

NONLOCAL BOUNDARY VALUE PROBLEMS WITH TRANSMISSION CONDITIONS

MUSTAFA KANDEMİR

ABSTRACT. We consider nonlocal boundary value problems which includes elliptic differential operator equations of the second order with discontinuous coefficients and nonlocal boundary conditions together with transmission conditions. We establish Fredholmness for these nonlocal boundary value problems.

1. INTRODUCTION

In classical theory of boundary-value problems for ordinary differential equations is usually considered for equations with continuous coefficients and for boundary conditions which contain only end-points of the considered interval. However, this paper deals with one non-classical boundary-value problem for ordinary differential equation with discontinuous coefficients and boundary conditions containing not only end-points of the considered interval, but also a point of discontinuity and internal points. Boundary conditions are called nonlocal conditions if they contain interior points of the considered interval. Local and nonlocal problems for discontinuous coefficients equations connected with different applied problems which include various transfer problems such as heat transfer in heterogeneous media. Naturally, transmission problems arise in various fields. For instance, these problems arise in problems of physical field as diffraction problems, elasticity problems, heat and mass transfer problems (see, e.g., A. A. Shkalikov [1] and references cited therein).

The systematic investigation of nonlocal problems was studied by A. Bitsadze and A. Samarskii in [4]. Note that, in the series of A. L. Skubachevskii's works, an abstract theory of the nonlocal boundary value problems has been constructed (see, for example, [2], [3]). Such type problems, also, can be found in investigations by P. L. Gurevich [14], M. S. Agronovich [10] and V. V. Podyapolskii [20].

Spectral properties and coercive solvability of boundary value problems in Sobolev spaces can be found in some works of V. B. Shakhmurov and B. A. Aliev (for example, [19], [5]). Some boundary-value problems for differential equations

Date: Received: Aug 21, 2014; Accepted: Oct 20, 2014.

* Corresponding author.

2010 *Mathematics Subject Classification.* 34A36, 34B05, 34B08, 34B10.

Key words and phrases. Boundary value problem, elliptic differential operator, transmission conditions, nonlocal problems, discontinuous coefficients.

with discontinuous coefficients were investigated by M. L. Rasulov in monographs (see, [9]). Note that, an abstract theory of the boundary value problems with continuous coefficients and an eigenvalue parameters in the boundary conditions have been constructed in the investigations by S. Yakubov and Ya. Yakubov (see, [17] and corresponding bibliography).

In recent years, boundary value problems with discontinuous coefficients and transmission conditions have been investigated by O. Sh. Mukhtarov and some others (see, for example [7], [8], [12], [13]).

2. PRELIMINARIES

In this section, we give some definitions, theorems and auxiliary results which are used through the paper.

Theorem 2.1. ([2], Theorem A 1) *Let H_1, H_2 be Hilbert spaces and $A : H_1 \rightarrow H_2$ be a closed linear operators with domain $D(A)$. Let $B : H_2 \rightarrow H_1$ be a bounded mapping H_2 onto $R(B)$ in a one-to-one manner, and $D(A) \subset R(B)$. Suppose that the operator $AB : H_2 \rightarrow H_2$ with domain $D(AB) = \{x \in H_2 : Bx \in D(A)\}$ is Fredholm. Then the operator $A : H_1 \rightarrow H_2$ is Fredholm, and $\text{ind} A = \text{ind} AB$.*

Theorem 2.2. (Rellich) *Let $Q \subset \mathbb{R}^n$ be a bounded domain, and $k > s \geq 0$. Then the inclusion map $W^k(Q) \rightarrow W^s(Q)$ is compact operator.*

Lemma 2.3. ([2], Lemma 1.2) *Let H_1, H_2 be Hilbert spaces, and let $L, L_0 : H_1 \rightarrow H_2$ be linear bounded operators. Assume that an operator L_0 has a bounded inverse $L_0^{-1} : H_2 \rightarrow H_1$ and that, for any $0 \leq \tau \leq 1$,*

$$C_1 \|L_\tau u\|_{H_2} \leq \|u\|_{H_1} \leq C_2 \|L_\tau u\|_{H_2}, \quad u \in H_1, \quad (2.1)$$

where $L_\tau = L_0 + \tau(L - L_0)$, $C_1, C_2 > 0$ do not depend on τ and u . Then the operator L has bounded a inverse $L^{-1} : H_2 \rightarrow H_1$.

Definition 2.4. (Direct sum) The Banach space $W_p^m(-1, 0) + W_p^m(0, 1)$, an integer $m \geq 0$, a real $p > 1$, is defined as

$$\begin{aligned} W_p^m(-1, 0) + W_p^m(0, 1) &= \left\{ u = \begin{cases} u_1(x), & \text{for } x \in (-1, 0) \\ u_2(x), & \text{for } x \in (0, 1) \end{cases}, \right. \\ &\quad u_1 \in W_p^m(-1, 0), \quad u_2 \in W_p^m(0, 1), \\ &\quad \left. \|u\| = \|u_1\|_{W_p^m(-1, 0)} + \|u_2\|_{W_p^m(0, 1)} \right\}. \end{aligned}$$

For convenience below, the direct sum $W_p^m(-1, 0) + W_p^m(0, 1)$ is denoted by $W_p^m(-1, 0, 1)$. Obviously, $W_p^0(-1, 0, 1) := L_p(-1, 0, 1) = L_p(-1, 1)$.

Definition 2.5. Let H_1 and H_2 be Hilbert spaces. The set

$$H_1 \oplus H_2 := \{(u, v) \mid u \in H_1, v \in H_2\}$$

is a Hilbert space according to norm $\|(u, v)\|_{H_1 \oplus H_2} := (\|u\|_{H_1}^2 + \|v\|_{H_2}^2)^{\frac{1}{2}}$ and which is called the orthogonal sum of H_1 and H_2 Hilbert spaces.

Let A_k be closed an operator in Hilbert space H . The domain of definition $D(A_k)$ of the operator A_k is

turned into a Hilbert space with respect to the norm

$$\|u_k\|_{H(A_k)} := (\|u_k\|^2 + \|A_k u_k\|^2)^{\frac{1}{2}},$$

where, $A_k = A_1$, $u_k = u_1$ for $x \in [-1, 0)$ and $A_k = A_2$, $u_k = u_2$ for $x \in (0, 1]$.

$W_p^m((-1, 0, 1); H)$ denotes a Banach space of functions $u(x)$ with values from H which have generalized derivatives up to the m -th order inclusive on $(-1, 0) \cup (0, 1)$ and the norm

$$\|u\|_{W_p^m((-1, 0, 1); H)} := \sum_{k=0}^m \left[\left(\int_{-1}^0 \|u_1^{(k)}(x)\|^p dx \right)^{\frac{1}{p}} + \left(\int_0^1 \|u_2^{(k)}(x)\|^p dx \right)^{\frac{1}{p}} \right]$$

is finite, where $p \geq 1$, m -integer, $u = u_1 + u_2$.

$W_p^2((-1, 0, 1); H(A_k), H) := L_p((-1, 1); H(A_k)) \cap W_p^2((-1, 0, 1); H)$ is a Banach space with the norm

$$\|u_k\|_{W_p^2((-1, 0, 1); H(A_k), H)} := \|A_k u_k\|_{L_p((-1, 1); H)} + \|u_k''\|_{L_p((-1, 1); H)}, \quad k = 1, 2.$$

Theorem 2.6. ([6], Theorem 1.15.2) Let A be a positive operator in a Banach space E .

