abstract. We will present in this paper the concept of matrix-product code 
$[C_1, C_2, \cdots, C_s]A$ where $C_1, C_2, \cdots, C_s$ are linear codes and matrix $A$ has full rank. The purpose of this paper is to give some properties of these codes that will be useful later, the two main points being the existence of a MacWilliams identity (which is one reason the existence of the phenomenon of duality formal) and the structure of the matrix-product code on $\mathbb{Z}_{p^k}$.

1. Introduction

In the coding theory an interesting and important approach is to construct a new codes from known ones. In [1], Blackmore and all. introduced the notion of matrix product codes over finite fields for generalize some construction known in the coding theory. For example, the Plotkin’s well-known $(u|u+v)$-construction, the ternary $(u+v+w|2u+v|u)$-construction, the $(a+x|b+x|a+b+x)$-construction, and the $(u + v|u − v)$-construction and etc. But the challenge problem is to determine the parameters of the new codes (length, dimension, minimal Hamming distance, $\cdots$ ) from the initial codes. To answer this question there are some other articles focusing on the study of decoding and the construction of matrix product codes ([3, 5]) have appeared.

2. Matrix Product Codes

Definition 2.1. Let $C_1, C_2, \cdots, C_s$ be linear codes of length $m$ over $\mathbb{F}_q$ and $A = (a_{ij})$ in $M_{s \times l}(\mathbb{F}_q)$ with $s \leq l$. A matrix product codes $C_A$ associated to $C_1, C_2, \cdots, C_s$ and $A$ is the linear code over $\mathbb{F}_q$ of length $ml$ defined by :

$$C_A = \{ (\sum_{i=1}^{s} a_{i1}x_i, \sum_{i=1}^{s} a_{i2}x_i, \cdots, a_{il}x_i) | x_i \in C_i \}$$

2.1. Description of $C_A$. If

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1l} \\ a_{21} & a_{22} & \cdots & a_{2l} \\ \vdots & \vdots & \ddots & \vdots \\ a_{s1} & a_{s2} & \cdots & a_{sl} \end{pmatrix}$$

Then :
Example 2.2. Let $C := [C_1, C_2, \ldots, C_s] = \{(x_1x_2 \cdots x_s) \in M_{m \times s}(\mathbb{F}_q) \mid x_i^t \in C_i\}$

We define:

$C^t := \{X^t \mid X \in C\}$ and $A^tC^t := \{A^tY \mid Y \in C^t\}$

Take $Z$ element of the $A^tC^t$, so $\exists Y \in C^t : Z = A^tY$, $Y \in C^t \iff \exists X \in C; X^t = Y$

on the other hand $X \in C \iff \exists x_i : x_i^t \in C_i$ and $X = (x_1x_2 \cdots x_s) \text{ so }$

$Z = \left(\begin{array}{cccc}
a_{11} & a_{21} & \cdots & a_{s1} \\
a_{12} & a_{22} & \cdots & a_{s2} \\
\vdots & \vdots & \ddots & \vdots \\
a_{1l} & a_{2l} & \cdots & a_{sl}
\end{array}\right) \times \left(\begin{array}{c}x_1 \\
x_2 \\
\vdots \\
x_s
\end{array}\right) = \left(\begin{array}{c}\sum_{i=1}^{s} a_{1i}x_i \\
\sum_{i=1}^{s} a_{2i}x_i \\
\vdots \\
\sum_{i=1}^{s} a_{si}x_i
\end{array}\right)$

$Z^t = (\sum_{i=1}^{s} a_{1i}x_i; \sum_{i=1}^{s} a_{2i}x_i; \ldots, a_{si}x_i) \in C_A$, so $(A^tC^t)^t = C_A$ that means $C.A = C_A$

**Example 2.2.** Let $C_1 = \{000, 111\}$ and $C_2 = \{000, 011\}$ two linear codes of length $m = 3$ and $s = 2$ and consider the two matrix $A = \left(\begin{array}{ccc}1 & 0 & 0 \\
0 & 1 & 0
\end{array}\right)$ and

$B = \left(\begin{array}{ccc}1 & 0 & 1 \\
0 & 1 & 1
\end{array}\right)$

$[C_1C_2] := \{(x_1x_2) \mid x_i \in C_i\} = \left\{\left(\begin{array}{c}0 \\
0
\end{array}\right); \left(\begin{array}{c}0 \\
0
\end{array}\right); \left(\begin{array}{c}1 \\
0
\end{array}\right); \left(\begin{array}{c}0 \\
1
\end{array}\right); \left(\begin{array}{c}1 \\
0
\end{array}\right); \left(\begin{array}{c}1 \\
1
\end{array}\right)\right\}$.

