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BÉZOUT-LIKE PROPERTIES IN AMALGAMATED ALGEBRA

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Abstract. In this paper, we investigate the transfer of the properties of "P-Bézout rings" and "2-Bézout rings" to the amalgamation of A and B along J with respect to f (denoted by $A \bowtie^f J$). Our aim is to provide necessary and sufficient conditions for $A \bowtie^f J$, to be a P-Bézout rings and 2-Bézout rings.

1. Introduction

All rings considered in this paper are commutative with identity elements, it is suitable to use "local" to refer to (not necessarily Noetherian) ring with a unique maximal ideal. A subring of a ring need not have the same unit.
Given a ring R, $nil(R)$ denotes the nil radical of R, that is, the set of all nilpotent elements of R. For an ideal J of R, $An(J)$ is denotes the annihilator of J.
R is called Bézout ring (resp., P-Bézout ring, resp., 2-Bézout ring) if every finitely generated ideal I of R is principal (resp., if every finitely generated prime ideal P of R is principal, resp., if every finitely presented ideal I of R is principal).
A Bézout ring is naturally a P-Bézout ring. The purpose of this paper is to study the transfer of the notions of "P-Bézout rings" and "2-Bézout rings" to the amalgamation of algebra along an ideal. Our aim is to provide new classes of commutative rings satisfying the above notions.

Let A and B be two rings with unity, let J be an ideal of B and let $f : A \to B$ be a ring homomorphism. In this setting, we can consider the following subring of $A \times B$:

$$A \bowtie^f J := \{(a, f(a) + j) \mid a \in A, j \in J\}$$

called *the amalgamation of A and B along J with respect to f*. In particular, they have studied amalgamations in the frame of pullbacks which allowed them to establish numerous (prime) ideal and ring-theoretic basic properties for this new construction. This construction is a generalization of *the amalgamated duplication of a ring along an ideal* (introduced and studied by D'Anna and Fontana in [7, 9, 10]). The interest of amalgamation resides, partly, in its ability to cover several basic constructions in commutative algebra, including pullbacks and trivial ring extensions (also called Nagata's idealizations)([14, page 2]) Moreover,

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other classical constructions (such as the $A + XB[X]$, $A + XB[[X]]$, and the $D + M$ constructions) can be studied as particular cases of the amalgamation ([8, Examples 2.5 and 2.6]) and other classical constructions, such as the CPI extensions (in the sense of Boisen and Sheldon [5]) are strictly related to it ([8, Example 2.7 and Remark 2.8]). In [8], the authors studied the basic properties of this construction (e.g., characterizations for $A \bowtie^f J$ to be a Noetherian ring, an integral domain, a reduced ring) and they characterized those distinguished pullbacks that can be expressed as an amalgamation. Moreover, in [10], they pursued the investigation on the structure of the rings of the form $A \bowtie^f J$, with particular attention to the prime spectrum, to the chain properties and to the Krull dimension.

2. Main results

Before we announce the main result of this section, we make the following useful remark.
Let $f : A \longrightarrow B$ be a ring homomorphism, J be an ideal of B and let n be a positive integer. Consider the function $f^n : A^n \longrightarrow B^n$ defined by $f^n((\alpha)_{i=1}^{i=n}) = (f(\alpha)_i)_{i=1}^{i=n}$. Obviously, f^n is a ring homomorphism and J^n is an ideal of B^n. This allows us to define $A^n \bowtie^{f^n} J^n$.
Moreover, let $\phi : (A \bowtie^f J)^n \longrightarrow A^n \bowtie^{f^n} J^n$ defined by
$\phi((a_i, f(a_i) + j_i)_{i=1}^{i=n}) = ((a_i)_{i=1}^{i=n}, f^n((a_i)_{i=1}^{i=n}) + (j_i)_{i=1}^{i=n})$ It is easily checked that ϕ is a ring isomorphism. So $(A \bowtie^f J)^n$ and $A^n \bowtie^{f^n} J^n$ are isomorphic as rings.
Let U be a submodule of A^n. Then $U \bowtie^{f^n} J^n = \{(u, f^n(u)+j) \in A^n \bowtie^{f^n} J^n / u \in U, j \in J^n\}$ is a submodule of $A^n \bowtie^{f^n} J^n$.