1) If $\alpha > 0$, then A^α is an isomorphism from $D(A^\alpha)$ onto E , from $D(A^{\alpha+\mu})$ onto $D(A^\mu)$ and from $(E, D(A^m))_{\frac{\alpha+\mu}{m}, p}$ onto $(E, D(A^m))_{\frac{\mu}{m}, p}$, where $\mu > 0$, $1 \leq p \leq \infty$, and $\alpha + \mu < m$, $m = 1, 2, \dots$

2) If $0 < \alpha < \beta < \infty$, $1 \leq p \leq \infty$ and $0 < \theta < 1$ then

$$(E, D(A^m))_{\theta, p} = (E, D(A^\beta))_{\frac{\alpha}{m}\theta, p}.$$

Lemma 2.7. ([17], Lemma 1.2.7/2) Let the following be satisfied:

- 1) a Banach space E is compactly embedded into a Banach space G ;
- 2) an operator B from E into a Banach space F is bounded and for any $\varepsilon > 0$

$$\|Bu\|_F \leq \varepsilon \|u\|_E + C(\varepsilon) \|u\|_G, \quad u \in E.$$

Then, The operator B from E into F is compact.

Theorem 2.8. ([6], Theorem 1.9.3./f) There exist a positive number c_θ , $0 < \theta < 1$, such that for all $a \in A_0 \cap A_1$

$$\|a\|_{[A_0, A_1]_\theta} \leq c_\theta \|a\|_{A_0}^{1-\theta} \|a\|_{A_1}^\theta.$$

3. STATEMENT OF THE PROBLEM WITH DISCONTINUOUS COEFFICIENTS AND TRANSMISSION CONDITIONS

$$L(D)u := -p(x)u''(x) + Au(x) - \lambda u(x) = f(x), \quad x \in [-1, 0) \cup (0, 1] \quad (3.1)$$

$$\begin{aligned} L_k u &= \alpha_k u(-1) + \beta_k u(-0) + \delta_k u(+0) + \gamma_k u(1) \\ &+ (B_{k1} u)(-1) + (B_{k2} u)(-0) + (C_{k1} u)(+0) \\ &+ (C_{k2} u)(1) + N_k u(x_k) = g_k, \quad k = 1, 2, 3, 4 \end{aligned} \quad (3.2)$$

where $\alpha_k, \beta_k, \delta_k, \gamma_k, g_k$ are complex numbers; $p(x) := p_1, A := A_1, f(x) := f_1(x)$ for $x \in [-1, 0)$; $p(x) := p_2, A := A_2, f(x) := f_2(x)$ for $x \in (0, 1]$; p_1 and p_2 are real valued functions, $f \in L_2(-1, 1)$ is a complex valued function and x_k are any inner points.

A, B_{kv}, C_{kv}, N_k ($k = 1, 2, 3, 4; v = 1, 2$) satisfies the following conditions:

Condition 1:

a) The operators A_1 and A_2 are properly elliptic for all $x \in [-1, 0)$ and $x \in (0, 1]$, respectively.

b) $A : W^1(-1, 0, 1) \rightarrow L_2(-1, 1), B_{kv} : L_2(-1, 1) \rightarrow L_2(-1, 1), C_{kv} : L_2(-1, 1) \rightarrow L_2(-1, 1), N_k : L_2(-1, 1) \rightarrow \mathbb{C}$ are linear bounded operators and there exists $\sigma > 0$ such that for any $u_v \in W^m(-1, 0, 1), (u = u_1 + u_2; m = 0, 2)$

$$\begin{aligned} \|B_{kv}u_v\|_{W^m(-1,0,1)} &\leq \varepsilon \|u_v\|_{W^m(-1+\sigma,-\sigma)+W^m(\sigma,1-\sigma)}, \\ \|C_{kv}u_v\|_{W^m(-1,0,1)} &\leq \varepsilon \|u_v\|_{W^m(-1+\sigma,-\sigma)+W^m(\sigma,1-\sigma)}, \\ k &= 1, 2, 3, 4, v = 1, 2. \end{aligned} \quad (3.3)$$

Condition 2:

a) The embedding $E \subset F$ is compact for Banach spaces E and F ,

b) The following inequalities be satisfied for any $\varepsilon > 0$ and for almost all $x_v \in [0, 1]$

$$\begin{aligned} \|(B_{kv}u_v)(x_v)\|_F &\leq \varepsilon \|u_v(x_v)\|_E + C(\varepsilon) \|u_v(x_v)\|_F, \\ \|(C_{kv}u_v)(x_v)\|_E &\leq \varepsilon \|u_v(x_v)\|_E + C(\varepsilon) \|u_v(x_v)\|_F, \\ k &= 1, 2, 3, 4, v = 1, 2. \end{aligned}$$

Denote:

1) $G_{\varepsilon,l} := \{\lambda \in \mathbb{C} \mid |\arg \lambda| \geq \varepsilon, |\lambda| \geq l\}$.

2) $\phi_k u := \alpha_k u(-1) + \beta_k u(-0) + \delta_k u(+0) + \gamma_k u(1)$.

3) $L_0(D)u := -p(x)u''(x)$.

In the Hilbert spaces $W^2(-1, 0, 1)$ and $L_2(-1, 1) \times \mathbb{C}^4$ we introduce the following equivalent norms depending on λ :

$$\begin{aligned} \sum_{j=1}^2 \|u_j\|_{W^m(-1,0,1)} &\leq \left(\sum_{j=1}^2 (\|u_j\|_{W^m(-1,0,1)}^2 + |\lambda|^2 \|u_j\|_{L_2(-1,1)}^2) \right)^{\frac{1}{2}}, \\ \sum_{j=1}^2 \|h\|_{L_2(-1,1) \times \mathbb{C}^4} &\leq \left(\sum_{j=1}^2 (\|f_j\|_{L_2(-1,1)}^2 + |\lambda|^{\frac{3}{2}} \sum_{j=1}^4 |g_j|^2) \right)^{\frac{1}{2}}, \end{aligned}$$

where $h := (f_v, g_1, g_2, g_3, g_4), v = 1, 2$.

4. SOLVABILITY AND SPECTRUM OF THE BOUNDARY VALUE PROBLEM

We consider the bounded linear operators $\mathbb{L}, \mathbb{L}_0, \tilde{\mathbb{L}} : W^2(-1, 0, 1) \rightarrow L_2(-1, 1) \times \mathbb{C}^4$ by the formulas

$$\mathbb{L}u := (L(D)u, L_1, L_2, L_3, L_4)$$

$$\mathbb{L}_0 u := (L_0(D)u - \lambda u, \phi_1, \phi_2, \phi_3, \phi_4),$$

$$\tilde{\mathbb{L}}u := \mathbb{L}_0 u + t(\mathbb{L} - \mathbb{L}_0)u, 0 \leq t \leq 1.$$

Lemma 4.1. *We assume that the condition 1.*

Then, for every $\varepsilon > 0$, there exist $l > 1$ such that, for $\lambda \in G_{\varepsilon,l}$ and $0 \leq t \leq 1$, we have that, for $u \in W^2(-1, 0, 1)$ the following a priori estimate is hold

$$c_1 \sum_{j=1}^2 \left\| \tilde{\mathbb{L}}u_j \right\|_{L_2(-1,1) \times \mathbb{C}^4} \leq \sum_{j=1}^2 \|u_j\|_{W^2(-1,0,1)} \leq c_2 \sum_{j=1}^2 \left\| \tilde{\mathbb{L}}u_j \right\|_{L_2(-1,1) \times \mathbb{C}^4} \quad (4.1)$$

where constants c_1, c_2 do not depend on λ , t , and u .

