\sigma(D)$ is a conic or a union of lines and since Y is connected and C is smooth, the only possibility is that C is the disjoint union of N and the strict transform of a line disjoint from R (and the residual curve to C in Y is a line that intersects each component of C). This is a contradiction since every line in a quadric cone containing R must intersect R. We conclude that $C = N = \xi f$ in this case.

Assume $N := C \cap E \in |\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1)|$ is a rational curve with class ξ^2 in E. Since $\xi^2 \cdot f \neq 0$, N is not contained in a fiber of $\pi : X \to \mathbb{P}^1$. By Lemma 3.1, we conclude that $C = N = \xi^2$ in this case. □

Proposition 3.4. *Let $\mathcal{E}$ be a globally generated vector bundle of rank r on X with $c_1 = \xi + f, c_2 = \xi^2$. If the associated curve C is contained in E and $\mathcal{E}$ has no trivial factor, then $\mathcal{E}$ exists if and only if $r = 2$.*

Proof. $|\mathcal{I}_C(\xi + f)|$ contains the strict transforms of the smooth quadrics defined by equations of the form $a_{00}x_0^2 + a_{01}x_0x_1 + a_{11}x_1^2 + f(x_2, x_3)x_0 + g(x_2, x_3)x_1 = 0$, where a_{00}, a_{01} and a_{11} are arbitrary and $f(x_2, x_3)$ and $g(x_2, x_3)$ are fixed. These equations show that $x_0 = x_1 = 0$, hence C is cut out by global sections. Therefore, $\mathcal{I}_C(\xi+f)$ is globally generated. Since $\bigwedge_C \cong \omega_C \otimes \mathcal{O}_X(2\xi)|_C \cong \mathcal{O}_C$ and $h^0(\mathcal{O}_C) = 1$, by [2, Theorem 1.1.], this case gives a rank r globally generated bundle if and only if $r = 2$. □

Proposition 3.5. *Let $\mathcal{E}$ be a globally generated vector bundle of rank r on X with $c_1 = \xi + f, c_2 = \xi f$. If the associated curve C is contained in E and $\mathcal{E}$ has no trivial factor, then $\mathcal{E}$ exists if and only if $r = 2, \mathcal{E} \cong \mathcal{O}_X(\xi) \oplus \mathcal{O}_X(f)$.*

Proof. Since $\bigwedge_C \cong \omega_C \otimes \mathcal{O}_X(2\xi)|_C \cong \mathcal{O}_C$ and $h^0(\mathcal{O}_C) = 1$, this curve gives a rank 2 vector bundle $\mathcal{E}$. We will prove that $\mathcal{E} \cong \mathcal{O}_X(\xi) \oplus \mathcal{O}_X(f)$. Since $h^0(\mathcal{O}_C(f)) = h^0(\mathcal{O}_C) = 1$ and $h^0(\mathcal{O}_X(f)) = 2$ by [8, Proposition 4.1], so $h^0(\mathcal{I}_C(f)) > 0$. By the exact sequence (2.1), we have $h^0(\mathcal{E}(-\xi)) > 0$ and so there is a non-zero map $m : \mathcal{O}_X(\xi) \to \mathcal{E}$. We have $h^0(\mathcal{E}(-2\xi)) = h^0(\mathcal{E}(-\xi - f)) = 0$ since $h^0(\mathcal{I}_C(-\xi + f)) = h^0(\mathcal{I}_C) = 0$, and so $\mathcal{O}_X(\xi)$ is saturated in $\mathcal{E}$. Hence the cokernel of m is torsion free, i.e. $\operatorname{coker}(m) \cong \mathcal{I}_T(f)$ with either $T = \emptyset$ or T is a locally complete intersection curve. Since $c_2(\mathcal{O}_X(\xi) \oplus \mathcal{O}_X(f)) = \xi f = C$, we have $T = \emptyset$. Since $h^1(\mathcal{O}_X(\xi - f)) = 0$, this implies $\mathcal{E} \cong \mathcal{O}_X(\xi) \oplus \mathcal{O}_X(f)$. □

Proposition 3.6. *Let $\mathcal{E}$ be a globally generated vector bundle of rank r on X with $c_1 = \xi + f$. If the associated curve $C = \bigcup_{i=1}^{s} C_i = \alpha\xi^2 + \beta\xi f$ has no connected component contained in E, and $\mathcal{E}$ has no trivial factor, then $\mathcal{E}$ exists if and only if r and $(s; \alpha, \beta)$ are as in the Main theorem.*

Proof. Let $\mathcal{E}$ be a globally generated vector bundle with $c_1 = \xi + f$, and assume that its associated curve C has no connected component contained in E. Since we assume $\mathcal{E}$ has no trivial factor, C is non-empty. Let $Y = \xi^2 + 2\xi f$ be a complete intersection of two general divisors in $|\mathcal{I}_C(\xi + f)|$. $Y = C \cup D$ with either $D = \emptyset$ or D a reduced curve containing no component of C and smooth outside $C \cap D$. We may assume Y does not have a component in E.

We have $(\alpha\xi^2 + \beta\xi f)f = \alpha\xi^2 f$. This implies that the line bundle $\mathcal{O}_X(f)$ restricted to C has degree α. Since $\mathcal{O}_X(f)$ is globally generated, it follows that $\alpha \geq 0$. The same argument applied to D implies that $\alpha \leq 1$. By assumption and $(\alpha\xi^2 + \beta\xi f)(\xi - f) = \beta\xi^2 f$, $C \cap E$ is a 0-dimensional scheme of degree β. It follows that $0 \leq \beta \leq 2$.