Now we study the transfer of the property of P-Bézout ring to the amalgamation algebra.

Theorem 2.1. *Let (A, B) be a pair of rings, $f : A \to B$ be a ring homomorphism and J be a proper ideal of B, then:*

1. *Assume that J is finitely generated ideal of $f(A) + J$, then:*
   If $A \bowtie^f J$ is a P-Bézout ring then so is A.
2. *Assume that $f^{-1}(J)$ is finitely generated ideal of A, then:*
   If $A \bowtie^f J$ is a P-Bézout ring then so is $f(A) + J$.
3. *Assume that J is finitely generated ideal of $f(A) + J$, $J \subset nil(B)$, and $\forall t \in A$ $f(t)J = J$ then:*
   $A \bowtie^f J$ is a P-Bézout ring if and only if so is A.
4. *Assume that J is not a finitely generated ideal of $f(A) + J$, $J \subset nil(B)$, and $\forall P \in Spec(A)$, $f(P) \subset J$ then:*
   $A \bowtie^f J$ is a P-Bézout ring.
5. *Assume that $f^{-1}(J) = 0$, then:*
   $A \bowtie^f J$ is P-Bézout ring if and only if so is $f(A) + J$.
6. *Assume that (A, M) is a local ring, $f(M) \subset An(J)$, $J \subset nil(B)$ and J is a finitely generated ideal of $f(A) + J$ then:*
   $A \bowtie^f J$ is a P-Bézout ring if and only if so is A.

Proof. (1) Assume that J is finitely generated ideal of $f(A)+J$.

- 1^{st} method:
  Let P be a finitely generated prime ideal of A, then $P\bowtie^f J$ is a finitely generated prime ideal of $A\bowtie^f J$ Since J is finitely generated ideal of $f(A)+J$. Hence $P\bowtie^f J=(A\bowtie^f J)(a,f(a)+j)$ since $A\bowtie^f J$ is P-Bézout ring, one may easily check that $P=Aa$, and hence A is P-Bézout ring.
- 2^{nd} method:
  J is finitely generated ideal of $f(A)+J$ implies that $\{0\}\times J$ is a finitely generated ideal of $A\bowtie^f J$ by [3. lemma 2.3 (1)], and since $\frac{A\bowtie^f J}{\{0\}\times J}\cong A$ then by [5.proposition 2.5] we have A is P-Bézout ring.

(2) Assume that $f^{-1}(J)$ is finitely generated ideal of A.

- 1^{st} method:
  Let Q be a finitely generated prime ideal of B, then $\overline{Q}^f=\{(a,f(a)+j)/f(a)+j\in Q\}$ is a finitely generated prime ideal of $A\bowtie^f J$. Hence $\overline{Q}^f=(A\bowtie^f J)(\alpha,f(\alpha)+j)$ since $A\bowtie^f J$ is P-Bézout ring, therefore $Q=(f(A)+J)(f(\alpha)+j)$ which implies that $f(A)+J$ is P-Bézout.
- 2^{nd} method:
  $f^{-1}(J)$ is finitely generated ideal of A implies that $f^{-1}(J)\times\{0\}$ is a finitely generated ideal of $A\bowtie^f J$ by [3.lemma 2.3 (1)], and since $\frac{A\bowtie^f J}{f^{-1}(J)\times\{0\}}\cong f(A)+J$ then by [5.proposition 2.5] we have $f(A)+J$ is P-Bézout ring.

(3) Assume that J is a finitely generated ideal of $f(A)+J$, $J\subset nil(B)$ and $\forall t\in A\ f(t)J=J$.
If $A\bowtie^f J$ is P-Bézout ring then so is A by 1). Conversely, since $J\subset nil(B)$ then $Spec(A\bowtie^f J)=\{P\bowtie^f J/P\in Spec(A)\}$. Let $P\bowtie^f J$ be a finitely generated prime ideal of $A\bowtie^f J$, then P is a finitely generated prime ideal of A, hence $P=A\alpha$ for some α in P. So we show that $P\bowtie^f J=(A\bowtie^f J)(\alpha,f(\alpha)$
Let $(x,f(x)+j)\in P\bowtie^f J$ then

$$(x,f(x)+j)=(a\alpha,f(a)f(\alpha)+f(\alpha)j')=(\alpha,f(\alpha))(a,f(a)+j')$$

for some $j'\in J$ and hence $P\bowtie^f J=(A\bowtie^f J)(\alpha,f(\alpha)$.