Proof. We consider $\tilde{\mathbb{L}}u = h$ and the operator

$$\mathbb{L}_0 u = h + Tu, \quad (4.2)$$

where

$$\begin{aligned} Tu : &= (T_1 u, T_2 u, T_3 u, T_4 u, T_5 u) = (-tAu, \\ &-t[(B_{11}u)(-1) + (B_{12}u)(-0) + C_{11}u(+0) + C_{12}u(1) + N_1u], \\ &-t[(B_{21}u)(-1) + (B_{22}u)(-0) + C_{21}u(+0) + C_{22}u(1) + N_2u], \\ &-t[(B_{31}u)(-1) + (B_{32}u)(-0) + C_{31}u(+0) + C_{32}u(1) + N_3u], \\ &-t[(B_{41}u)(-1) + (B_{42}u)(-0) + C_{41}u(+0) + C_{42}u(1) + N_4u]), \end{aligned}$$

$h = (f_v, g_1, g_2, g_3, g_4)$, $v = 1, 2$. Now, first, from ([6], Theorem 2, formula 2.4.2./16) we have the equality

$$W_p^s(-1, 0, 1) = (L_p(-1, 1), W_p^k(-1, 0, 1))_{\frac{s}{k}, p},$$

where, $0 \leq s \leq k$, $1 < p < \infty$. Hence, from Theorem 2.8 we obtain

$$\|u\|_{W^k(-1,0,1)} \leq C \|u\|_{L_2(-1,1)}^{1-\frac{s}{k}} \|u\|_{W^k(-1,0,1)}^{\frac{s}{k}}.$$

Using the Young inequality, we have

$$\begin{aligned} |\lambda|^{k-s} \|u\|_{W^s(-1,0,1)} &\leq C(|\lambda|^k \|u\|_{L_2(-1,1)})^{1-\frac{s}{k}} \|u\|_{W^k(-1,0,1)}^{\frac{s}{k}} \\ &\leq C(|\lambda|^k \|u\|_{L_2(-1,1)} + \|u\|_{W^k(-1,0,1)}), \quad \lambda \in \mathbb{C}. \end{aligned} \quad (4.3)$$

Further, from ([10], ch.1, sec. 1) we obtain the inequality

$$|\lambda|^{\frac{1}{2}} \|u\|_{L_2(-1,1)} \leq C(|\lambda| \|u\|_{L_2(-1,1)} + \|u\|_{W^1(-1,0,1)}), \quad \lambda \in \mathbb{C}. \quad (4.4)$$

In view of ([2], Theorem C.7) there exist $l_0 = l_0(\varepsilon) > 1$ such that for $\lambda \in G_{\varepsilon,l}$ we have the following inequality for the solution of the problem (4.2)

$$\sum_{j=1}^2 \|u_j\|_{W^2(-1,0,1)} \leq c_1 \|h + Tu\|_{L_2(-1,1) \times \mathbb{C}^4}. \quad (4.5)$$

In view of condition 1 and (4.3), we have the inequalities

$$\sum_{j=1}^2 \|A_j u_j\|_{L_2(-1,1)} \leq c_2 |\lambda|^{-\frac{1}{2}} \sum_{j=1}^2 \|u_j\|_{W^2(-1,0,1)}, \quad (4.6)$$

and

$$\begin{aligned}
\sum_{j=1}^2 |\lambda|^{\frac{3}{4}} |t C_k u_j| &\leq c_3 |\lambda|^{\frac{3}{4}} \sum_{j=1}^2 \|u_j\|_{L_2(-1,1)} \\
&\leq c_3 |\lambda|^{-\frac{1}{4}} \sum_{j=1}^2 \|u_j\|_{W^2(-1,0,1)}, \quad k = 1, 2, 3, 4.
\end{aligned} \tag{4.7}$$

From inequalities (3.4), (4.3) and (4.4) it follows that

$$\begin{aligned}
&\sum_{j=1}^2 |\lambda|^{\frac{3}{4}} |t [B_{k1} u_j(-1) + C_{k1} u_j(+0)]| \\
&\leq c_4 |\lambda|^{\frac{1}{2}} \left(\sum_{j=1}^2 \|[B_{k1} u_j(-1) + C_{k1} u_j(+0)]\|_{W^1(-1,0,1)} \right. \\
&\quad \left. + |\lambda|^{\frac{1}{2}} \sum_{j=1}^2 \|[B_{k1} u_j(-1) + C_{k1} u_j(+0)]\|_{L_2(-1,1)} \right) \\
&\leq c_5 \left(\sum_{j=1}^2 \|[B_{k1} u_j(-1) + C_{k1} u_j(+0)]\|_{W^2(-1,0,1)} \right. \\
&\quad \left. + |\lambda| \sum_{j=1}^2 \|[B_{k1} u_j(-1) + C_{k1} u_j(+0)]\|_{L_2(-1,1)} \right) \\
&\leq c_5 \left(\sum_{j=1}^2 \|u_j(-1) + u_j(+0)\|_{W^2(-1+\sigma, -\sigma) + W^2(\sigma, 1-\sigma)} \right. \\
&\quad \left. + \sum_{j=1}^2 \|u_j(-1) + u_j(+0)\|_{L_2(-1+\sigma, -\sigma) + L_2(\sigma, 1-\sigma)} \right),
\end{aligned} \tag{4.8}$$

$k = 1, 2, 3, 4$.

We can construct the function $\xi(x) \in C^\infty[-1, 1]$ such that

$$\xi(x) = \begin{cases} 1, & x \in [-1 + \sigma, -\sigma] \cup [\sigma, 1 - \sigma] \\ 0, & x \in [-1, -1 + \frac{\sigma}{2}] \cup [-\frac{\sigma}{2}, \frac{\sigma}{2}] \cup [1 - \frac{\sigma}{2}, 1] \end{cases}$$

and $0 \leq \xi(x) \leq 1$ for all $x \in [-1, 1]$. From the inequalities (4.8) and (4.3) for $B_{k1}u(-1)$ and $C_{k1}u_j(+0)$ we have

$$\begin{aligned}
& \sum_{j=1}^2 |\lambda|^{\frac{3}{4}} |t [B_{k1}u_j(-1) + C_{k1}u_j(+0)]| \\
& \leq c_6 \sum_{j=1}^2 \|u_j(-1) + u_j(+0)\|_{W^2(-1+\sigma, -\sigma) + W^2(\sigma, 1-\sigma)} \\
& \leq c_6 \sum_{j=1}^2 \|\xi(u_j(-1) + u_j(+0))\|_{W^2(-1, 0, 1)} \\
& \leq c_7 \sum_{j=1}^2 \|(L_{0j} - \lambda I)\xi(u_j(-1) + u_j(+0))\|_{L_2(-1, 1)} \\
& \leq c_8 \left(\sum_{j=1}^2 \|(L_{0j} + tA_j - \lambda I)(u_j(-1) + u_j(+0))\|_{L_2(-1, 1)} \right. \\
& \quad \left. + \sum_{j=1}^2 \|u_j(-1) + u_j(+0)\|_{W^1(-1, 0, 1)} \right) \\
& \leq c_9 \left(\sum_{j=1}^2 \|(L_{0j} + tA_j - \lambda I)(u_j(-1) + u_j(+0))\|_{L_2(-1, 1)} \right. \\
& \quad \left. + |\lambda|^{\frac{1}{2}} \sum_{j=1}^2 \|u_j(-1) + u_j(+0)\|_{W^2(-1, 0, 1)} \right). \tag{4.9}
\end{aligned}$$

For $B_{k2}u(-0)$ and $C_{k2}u(1)$, similarly, we get

$$\begin{aligned}
& \sum_{j=1}^2 |\lambda|^{\frac{3}{4}} |t [B_{k2}u_j(-0) + C_{k2}u_j(1)]| \\
& \leq c_9 \left(\sum_{j=1}^2 \|(L_{0j} + tA_j - \lambda I)(u_j(-1) + u_j(+0))\|_{L_2(-1, 1)} \right. \\
& \quad \left. + |\lambda|^{\frac{1}{2}} \sum_{j=1}^2 \|u_j(-1) + u_j(+0)\|_{W^2(-1, 0, 1)} \right). \tag{4.10}
\end{aligned}$$