Assume $(\alpha, \beta) = (1, 0)$. Then $\sigma(C)$ is a line disjoint from R. Therefore, $\sigma(C) \cup R$ is cut out by quadrics since $\sigma(C) \cup R$ is the intersection of the unions of a plane containing $\sigma(C)$ and a plane containing R. This implies $\mathcal{I}_{\sigma(C)\cup R|\mathbb{P}^3}(2)$ is globally generated, hence the same for its pull-back $\mathcal{I}_C(2\tilde{H} - E) = \mathcal{I}_C(\xi + f)$. Since C is connected, rational and $\deg(\omega_C \otimes \mathcal{O}_X(2\xi)|_C) = 0$, we have $\bigwedge_C \cong \mathcal{O}_C$ and $h^0(\bigwedge_C) = 1$. By [2, Theorem 1.1.], this case gives a rank r globally generated vector bundle if and only if $r = 2$.

Assume $(\alpha, \beta) = (0, 1)$. Then $\sigma(C)$ is a line intersecting R at a point. Therefore, $\sigma(C) \cup R$ is a reducible conic and so $\mathcal{I}_{\sigma(C)\cup R|\mathbb{P}^3}(2)$ is globally generated. This implies $\mathcal{I}_C(2\tilde{H} - E)$ is also globally generated. Since C is rational and $\deg(\omega_C \otimes \mathcal{O}_X(2\xi)|_C) = 0$, we have $\bigwedge_C \cong \mathcal{O}_C$ and $h^0(\bigwedge_C) = 1$. Hence this case gives a rank r globally generated vector bundle if and only if $r = 2$. In this case, $\sigma(C)$ is in a plane containing R, hence C is contained in a fiber f of π. So we have $h^0(\mathcal{I}_C(f)) > 0$ and we can show $\mathcal{E} \cong \mathcal{O}_X(\xi) \oplus \mathcal{O}_X(f)$ as in the proof of Proposition 3.5.

Assume $(\alpha, \beta) = (1, 1)$ and $s = 1$. Then $\sigma(C)$ is a conic intersecting R at a point. Therefore, $\sigma(C) \cup R$ is cut out by quadrics since $\sigma(C) \cup R$ is the intersection

of the unions of a plane containing $\sigma(C)$ and a plane containing R, and the quadric cones containing $\sigma(C)$ and R. This implies that $\mathcal{I}_{\sigma(C)\cup R|\mathbb{P}^3}(2)$ is globally generated, therefore $\mathcal{I}_C(2\tilde{H}-E)$ is also globally generated. Since C is rational and $\deg(\omega_C\otimes\mathcal{O}_X(2\xi)|_C)=2$, so $\bigwedge_C$ is globally generated, non-trivial and we have $h^0(\bigwedge_C)=3$. Hence this case gives a rank r globally generated vector bundle if and only if $3\leq r\leq 4$.

Assume $(\alpha,\beta)=(1,1)$ and $s=2$. Then $\sigma(C)$ is a disjoint union of two lines, intersecting R at a point of one of its components. We write $\sigma(C)=L_1\cup L_2$, where L_1, L_2 are lines and L_2 intersects R. Let $\langle L_2,R\rangle$ be the plane spanned by L_2 and R. Since L_1 intersects $\langle L_2,R\rangle$ at a point in $\langle L_2,R\rangle\setminus(L_2\cup R)$, every quadric vanishing along $L_1\cup L_2\cup R$ must vanish on $\langle L_2,R\rangle$. Therefore $\langle L_2,R\rangle$ is contained in the base locus of $\mathcal{I}_{\sigma(C)\cup R|\mathbb{P}^3}(2)$ and so its strict transform is contained in the base locus of $\mathcal{I}_C(2\tilde{H}-E)$. Thus $\mathcal{I}_C(2\tilde{H}-E)$ is not globally generated.

Assume $(\alpha,\beta)=(0,2)$ and $s=1$. $\sigma(C)$ is a conic intersecting R at two points. The plane spanned by $\sigma(C)$ also contains R, so every quadric vanishing along $\sigma(C)\cup R$ must vanish on this plane. Thus, the strict transform of this plane is contained in the base locus of $\mathcal{I}_C(2\tilde{H}-E)$.

Assume $(\alpha,\beta)=(0,2)$ and $s=2$. $\sigma(C)$ is a disjoint union of two lines, each of which intersects R at a point. We write $\sigma(C)=L_1\cup L_2$, where L_1, L_2 are the two lines. Then $\sigma(C)\cup R$ is cut out by quadrics since it is the intersection of the unions of a plane containing L_1 and the plane $\langle L_2,R\rangle$, and the unions of a plane containing L_1 and the plane $\langle L_2,R\rangle$. This implies $\mathcal{I}_{\sigma(C)\cup R|\mathbb{P}^3}(2)$ is globally generated, hence $\mathcal{I}_C(2\tilde{H}-E)$ is also globally generated. Since C has two components and each component is rational, $\bigwedge_C\cong\mathcal{O}_C\cong\mathcal{O}_{\mathbb{P}^1}^2$ is globally generated, trivial and we have $h^0(\bigwedge_C)=2$. Hence this case gives a rank r globally generated vector bundle if and only if $2\leq r\leq 3$.

Assume $(\alpha,\beta)=(1,2)$. In this case, $C=Y$ and $\sigma(C)$ is a twisted cubic having R as a bisecant line. C is rational and $\mathcal{I}_C(2\tilde{H}-E)$ is globally generated. Since $\deg(\omega_C\otimes\mathcal{O}_X(2\xi)|_C)=4$, $\bigwedge_C$ is globally generated, non-trivial and we have $h^0(\bigwedge_C)=5$. Hence this case gives a rank r globally generated vector bundle if and only if $3\leq r\leq 6$. □

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