(4) Assume that J is not a finitely generated ideal of $f(A)+J$, $J\subset nil(B)$, and $\forall P\in Spec(A)$, $f(P)\subset J$ then:
Clearly, $Spec(A\bowtie^f J)=\{P\bowtie^f J/P\in Spec(A)\}$
Our aim is to show that there exists non proper finitely generated prime ideal of $A\bowtie^f J$. Deny. Let $P\bowtie^f J=\sum_{i=1}^{i=n}(A\bowtie^f J)(a_i,f(a_i)+j_i)$ a finitely generated ideal of $A\bowtie^f J$, where $(a_i,f(a_i)+j_i)\in P\bowtie^f J$ for each $i=1,....,n$. It is easily seen that $J=\sum_{i=1}^{i=n}(f(A)+J)(f(a_i)+j_i)$ since $f(P)\subset J$, which is a contradiction since J is not a finitely generated ideal of $f(A)+J$.

(5) Suppose that $f^{-1}(J) = 0$,
Since $\frac{A\bowtie^f J}{f^{-1}(J)\times(0)} \simeq f(A) + J$, then $A \bowtie^f J \simeq f(A) + J$. The conclusion is straightforward.

(6) Assume that (A, M) is a local ring, $f(M) \subset An(J)$, $J \subset nil(B)$ and J is a finitely generated ideal of $f(A) + J$ then:
If $A \bowtie^f J$ is a P-Bézout ring then so is A by 1). Conversely, since $J \subset nil(B)$ then $Spec(A \bowtie^f J) = \{P \bowtie^f J/P \in Spec(A)\}$.
Let $P \bowtie^f J$ a finitely generated prime ideal of $A \bowtie^f J$, hence P is a finitely generated prime ideal of A. Therefore $P = A\alpha$ where $\alpha \in P$ since A is a P-Bézout ring. One may easily check that $P \bowtie^f J = (A \bowtie^f J)(\alpha, f(\alpha))$ and this complete the proof.

□

Now we study the transfer of the property of 2-Bézout ring to the amalgamation algebra.

Theorem 2.2. *Let (A, B) be a pair of rings, $f : A \to B$ be a ring homomorphism and J be a proper ideal of B, then:*

(1) *Assume that (A, M) is local and $f(M) \subset An(J)$, then:*
- *If $A \bowtie^f J$ is 2-Bézout ring then so is A.*
- *$A \bowtie^f J$ is 2-Bézout ring if and only if so is A provided $J^2 = 0$.*

(2) *Assume that (B, J) is local and J is a finitely generated ideal of $f(A)+J$, then:*
- *If $A \bowtie^f J$ is 2-Bézout ring then so is A.*
- *$A \bowtie^f J$ is 2-Bézout ring if and only if so is A provided $J^2 = 0$.*

(3) *Assume that $f^{-1}(J) = 0$, then:*
$A \bowtie^f J$ is 2-Bézout ring if and only if so is $f(A) + J$.

(4) *Assume that (A, M) is local, $f(M) \subset J$ and $J^2 = 0$, then $A \bowtie^f J$ is 2-Bézout ring.*

Proof. (1) Assume that (A, M) is local and $f(M) \subset An(J)$:

- If $A \bowtie^f J$ is 2-Bézout ring, we show that A is 2-Bézout ring. Let $I = \sum_{i=1}^{i=n} Ab_i$ a finitely presented ideal of A. Consider the exact sequence of A-module:

$$0 \longrightarrow \ker v \longrightarrow A^n \longrightarrow I \longrightarrow 0$$