From inequalities (4.5)-(4.10) we obtain

$$\sum_{j=1}^2 \|u_j\|_{W^2(-1, 0, 1)} \leq c_{10} \sum_{j=1}^2 \left\| \widetilde{\mathbb{L}}u_j \right\|_{L_2(-1, 1) \times \mathbb{C}^4} + C(\lambda) \sum_{j=1}^2 \|u_j\|_{W^2(-1, 0, 1)}$$

Therefore, choosing $l_1 > l_0$ such that $C(l_1) < \frac{1}{2}$, we have the inequality

$$\sum_{j=1}^2 \|u_j\|_{W^2(-1,0,1)} \leq C \sum_{j=1}^2 \|\tilde{\mathbb{L}}u_j\|_{L_2(-1,1) \times \mathbb{C}^4}.$$

So, we proved the right side of the inequality (4.1). The left side of the inequality (4.1) can be obtained from condition 1 and inequalities (4.6)-(4.9). $\square$

Theorem 4.2. *We assume that the condition 1 is satisfying. Then we have:*

1) *For any $\varepsilon > 0$, there exist l_1 such that for $\lambda \in G_{\varepsilon, l_1}$ the operator $\mathbb{L}(\lambda)$ has a bounded inverse $\mathbb{L}^{-1}(\lambda) : L_2(-1, 1) \times \mathbb{C}^4 \rightarrow W^2(-1, 0, 1)$.*

2) *$\mathbb{L}(\lambda) : W^2(-1, 0, 1) \rightarrow L_2(-1, 1) \times \mathbb{C}^4$ is a Fredholm operator, and $\text{ind} \mathbb{L}(\lambda) = 0$ for all $\lambda \in \mathbb{C}$.*

Proof. 1) By virtue of Lemma 2.3, Lemma 4.1 and ([10], Chapter 1) we have that $\mathbb{L}(\lambda)$ has a bounded inverse for $\lambda \in G_{\varepsilon, l_1}$.

2) For $\mu \in G_{\varepsilon, l_1}$ and $\lambda \in \mathbb{C}$ we have

$$\mathbb{L}(\lambda)\mathbb{L}^{-1}(\mu) = I + (\mathbb{L}(\lambda) - \mathbb{L}(\mu))\mathbb{L}^{-1}(\mu),$$

where I identity operator in $L_2(-1, 1) \times \mathbb{C}^4$. In view of condition 1 the operator

$$\mathbb{L}(\lambda) - \mathbb{L}(\mu) : W^1(-1, 0, 1) \rightarrow L_2(-1, 1) \times \mathbb{C}^4$$

is bounded. From the compactness of imbedding of $W^2(-1, 0, 1)$ into $W^1(-1, 0, 1)$ and by virtue of Theorem 2.2 it follows that the operator

$$(\mathbb{L}(\lambda) - \mathbb{L}(\mu))\mathbb{L}^{-1}(\mu) : L_2(-1, 1) \times \mathbb{C}^4 \rightarrow L_2(-1, 1) \times \mathbb{C}^4$$

is compact. So, the operator

$$I + (\mathbb{L}(\lambda) - \mathbb{L}(\mu))\mathbb{L}^{-1}(\mu)$$

is a Fredholm operator and

$$\text{ind}[I + (\mathbb{L}(\lambda) - \mathbb{L}(\mu))\mathbb{L}^{-1}(\mu)] = 0.$$

Therefore, $\mathbb{L}(\lambda)\mathbb{L}^{-1}(\mu)$ is a Fredholm operator. $\mathbb{L}(\lambda)$ is a Fredholm since the operator $\mathbb{L}^{-1}(\lambda)$ has a bounded inverse. So, by Theorem 2.1, we have

$$\text{ind}(\mathbb{L}(\lambda)\mathbb{L}^{-1}(\mu)) = \text{ind}(\mathbb{L}(\lambda)) = 0.$$

$\square$

4.1. Differential operator for homogeneous boundary conditions.

Let consider boundary value problem

$$L_{01}(D)u := -p(x)u''(x) + Au(x) = f(x), \quad x \in [-1, 0) \cup (0, 1]$$

$$\begin{aligned} L_k u &= \alpha_k u(-1) + \beta_k u(-0) + \delta_k u(+0) + \gamma_k u(1) \\ &\quad + (B_{k1}u)(-1) + (B_{k2}u)(-0) + (C_{k1}u)(+0) + (C_{k2}u)(1) + N_k u(x_k) = 0, \\ &\quad k = 1, 2, 3, 4 \end{aligned}$$

and unbounded operator $\mathcal{L} : L_2(-1, 1) \rightarrow L_2(-1, 1)$ defined as follows

$$\mathcal{L}u := L_{01}(D)u,$$

$$\mathcal{D}(\mathcal{L}) := W^2(-1, 0, 1) = \{u \in W^2(-1, 0, 1) \mid L_k u = 0, k = 1, 2, 3, 4\}.$$

Theorem 4.3. *We assume that the condition 1 is satisfying. Then we have:*

- 1) $\mathcal{L}$ is a Fredholm operator, and $\text{ind}\mathcal{L} = 0$.
- 2) For $\lambda \notin \sigma(\mathcal{L})$, the resolvent $R(\lambda, \mathcal{L}) : L_2(-1, 1) \rightarrow L_2(-1, 1)$ is a compact operator.
- 3) The spectrum $\sigma(\mathcal{L})$ is discrete.
- 4) For every $0 < \varepsilon < \pi$, all points of the spectrum $\sigma(\mathcal{L})$, except, possibly, for a finite number of them, belong to the sector of complex plane defined by $|\arg \lambda| < \varepsilon$.

Proof. 1) From Theorem 2.1 we obtain statement (1).

2) By virtue of Theorem 4.2 there exists a bounded operator

$$(\mathcal{L} - \lambda I)^{-1} : L_2(-1, 1) \rightarrow W^2(-1, 0, 1).$$

Therefore, from the compactness of imbedding of $W^2(-1, 0, 1)$ into $L_2(-1, 1)$ it follows that the operator

$$(\mathcal{L} - \lambda I)^{-1} : L_2(-1, 1) \rightarrow L_2(-1, 1)$$

is compact.

3) By virtue of Theorem 6.29 in ([18], chp. III, sec. 6) the spectrum $\sigma(\mathcal{L})$ consists entirely of isolated eigenvalues of finite algebraic multiplicity without any finite cluster point since $R(\lambda, \mathcal{L})$ is compact.

4) By (3) and from ([11] chp. 15, sec.6), for every $\lambda > 0$, all points of the spectrum $\sigma(\mathcal{L})$, except, possibly, for a finite number of these points, belong to the sector $|\arg \lambda| < \varepsilon$. $\square$

5. GENERAL BOUNDARY VALUE PROBLEM

In this section, we will consider a discontinuous boundary value problem with inner point and transmission conditions in direct sum of Sobolev spaces. This problem is defined in the form

$$L(D)u := -u''(x) + (A - \lambda)u(x) = f(x), \quad x \in [-1, 0) \cup (0, 1] \quad (5.1)$$

$$\begin{aligned} L_{kv}u &= \alpha_{kv}u^{(m_k)}(-1) + \beta_{kv}u^{(m_k)}(-0) + \delta_{kv}u^{(m_k)}(+0) + \gamma_{kv}u^{(m_k)}(1) \\ &\quad + (B_{kv}^1 u)(-1) + (B_{kv}^2 u)(-0) + (C_{kv}^1 u)(+0) + (C_{kv}^2 u)(1) \\ &\quad + N_{kv}u^{(m_k)}(x_{kv}) = g_{kv}, \\ v &= 1 \text{ for } k = 1, 2 \text{ and } v = 2 \text{ for } k = 1, 2, \end{aligned} \quad (5.2)$$

where λ is a complex parameter; $m_k \geq 0$; α_{kv} , β_{kv} , δ_{kv} , γ_{kv} , g_{kv} are complex numbers; $f(x) = f_1(x)$, $A = A_1$, for $x \in [-1, 0)$; $f(x) = f_2(x)$, $A = A_2$, for $x \in (0, 1]$; $x_{k1} \in (-1, 0)$, $x_{k2} \in (0, 1)$ and A_k is an operator in H .