Then the matrix product codes of $C_1$, $C_2$ and $A$:

$C_A = [C_1C_2].A$

- $\left(\begin{array}{c}0 \\
0
\end{array}\right) \times A = \left(\begin{array}{c}0 \\
0
\end{array}\right) = (000, 000, 000, 000)$
- $\left(\begin{array}{c}0 \\
0
\end{array}\right) \times A = \left(\begin{array}{c}0 \\
0
\end{array}\right) = (000, 011, 000, 011)$
- $\left(\begin{array}{c}1 \\
0
\end{array}\right) \times A = \left(\begin{array}{c}1 \\
0
\end{array}\right) = (111, 000, 000, 000)$
- $\left(\begin{array}{c}1 \\
0
\end{array}\right) \times A = \left(\begin{array}{c}1 \\
0
\end{array}\right) = (111, 001, 000, 100)$
- $\left(\begin{array}{c}1 \\
1
\end{array}\right) \times A = \left(\begin{array}{c}1 \\
1
\end{array}\right) = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$
Proposition 3.1. Let \( G_1, G_2, \ldots, G_s \) be the generator matrix of the code \( C_1, C_2, \ldots, C_s \) respectively and \( A = (a_{ij}) \in M(\mathbb{F}_q, s \times l) \). Then the generator matrix \( G \) of the matrix product code \( C = [C_1, C_2, \ldots, C_s]A \) is given by:

\[
G = \begin{pmatrix}
a_{11}G_1 & a_{12}G_1 & \cdots & a_{1l}G_1 \\
a_{21}G_2 & a_{22}G_2 & \cdots & a_{2l}G_2 \\
\vdots & \vdots & \ddots & \vdots \\
a_{sl}G_s & a_{s2}G_s & \cdots & a_{sl}G_s
\end{pmatrix}
\]

Proof. Let \( c \in C = [C_1, C_2, \ldots, C_s]A \), the matrix-product code \( C \) by definition is the set of all matrix product \( c = (c_1, c_2, \ldots, c_s)A \) where \( c_i \in C_i \), then:

\[
c = (c_1, c_2, \ldots, c_s)A = (c_1, c_2, \ldots, c_s) \begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1l} \\
a_{21} & a_{22} & \cdots & a_{2l} \\
\vdots & \vdots & \ddots & \vdots \\
a_{s1} & a_{s2} & \cdots & a_{sl}
\end{pmatrix}
\]

And the codewords can be viewed as vectors of length \( ml \),

\[
c = 2 \sum_{j=1}^{s} a_{j1}c_j, \quad \sum_{j=1}^{s} \sum_{j=1}^{s} a_{j2}c_j, \cdots, \sum_{j=1}^{s} a_{jlc_j} \in \mathbb{F}_q^{ml}
\]  \hspace{1cm} (3.1)

or \( c_i \in C_i \), then \( c_1 = (u_1, u_2, \ldots, u_{k_1}), G_1 \), \( c_2 = (v_1, v_2, \ldots, v_{k_2}), G_2 \), \ldots, \( c_s = (h_1, h_2, \ldots, h_{k_s}), G_s \). If we replaced all \( c_i \) in (1) we have:

\[
c = \sum_{j=1}^{s} a_{j1}(u_1, u_2, \ldots, u_{k_j}), G_j, \quad \sum_{j=1}^{s} a_{j2}(u_1, u_2, \ldots, u_{k_j}), G_j, \cdots, \sum_{j=1}^{s} a_{jl}(u_1, u_2, \ldots, u_{k_j}), G_j
\]
We can write this expression in matrix form:

\[
c = (u_1, u_2, \ldots, u_{k_1}, v_1, v_2, \ldots, v_{k_2}, \ldots, h_1, h_2, \ldots, h_{k_s}).
\]

\[
\begin{pmatrix}
a_{11}G_1 & a_{12}G_1 & \cdots & a_{1l}G_1 \\
a_{21}G_2 & a_{22}G_2 & \cdots & a_{2l}G_2 \\
\vdots & \vdots & \ddots & \vdots \\
a_{s1}G_s & a_{s2}G_s & \cdots & a_{sl}G_s
\end{pmatrix}
\]