Where $v((a_i)_{i=1,\dots,n}) = \sum_{i=1}^{i=n} a_i b_i$. We show that I is principal.
Let $K = \sum_{i=1}^{i=n}(A \bowtie^f J)(b_i, f(b_i))$, consider the exact sequence of $(A \bowtie^f J)$-module:

$$0 \longrightarrow \ker u \longrightarrow (A \bowtie^f J)^n \longrightarrow K \longrightarrow 0$$

Where $u((a_i, f(a_i) + j_i)_{i=1,\dots,n}) = \sum_{i=1}^{i=n}(a_i, f(a_i + j_i)(b_i, f(b_i)$, it is easily seen that $keru = kerv \bowtie^{f^n} J^n$. Since I is a finitely presented then $kerv$ is finitely generated submodule of A^n. Or J is finitely

generated ideal then $keru = kerv \bowtie^{f^n} J^n$ is a finitely generated submodule of $A^n \bowtie^{f^n} J^n$, hence K is a finitely presented ideal of $A \bowtie^f J$, therefore $K = (A \bowtie^f J)(\alpha, f(\alpha) + t)$ since $A \bowtie^f J$ is a 2-Bézout ring.
$(\alpha, f(\alpha)+t) \in K = \sum_{i=1}^{i=n}(A \bowtie^f J)(b_i, f(b_i))$ implies that $\alpha \in I$ then $A\alpha \subset I$.
On the other hand, let $x \in I = \sum_{i=1}^{i=n} Ab_i$ then:

$$(x, f(x)) = \sum_{i=1}^{i=n}(a_i, f(a_i)(b_i, f(b_i)) \in K = (a, f(a) + t)(\alpha, f(\alpha) + t)$$

Which implies that $x = a\alpha$ hence $I \subset A\alpha$ and this complete the proof.

- If $A \bowtie^f J$ is 2-Bézout ring then so is A by 1).
Conversely, assume that A is 2-Bézout ring and $J^2 = 0$, we show that $A \bowtie^f J$ is 2-Bézout ring.
Let $K = \sum_{i=1}^{i=n}(A \bowtie^f J)(b_i, f(b_i) + k_i)$ a finitely presented ideal of $A \bowtie^f J$, consider the exact sequence

$$0 \longrightarrow \ker u \longrightarrow (A \bowtie^f J)^n \longrightarrow K \longrightarrow 0$$

Since K is a finitely presented ideal of $A \bowtie^f J$ then $keru$ is a finitely submodule of $(A \bowtie^f J)^n$. Let $I = \sum_{i=1}^{i=n} Ab_i$ and the exact sequence

$$0 \longrightarrow \ker v \longrightarrow A^n \longrightarrow I \longrightarrow 0$$

Its easily seen that $keru = kerv \bowtie^{f^n} J^n$ that is a finitely generated submodule, then $kerv$ is a finitely generated submodule of A^n and hence I is a finitely presented ideal of A. Therefore $I = A\alpha$ since A is 2-Bézout ring. One may easily check that $K = (A \bowtie^f J)(\alpha; f(\alpha))$ and this completes the proof.

(2) Assume that (B, J) is a local ring and J is a finitely generated ideal of $f(A) + J$:

- Assume that $A \bowtie^f J$ is a 2-Bézout ring, we show that A is a 2-Bézout ring.
Let $I = \sum_{i=1}^{i=n} Ab_i$ a finitely presented ideal of A. Consider the exact sequence of A-module:

$$0 \longrightarrow \ker v \longrightarrow A^n \longrightarrow I \longrightarrow 0$$

Where $v((a_i)_{i=1,\dots,n}) = \sum_{i=1}^{i=n} a_i b_i$. We show that I is principal.
Since $f(I) \subset J$ then $I \times \{0\}$ is an ideal of $A \bowtie^f J$. Consider the exact sequence of $(A \bowtie^f J)$-module:

$$0 \longrightarrow \ker u \longrightarrow (A \bowtie^f J)^n \longrightarrow I \times \{0\} \longrightarrow 0$$

Where $u((a_i, f(a_i) + j_i)_{i=1,\dots,n}) = \sum_{i=1}^{i=n}(a_i, f(a_i + j_i)(b_i, 0)$.
It is easily seen that $keru = kerv \bowtie^{f^n} J^n$. Since I is a finitely presented ideal of A then $kerv$ is a finitely generated submodule of A^n, therefore $keru$ is a finitely generated submodule of $(A \bowtie^f J)^n$ since

J is a finitely generated ideal. Hence $I \times \{0\}$ is a finitely presented ideal of $A \bowtie^f J$, then $I \times \{0\} = (A \bowtie^f J)(\alpha, 0)$ which implies that $I = A\alpha$ and this completes the proof.