$$\begin{aligned} \text{Denote: } \phi_{kv} &:= \alpha_{kv}u^{(m_k)}(-1) + \beta_{kv}u^{(m_k)}(-0) + \delta_{kv}u^{(m_k)}(+0) + \gamma_{kv}u^{(m_k)}(1), \\ \mathcal{L}_{kv} &:= (B_{kv}^1 u)(-1) + (B_{kv}^2 u)(-0) + (C_{kv}^1 u)(+0) + (C_{kv}^2 u)(1). \end{aligned}$$

5.1. Homogeneous boundary value problem.

$$L(\lambda, D)u := -u''(x) + (A - \lambda)u(x) = 0, \quad x \in [-1, 0) \cup (0, 1] \quad (5.3)$$

$$\begin{aligned} L_{kv}u &= \alpha_{kv}u^{(m_k)}(-1) + \beta_{kv}u^{(m_k)}(-0) + \delta_{kv}u^{(m_k)}(+0) + \gamma_{kv}u^{(m_k)}(1) \\ &\quad + N_{kv}u^{(m_k)}(x_{kv}) = g_{kv}, \\ v &= 1 \text{ for } k = 1, 2 \text{ and } v = 2 \text{ for } k = 1, 2, \end{aligned} \quad (5.4)$$

$u(x)$ is a solution of equation (5.3) and $u(x) := u_1(x) + u_2(x)$; $u_1(x) := u_{11}(x) + u_{21}(x)$ for $x \in [-1, 0)$; $u_2(x) := u_{12}(x) + u_{22}(x)$ for $x \in (0, 1]$.

Denote:

$$\theta := \begin{vmatrix} (-1)^{m_1}\beta_{11} & \alpha_{11} & (-1)^{m_1}\delta_{11} & \gamma_{11} \\ (-1)^{m_2}\beta_{21} & \alpha_{21} & (-1)^{m_2}\delta_{21} & \gamma_{21} \\ (-1)^{m_1}\beta_{12} & \alpha_{12} & (-1)^{m_1}\delta_{12} & \gamma_{12} \\ (-1)^{m_2}\beta_{22} & \alpha_{22} & (-1)^{m_2}\delta_{22} & \gamma_{22} \end{vmatrix}.$$

Theorem 5.1. *We assume that the following conditions be satisfied:*

1) A_k is a closed, densely defined operator in a Hilbert space H and

$$R(\lambda, A_k I) \leq C(1 + |\lambda|)^{-1}, \quad |\arg \lambda| \geq \pi - \varphi, \quad k = 1, 2, \quad 0 \leq \varphi < \pi$$

where $R(\lambda, A_k I) := (\lambda I - A_k)^{-1}$ is the resolvent of the operator A_k ;

2) $\theta \neq 0$.

Then problem (5.3)-(5.4) for $f_{kv} \in (H(A_k), H)_{\theta_{kv}, p}$, where $\theta_{kv} := \frac{m_k}{2} + \frac{1}{2p}$, $v = 1$ for $k = 1, 2$ and $v = 2$ for $k = 1, 2$; $p \in (1, \infty)$, $|\arg \lambda| \leq \varphi$ and $|\lambda| \rightarrow \infty$, has a unique solution that belongs to the space $W_p^2((-1, 0, 1); H(A_k), H)$.

Proof. In view of condition (1) in the Theorem and ([17], Lemma 5.3.2/1) for $|\arg \lambda| \leq \varphi$, there exists semigroups $e^{-x(A_1 - \lambda I)^{\frac{1}{2}}}$, $e^{-x(A_2 - \lambda I)^{\frac{1}{2}}}$ which are the holomorphic and strongly continuous for $x < 0$ and $x > 0$, respectively. By virtue of S. G. Krein ([15], p. 252) and ([17], section 5.4.3), an arbitrary solution of equation (5.3) in the space $W_p^2((-1, 0, 1); H(A_k), H)$, $k = 1, 2$, has the form

$$\begin{aligned} u(x) &= u_1(x) + u_2(x) \\ &= e^{-x(A_1 - \lambda I)^{\frac{1}{2}}} d_1 + e^{(1+x)(A_1 - \lambda I)^{\frac{1}{2}}} d_2 \\ &\quad + e^{-x(A_2 - \lambda I)^{\frac{1}{2}}} d_3 + e^{-(1-x)(A_2 - \lambda I)^{\frac{1}{2}}} d_4, \end{aligned} \quad (5.5)$$

where

$$\begin{aligned} u_1(x) &: = e^{-x(A_1 - \lambda I)^{\frac{1}{2}}} d_1 + e^{(1+x)(A_1 - \lambda I)^{\frac{1}{2}}} d_2, \\ u_2(x) &: = e^{-x(A_2 - \lambda I)^{\frac{1}{2}}} d_3 + e^{-(1-x)(A_2 - \lambda I)^{\frac{1}{2}}} d_4; \end{aligned}$$

$d_1, d_2 \in (H(A_1), H)_{\frac{1}{2p}, p}$ and $d_3, d_4 \in (H(A_2), H)_{\frac{1}{2p}, p}$.

Let $u \in W_p^2((-1, 0, 1); H(A_k), H)$ be a solution of equation (5.3). So, from (5.3) we have

$$[D - (A_k - \lambda I)^{\frac{1}{2}}][D + (A_k - \lambda I)^{\frac{1}{2}}]u_k(x) = 0, \quad k = 1, 2.$$

Denote:

$$v_k(x) := [D + (A_k - \lambda I)^{\frac{1}{2}}]u_k(x), k = 1, 2. \quad (5.6)$$

Therefore, $v_k \in W_p^2((-1, 0, 1); H(A_k^{\frac{1}{2}}), H)$ and

$$[D - (A_k - \lambda I)^{\frac{1}{2}}]v_k(x) = 0.$$

So, we have

$$v_1(x) = e^{(1+x)(A_1 - \lambda I)^{\frac{1}{2}}}v_1(-1) \text{ and } v_2(x) = e^{-(1-x)(A_2 - \lambda I)^{\frac{1}{2}}}v_2(1), \quad (5.7)$$

where, in view of ([6], Theorem 1.8.2) $v_1(-1) \in (H(A_1^{\frac{1}{2}}), H)_{\frac{1}{p}, p}$ and $v_2(1) \in (H(A_2^{\frac{1}{2}}), H)_{\frac{1}{p}, p}$. From (5.6) and (5.7) we have

$$\begin{aligned} u_1(x) &= e^{-x(A_1 - \lambda I)^{\frac{1}{2}}}u_1(-0) + \frac{1}{2}(A_1 - \lambda I)^{-\frac{1}{2}}[e^{(1+x)(A_1 - \lambda I)^{\frac{1}{2}}} \\ &\quad - e^{(1-x)(A_1 - \lambda I)^{\frac{1}{2}}}]v_1(-1), \\ u_2(x) &= e^{-x(A_2 - \lambda I)^{\frac{1}{2}}}u_2(+0) + \frac{1}{2}(A_2 - \lambda I)^{-\frac{1}{2}}[e^{-(1-x)(A_2 - \lambda I)^{\frac{1}{2}}} \\ &\quad - e^{-(1+x)(A_2 - \lambda I)^{\frac{1}{2}}}]v_2(1). \end{aligned}$$