Therefore, the generator matrix \( G \) of the matrix product code \( C \)

\[
G = \begin{pmatrix} a_{11}G_1 & a_{12}G_1 & \cdots & a_{1l}G_1 \\ a_{21}G_2 & a_{22}G_2 & \cdots & a_{2l}G_2 \\ \vdots & \vdots & \ddots & \vdots \\ a_{s1}G_s & a_{s2}G_s & \cdots & a_{sl}G_s \end{pmatrix}
\]

\[\square\]

**Proposition 3.2.** Let \( C_i \) be a \([m, k_i, d_i]\) linear codes over \( \mathbb{F}_q \), \( i \in \{1, \ldots, s\} \) and \( A = (a_{ij}) \in M(\mathbb{F}_q, s \times l) \). Then the matrix product code \( C = [C_1, C_2, \ldots, C_s]A \) is a linear code over \( \mathbb{F}_q \) with length \( lm \) and dimension.

\[ k < k_1 + k_2 + \cdots + k_s \quad (3.2) \]

If the matrix \( A \) has full rank the bound (2) is achieved, so the dimension of the matrix product code \( C = [C_1, C_2, \ldots, C_s]A \) is

\[ k = k_1 + k_2 + \cdots + k_s \quad (3.3) \]

**Proof.** Let \( A = (a_{ij}) \in M(\mathbb{F}_q, s \times l) \) matrix with \( s \leq l \) which has full rank. Any codeword of \( C \) is of the form \( c = (c_1, c_2, \ldots, c_s)A \). Let us suppose that \( (c_1, c_2, \ldots, c_s) \neq [0, 0, \ldots, 0] \) for \( i = 1, 2, \ldots, s \) and \( c_i \in C_i \). Since \( A \) has full rank and \( s \leq l \), then rank of \( A \) is equal to \( s \), then \( c = (c_1, c_2, \ldots, c_s)A \neq [0, 0, \ldots, 0] \). Therefore \( \#C = (\#C_1)(\#C_2)\cdots(\#C_s) = q^{k_1+k_2+\cdots+k_s}. \)

\[\square\]

**Proposition 3.3.** Let \( R_i = (a_{i1}, a_{i2}, \cdots, a_{il}) \) denote the \( i \)-row of the matrix \( A \) for \( i = 1, 2, \ldots, s \) and denote by \( D_i \) the minimum distance of the code \( C_{R_i} \) code generated by \(< R_1, \ldots, R_i > \) in \( \mathbb{F}_q^l \). In [6] the following lower bound for the minimum distance of the matrix product code \( C \) is obtained

\[ d(C) = d = \min\{d_1D_1, d_2D_2, \cdots, d_sD_s\} \quad (3.4) \]

where \( d_i \) is the minimum distance of \( C_i \)

**Proof.** Demonstration of this proposition is given in [6]

\[\square\]

**Lemma 3.4.** If \( C_1, C_2, \ldots, C_s \) are nested codes \( C_1 \supset C_2 \supset \cdots \supset C_s \) the bound in (4) is sharp [3]

**Proof.** The proof of this lemma is given in [3]

\[\square\]
Example 3.5. If $C_1, C_2$ are two linear codes over $\mathbb{F}_2$, with generator matrices respectively $G_1, G_2$ and $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Then the matrix-product code

$$C = [C_1, C_2].A = \{ (c_1 | c_1 + c_2) : c_1 \in C_1, c_2 \in C_2 \}$$

and the generator matrix $G$ of the matrix product code $C$ is $G = \begin{pmatrix} G_1 & G_1 \\ 0 & G_2 \end{pmatrix}$

Example 3.6. Consider the linear codes $C_1, C_2, C_3$ of length 3 over $\mathbb{F}_3$, with generator matrices respectively:

$$G_1 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad G_2 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad G_3 = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$$

and $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

Then the matrix-product code $C = [C_1, C_2, C_3].A$ is given by its matrix generator

$$G = \begin{pmatrix} G_1 & G_1 & G_1 \\ 0 & 2G_2 & G_2 \\ 0 & 0 & G_3 \end{pmatrix}$$

The parameters of these codes are $[3, 3, 1], [3, 2, 2]$, and $[3, 1, 3]$, respectively, and the codes are nested, the matrix-product code $C = [C_1, C_2, C_3].A$ is a $[9, 6, 3]$ linear code.

As conclusion of this work, we aim to study the matrix-product code and give some Bound on the parameters of the matrix-product code where the codes $C_1, \cdots, C_s$ are linear and the matrix $A$ has full rank.

References


2. Y. Fan; S. Ling; H. Liu; *Matrix product codes over finite commutative Frobenius rings*, (to appear in Des Codes Cryptogr). DOI: 10.1007/s10623-012-9726-y