- If $A \bowtie^f J$ is 2-Bézout ring then so is A by 1).
  Conversely, assume that A is 2-Bézout ring and $J^2 = 0$, we show that $A \bowtie^f J$ is 2-Bézout ring.
  Let $K = \sum_{i=1}^{i=n}(A \bowtie^f J)(b_i, f(b_i) + k_i)$ a finitely presented ideal of $A \bowtie^f J$, consider the exact sequence

$$0 \longrightarrow \ker u \longrightarrow (A \bowtie^f J)^n \longrightarrow K \longrightarrow 0$$

  Since K is a finitely presented ideal of $A \bowtie^f J$ then $keru$ is a finitely submodule of $(A \bowtie^f J)^n$. Let $I = \sum_{i=1}^{i=n} Ab_i$ and the exact sequence

$$0 \longrightarrow \ker v \longrightarrow A^n \longrightarrow I \longrightarrow 0$$

  It is easily seen that $keru = kerv \bowtie^{f^n} J^n$ that is a finitely generated submodule, then $kerv$ is a finitely generated submodule of A^n and hence I is a finitely presented ideal of A. Therefore $I = A\alpha$ since A is 2-Bézout ring. One may easily check that $K = (A \bowtie^f J)(\alpha; f(\alpha))$ and this completes the proof.

(3) Suppose that $f^{-1}(J) = 0$,
Since $\frac{A \bowtie^f J}{f^{-1}(J) \times (0)} \simeq f(A) + J$, then $A \bowtie^f J \simeq f(A) + J$. The conclusion is straightforward.

(4) Assume that (A, M) is local, $f(M) \subset J$ and $J^2 = 0$.
Our aim is to show that there exists no finitely presented ideal of $A \bowtie^f J$.
Let $K = \sum_{i=1}^{i=n} A \bowtie^f J(b_i, f(b_i) + k_i)$ a finitely presented ideal of $A \bowtie^f J$.
Consider the exact sequence

$$0 \longrightarrow \ker u \longrightarrow (A \bowtie^f J)^n \longrightarrow K \longrightarrow 0$$

Where $u((a_i, f(a_i) + j_i)_{i=1,\dots,n}) = \sum_{i=1}^{i=n}(a_i, f(a_i) + j_i)(b_i, f(b_i) + k_i)$.
It is easily seen that $\ker u = v \bowtie^{f^n} J^n$ where $v = \{(a_i)_{i=1,2,\dots n} / \sum_{i=1}^{i=n} a_i b_i = 0\}$. On the other hand $\ker u \subset M^n \bowtie^{f^n} J^n$ ($A \bowtie^f J, M \bowtie^f J$ is a local ring) and $\ker u$ is a finitely generated ideal of $A \bowtie^f J$. Let $\{(y_i, f^n(y_i) + t_i)_{i=1,2,\dots,m}\}$ be a minimal generating set of $\ker u$ and let $(j_i)_{i=1,2,\dots,n} \in J^n$. Since $(0, (j_i)_{i=1,2,\dots,n}) \in \ker u$, then

$$(0, (j_i)_{i=1,2,\dots,n}) = \sum_{i=1}^{i=m}(a_i, f(a_i) + l_i)(y_i, f^n(y_i) + t_i) = (\sum_{i=1}^{i=m} a_i y_i, f(\sum_{i=1}^{i=m} a_i y_i)) = 0$$

which implies that $(j_i)_{i=1,2,\dots,n} = 0$ hence $J = 0$, which is a contradiction. Consequently $A \bowtie^f J$ is 2-Bézout ring.

□

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