Hence, we can write

$$\begin{aligned} u(x) &= e^{-x(A_1 - \lambda I)^{\frac{1}{2}}}u_1(-0) + e^{-x(A_2 - \lambda I)^{\frac{1}{2}}}u_2(+0) \\ &\quad + \frac{1}{2}e^{(1+x)}[(A_1 - \lambda I)^{-\frac{1}{2}}e^{(A_1 - \lambda I)^{\frac{1}{2}}}v_1(-1) \\ &\quad - (A_2 - \lambda I)^{-\frac{1}{2}}e^{-(A_2 - \lambda I)^{\frac{1}{2}}}v_2(1)] \\ &\quad + \frac{1}{2}e^{-(1-x)}[(A_2 - \lambda I)^{-\frac{1}{2}}e^{(A_2 - \lambda I)^{\frac{1}{2}}}v_2(1) \\ &\quad - (A_1 - \lambda I)^{-\frac{1}{2}}e^{-(A_1 - \lambda I)^{\frac{1}{2}}}v_1(-1)], \end{aligned} \quad (5.8)$$

where, from ([6], Theorem 1.8.2) $u_1(-0) \in (H(A_1), H)_{\frac{1}{2p}, p}$ and

$u_2(+0) \in (H(A_2), H)_{\frac{1}{2p}, p}$. By virtue of ([6], Theorem 1.15.2), the operator $A_k^{\frac{1}{2}}$ is an isomorphism from $(H(A_k), H)_{\frac{1}{2p}, p}$ onto $(H(A_k^{\frac{1}{2}}), H)_{\frac{1}{p}, p}$. Hence, (5.8) has form (5.5). Let us prove that the function (5.7) belongs to the space $W_p^2((-1, 0, 1); H(A_k), H)$. Let us now write (5.5) in boundary conditions (5.4)

$$\begin{aligned} &(-1)^{m_{kv}}[\alpha_{kv}e^{(A_1 - \lambda I)^{\frac{1}{2}}} + \beta_{kv} + N_{11}e^{-x_{11}(A_1 - \lambda I)^{\frac{1}{2}}}](A_1 - \lambda I)^{\frac{m_k}{2}}d_1 \\ &+ [\alpha_{kv} + \beta_{kv}e^{(A_1 - \lambda I)^{\frac{1}{2}}} + N_{21}e^{(1+x_{21})(A_1 - \lambda I)^{\frac{1}{2}}}](A_1 - \lambda I)^{\frac{m_k}{2}}d_2 \\ &+ (-1)^{m_k}[\eta_{kv} + \gamma_{kv}e^{-(A_2 - \lambda I)^{\frac{1}{2}}} + N_{12}e^{-x_{12}(A_2 - \lambda I)^{\frac{1}{2}}}](A_2 - \lambda I)^{\frac{m_k}{2}}d_3 \\ &+ [\eta_{kv}e^{-(A_2 - \lambda I)^{\frac{1}{2}}} + \gamma_{kv} + N_{22}e^{-(1-x_{22})(A_2 + \lambda I)^{\frac{1}{2}}}](A_2 - \lambda I)^{\frac{m_k}{2}}d_4 \\ &= g_{kv}, \quad v = 1 \text{ for } k = 1, 2 \text{ and } v = 2 \text{ for } k = 1, 2. \end{aligned} \quad (5.9)$$

Denote: $w_1 := (A_1 - \lambda I)^{\frac{m}{2}}d_1$, $w_2 := (A_1 - \lambda I)^{\frac{m}{2}}d_2$,

$w_3 := (A_2 - \lambda I)^{\frac{m}{2}} d_3$, $w_4 := (A_2 - \lambda I)^{\frac{m}{2}} d_4$, where $m = \max\{m_1, m_2\}$.
System (5.9) has the form

$$\begin{aligned}
& \left[\begin{pmatrix} (-1)^{m_1} \beta_{11} & \alpha_{11} & (-1)^{m_1} \delta_{11} & \gamma_{11} \\ (-1)^{m_2} \beta_{21} & \alpha_{21} & (-1)^{m_2} \delta_{21} & \gamma_{21} \\ (-1)^{m_1} \beta_{12} & \alpha_{12} & (-1)^{m_1} \delta_{12} & \gamma_{12} \\ (-1)^{m_2} \beta_{22} & \alpha_{22} & (-1)^{m_2} \delta_{22} & \gamma_{22} \end{pmatrix} + R(\lambda) \right] \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix} \\
&= \begin{pmatrix} (A_1 - \lambda I)^{-\frac{m_1-m}{2}} & 0 & 0 \\ 0 & (A_1 - \lambda I)^{-\frac{m_2-m}{2}} & 0 \\ 0 & 0 & (A_2 - \lambda I)^{-\frac{m_1-m}{2}} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ (A_2 - \lambda I)^{-\frac{m_2-m}{2}} \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{pmatrix}, \tag{5.10}
\end{aligned}$$

where,

$$\begin{aligned}
& (R(\lambda)w)_1 \\
&= (-1)^{m_1} [\alpha_{11} e^{(A_1 - \lambda I)^{1/2}} + N_{11} e^{-x_{11}(A_1 - \lambda I)^{1/2}}] w_1 \\
&+ [\beta_{11} e^{(A_1 - \lambda I)^{1/2}} + N_{21} e^{(1+x_{21})(A_1 - \lambda I)^{1/2}}] w_2 \\
&+ (-1)^{m_1} [\gamma_{11} e^{-(A_2 - \lambda I)^{1/2}} + N_{12} e^{-x_{12}(A_2 - \lambda I)^{1/2}}] w_3 \\
&+ [\delta_{11} e^{-(A_2 - \lambda I)^{1/2}} + N_{22} e^{-(1-x_{22})(A_2 - \lambda I)^{1/2}}] w_4,
\end{aligned}$$

$$\begin{aligned}
& (R(\lambda)w)_2 \\
&= (-1)^{m_2} [\alpha_{21} e^{(A_1 - \lambda I)^{1/2}} + N_{11} e^{-x_{11}(A_1 - \lambda I)^{1/2}}] w_1 \\
&+ [\beta_{21} e^{(A_1 - \lambda I)^{1/2}} + N_{21} e^{(1+x_{21})(A_1 - \lambda I)^{1/2}}] w_2 \\
&+ (-1)^{m_2} [\gamma_{21} e^{-(A_2 - \lambda I)^{1/2}} + N_{12} e^{-x_{12}(A_2 - \lambda I)^{1/2}}] w_3 \\
&+ [\eta_{21} e^{-(A_2 - \lambda I)^{1/2}} + N_{22} e^{-(1-x_{22})(A_2 - \lambda I)^{1/2}}] w_4,
\end{aligned}$$

$$\begin{aligned}
& (R(\lambda)w)_3 \\
&= (-1)^{m_1} [\alpha_{12} e^{(A_1 - \lambda I)^{1/2}} + N_{11} e^{-x_{11}(A_1 - \lambda I)^{1/2}}] w_1 \\
&+ [\beta_{12} e^{(A_1 - \lambda I)^{1/2}} + N_{21} e^{(1+x_{21})(A_1 - \lambda I)^{1/2}}] w_2 \\
&+ (-1)^{m_1} [\gamma_{12} e^{-(A_2 - \lambda I)^{1/2}} + N_{12} e^{-x_{12}(A_2 - \lambda I)^{1/2}}] w_3 \\
&+ [\delta_{12} e^{-(A_2 - \lambda I)^{1/2}} + N_{22} e^{-(1-x_{22})(A_2 - \lambda I)^{1/2}}] w_4,
\end{aligned}$$

$$\begin{aligned}
& (R(\lambda)w)_4 \\
&= (-1)^{m_2} [\alpha_{22} e^{(A_1 - \lambda I)^{1/2}} + N_{11} e^{-x_{11}(A_1 - \lambda I)^{1/2}}] w_1 \\
&+ [\beta_{22} e^{(A_1 - \lambda I)^{1/2}} + N_{21} e^{(1+x_{21})(A_1 - \lambda I)^{1/2}}] w_2 \\
&+ (-1)^{m_2} [\gamma_{22} e^{-(A_2 - \lambda I)^{1/2}} + N_{12} e^{-x_{12}(A_2 - \lambda I)^{1/2}}] w_3 \\
&+ [\delta_{22} e^{-(A_2 - \lambda I)^{1/2}} + N_{22} e^{-(1-x_{22})(A_2 - \lambda I)^{1/2}}] w_4.
\end{aligned}$$

By virtue of ([17], Lemma 5.4.2/6) for $|\arg \lambda| \leq \varphi$ and $|\lambda| \rightarrow \infty$

$$\|R(\lambda)\|_{B(H^2)} \rightarrow 0, \|R(\lambda)\|_{B(H[(A_k)^2])} \rightarrow 0. \quad (5.11)$$

Therefore, by $\theta \neq 0$ and Neumann identity for $|\arg \lambda| \leq \varphi$ and $|\lambda| \rightarrow \infty$,

$$\begin{aligned}
& \left[\begin{pmatrix} (-1)^{m_1} \beta_{11} & \alpha_{11} & (-1)^{m_1} \delta_{11} & \gamma_{11} \\ (-1)^{m_2} \beta_{21} & \alpha_{21} & (-1)^{m_2} \delta_{21} & \gamma_{21} \\ (-1)^{m_1} \beta_{12} & \alpha_{12} & (-1)^{m_1} \delta_{12} & \gamma_{12} \\ (-1)^{m_2} \beta_{22} & \alpha_{22} & (-1)^{m_2} \delta_{22} & \gamma_{22} \end{pmatrix} + R(\lambda) \right]^{-1} \\
&= \begin{pmatrix} (-1)^{m_1} \beta_{11} & \alpha_{11} & (-1)^{m_1} \delta_{11} & \gamma_{11} \\ (-1)^{m_2} \beta_{21} & \alpha_{21} & (-1)^{m_2} \delta_{21} & \gamma_{21} \\ (-1)^{m_1} \beta_{12} & \alpha_{12} & (-1)^{m_1} \delta_{12} & \gamma_{12} \\ (-1)^{m_2} \beta_{22} & \alpha_{22} & (-1)^{m_2} \delta_{22} & \gamma_{22} \end{pmatrix}^{-1} \\
&\times \sum_{k=0}^{\infty} \left[-R(\lambda) \begin{pmatrix} (-1)^{m_1} \beta_{11} & \alpha_{11} & (-1)^{m_1} \delta_{11} & \gamma_{11} \\ (-1)^{m_2} \beta_{21} & \alpha_{21} & (-1)^{m_2} \delta_{21} & \gamma_{21} \\ (-1)^{m_1} \beta_{12} & \alpha_{12} & (-1)^{m_1} \delta_{12} & \gamma_{12} \\ (-1)^{m_2} \beta_{22} & \alpha_{22} & (-1)^{m_2} \delta_{22} & \gamma_{22} \end{pmatrix}^{-1} \right]^k. \quad (5.12)
\end{aligned}$$

So, system (5.10) has a unique solution for $|\arg \lambda| \leq \varphi$ and $|\lambda|$ sufficiently large, and the solution can be expressed in the form

$$\begin{aligned}
w_j &= \sum_{k=1}^2 [C_{jk} + R_{jk}(\lambda)] (A_1 - \lambda I)^{\frac{m-m_k}{2}} g_{k1}, j = 1, 2, \\
w_j &= \sum_{k=1}^2 [C_{jk} + R_{jk}(\lambda)] (A_2 - \lambda I)^{\frac{m-m_k}{2}} g_{k2}, j = 3, 4,
\end{aligned}$$

where C_{jk} are complex numbers and in view of (5.12) for $|\arg \lambda| \leq \varphi$ and $|\lambda| \rightarrow \infty$

$$\begin{aligned}
& \|R_{jk}(\lambda)\|_{B(H)} \rightarrow 0, \|R_{jk}(\lambda)\|_{B(H(A_k))} \rightarrow 0, k = 1, 2 \\
& \text{for every } j, j = 1, 2, 3, 4.
\end{aligned} \quad (5.13)$$

Therefore, we have

$$d_j = \sum_{k=1}^2 [C_{jk} + R_{jk}(\lambda)] (A_1 - \lambda I)^{\frac{-m_k}{2}} g_{k1}, j = 1, 2,$$

$$d_j = \sum_{k=1}^2 [C_{jk} + R_{jk}(\lambda)](A_2 - \lambda I)^{-\frac{m_k}{2}} g_{k2}, \quad j = 3, 4. \quad (5.14)$$

Substituting (5.14) into (5.5) we have

$$\begin{aligned} u(x) &= \sum_{k=1}^2 [(C_{1k} + R_{1k}(\lambda))e^{-x(A_1 - \lambda I)^{1/2}} \\ &\quad + (C_{2k} + R_{2k}(\lambda))e^{(1+x)(A_1 - \lambda I)^{1/2}}](A_1 - \lambda I)^{-\frac{m_k}{2}} g_{k1} \\ &\quad + \sum_{k=1}^2 [(C_{3k} + R_{3k}(\lambda))e^{-x(A_2 - \lambda I)^{1/2}} \\ &\quad + (C_{4k} + R_{4k}(\lambda))e^{-(1-x)(A_2 - \lambda I)^{1/2}}](A_2 - \lambda I)^{-\frac{m_k}{2}} g_{k2}. \end{aligned} \quad (5.15)$$

□

5.2. Nonhomogeneous boundary value problem.

Let consider the boundary value problem

$$L_0(\lambda, D) := -u''(x) + (A - \lambda)u(x) = f(x), \quad x \in [-1, 0) \cup (0, 1] \quad (5.16)$$

$$\begin{aligned} L_{kv}u &= L_{kv}u = \alpha_{kv}u^{(m_k)}(-1) + \beta_{kv}u^{(m_k)}(-0) + \delta_{kv}u^{(m_k)}(+0) + \gamma_{kv}u^{(m_k)}(1) \\ &\quad + N_{kv}u^{(m_k)}(x_{kv}) = g_{kv}, \end{aligned} \quad (5.17)$$

$$v = 1 \text{ for } k = 1, 2 \text{ and } v = 2 \text{ for } k = 1, 2.$$

Theorem 5.2. *We assume that the following conditions be satisfied:*

1) A_k is a closed, densely defined operator in a Hilbert space H and

$$R(\lambda, A_k I) \leq C(1 + |\lambda|)^{-1}, |\arg \lambda| \geq \pi - \varphi, \quad k = 1, 2, \quad 0 \leq \varphi < \pi$$

where $R(\lambda, A_k I) := (\lambda I - A_k)^{-1}$ is the resolvent of the operator A_k ;

2) $\theta \neq 0$;

Then the operator $\mathbb{L}_0(\lambda) : u \rightarrow \mathbb{L}_0(\lambda)u := (L_0(\lambda)u, L_1, L_2, L_3, L_4)$, for $|\arg \lambda| \leq \varphi$ and $|\lambda| \rightarrow \infty$, is an isomorphism from $W_p^2((-1, 0, 1); H(A_k), H)$ onto

$$\begin{aligned} &(L_p(-1, 1); H) + (H(A_1), H)_{\theta_{11}, p} + (H(A_1), H)_{\theta_{21}, p} + (H(A_2), H)_{\theta_{12}, p} \\ &+ (H(A_2), H)_{\theta_{22}, p}, \end{aligned}$$

$$\theta_{kv} := \frac{m_k}{2} + \frac{1}{2p}, \quad v = 1 \text{ for } k = 1, 2 \text{ and } v = 2 \text{ for } k = 1, 2, \quad p \in (1, \infty).$$

Proof. We have proved the uniqueness of solution for the boundary value problem in Theorem 7. Let us define $f(x) := f_1(x) := \tilde{f}_1(x)$, at $x \in [-1, 0)$; $f(x) := f_2(x) := \tilde{f}_2(x)$, at $x \in (0, 1]$, $f(x) := f_1(x) := 0$, at $x \notin [-1, 0)$; $f(x) := f_2(x) := 0$, at $x \notin (0, 1]$. We will show that a solution of problem (4.1)-(4.2) which belongs to $W_p^2((-1, 0, 1); H(A_k), H)$ as a sum in the following form

$$\begin{aligned} u(x) &= u_1(x) + u_2(x) \\ &= u_{11}(x) + u_{21}(x) + u_{12}(x) + u_{22}(x), \end{aligned}$$

where

$$\begin{aligned} u_1(x) &= u_{11}(x) + u_{21}(x) \text{ for } x \in [-1, 0), \\ u_2(x) &= u_{12}(x) + u_{22}(x) \text{ for } x \in (0, 1], \end{aligned}$$

besides,

$$\begin{aligned} u_{10}(x) &: = u_{11}(x) + u_{12}(x), \\ \tilde{u}_{10}(x) &: = \tilde{u}_{11}(x) + \tilde{u}_{12}(x), \\ u_{20}(x) &: = u_{21}(x) + u_{22}(x), \end{aligned}$$

where $u_{11}(x)$ and $u_{12}(x)$ are the restriction on $[-1, 0)$ and $(0, 1]$, respectively, of the solutions $\tilde{u}_{11}(x)$ and $\tilde{u}_{12}(x)$ of the equation

$$L_0(\lambda)\tilde{u}_{10} = \tilde{f}(x), \quad x \in \mathbb{R}, \quad (5.18)$$

and $u_{20}(x)$ is a solution of the boundary value problem

$$\begin{aligned} L_0(\lambda)u_{20} &= 0, \\ L_{kv0}(\lambda)u_{20} &= g_{kv} - L_{kv0}(\lambda)u_{10}, \end{aligned} \quad (5.19)$$

$v = 1$ for $k = 1, 2$ and $v = 2$ for $k = 1, 2$.

A solution of equation (5.19) has in the form

$$\tilde{u}_{10}(x) = \frac{1}{2\pi} \sum_{k=1}^2 \int_{\mathbb{R}} e^{i\mu_k x} L_0(\lambda, i\mu_k)^{-1} F_k \tilde{f}_k(\mu_k) d\mu_k, \quad (5.20)$$

where $F_k \tilde{f}_k$ is the Fourier transform of the function $\tilde{f}_k(x)$, and $L_0(\lambda, \sigma_k)$ is a characteristic operator pencil of equation (5.18), that is, $L_0(\lambda, \sigma_k) = -\sigma_k^2 I + A_k - \lambda I$, $k = 1, 2$. It is shown that ([17], sec. 5.4.4) there exists the solution of equation (5.18) in $W_p^2(\mathbb{R}; H(A_k), H)$ and it satisfies the inequality

$$\sum_{k=1}^2 (|\lambda| \|\tilde{u}_{1k}\|_{L_p(\mathbb{R}; H)} + \|\tilde{u}_{1k}\|_{W_p^2(\mathbb{R}; H(A_k), H)}) \leq C \sum_{k=1}^2 \|\tilde{f}_{1k}\|_{L_p(\mathbb{R}; H)}, \quad |\arg \lambda| \leq \varphi \quad (5.21)$$

and, hence, $u_{10}(x) \in W_p^2((-1, 0, 1); H(A_k), H)$. In view of ([6], Teorem 1.8.2) and inequality (5.21), we have $u_{10}^{(m_k)}(-0) \in W_p^2((-1, 0); H(A_1), H)_{\frac{m_k}{2} + \frac{1}{2p}, p}$, $u_{10}^{(m_k)}(+0) \in W_p^2((0, 1); H(A_2), H)_{\frac{m_k}{2} + \frac{1}{2p}, p}$. Therefore, $L_{k0}u_{10} \in (H(A_k), H)_{\theta_k, p}$. Hence, problem (5.19) has a unique solution $u_{20}(x)$ that belongs to $W_p^2((-1, 0, 1); H(A_k), H)$. $\square$

Theorem 5.3. *We assume that the following condition be fulfilled:*

1) A_k is a closed, densely defined operator in a Hilbert space H and

$$R(\lambda, A_k I) \leq C(1 + |\lambda|)^{-1}, \quad \arg \lambda = \pi, \quad |\lambda| \rightarrow \infty, \quad k = 1, 2,$$

where $R(\lambda, A_k I) := (\lambda I - A_k)^{-1}$ is the resolvent of the operator A_k ;

2) the embedding $H(A_k) \subset H$ is compact;

3) $\theta \neq 0$;

Then, for the problem (5.1)-(5.2) the operator

$$\mathbb{L}(\lambda) : u_k \rightarrow \mathbb{L}(\lambda)u_k := (L(D)u_k, L_{11}u_k, L_{21}u_k, L_{12}u_k, L_{22}u_k), \quad k = 1, 2$$

from $W_p^2((-1, 0, 1); H(A_k), H)$ into

$$\begin{aligned} &L_p((-1, 1); H) \dot{+} (H(A_1), H)_{\theta_{11}, p} \dot{+} (H(A_1), H)_{\theta_{21}, p} \dot{+} (H(A_2), H)_{\theta_{12}, p} \\ &\dot{+} (H(A_2), H)_{\theta_{22}, p} \end{aligned}$$

is Fredholm.

Proof. The operator $\mathbb{L}$ can be written in the form

$$\mathbb{L} = \mathbb{L}_0 + \mathbb{L}_1$$

where

$$\begin{aligned} \mathbb{L}_0 u_k &:= (L(D)u_k, (\phi_{11} + N_{11})u_k, (\phi_{21} + N_{21})u_k, (\phi_{12} + N_{12})u, (\phi_{22} + N_{22})u_k) \\ \mathbb{L}_1 u &= (-\lambda u_k, \mathcal{L}_{11}u_k, \mathcal{L}_{21}u_k, \mathcal{L}_{12}u_k, \mathcal{L}_{22}u_k), \quad k = 1, 2. \end{aligned}$$

We can conclude from Theorem 8 that $\mathbb{L}_0$ from $W_p^2((-1, 0, 1); H(A_k), H)$ onto

$$\begin{aligned} &L_p((-1, 1); H) \dot{+} (H(A_1), H)_{\theta_{11}, p} \dot{+} (H(A_1), H)_{\theta_{21}, p} \dot{+} (H(A_2), H)_{\theta_{12}, p} \\ &\dot{+} (H(A_2), H)_{\theta_{22}, p} \end{aligned}$$

has an inverse. From condition 2 for $\varepsilon > 0$ and $u_k \in W_p^2((-1, 0, 1); H(A_k), H)$ we have

$$\begin{aligned} \|(B_{kv}u_k)(x_k)\|_{(H(A), H)_{\theta_{kv}, p}} &\leq \varepsilon \|u_k(x_k)\|_{W_p^2((0, 1); H(A), H)} \\ &\quad + C(\varepsilon) \|u_k(x_k)\|_{L_p((0, 1); H)}, \\ \|(C_{kv}u_k)(x_k)\|_{(H(A), H)_{\theta_{kv}, p}} &\leq \varepsilon \|u_k(x_k)\|_{W_p^2((0, 1); H(A), H)} \\ &\quad + C(\varepsilon) \|u_k(x_k)\|_{L_p((0, 1); H)} \\ &\quad k = 1, 2, 3, 4, \quad v = 1, 2, \quad x_k \in [0, 1]. \end{aligned}$$

Therefore, by virtue of Lemma 2.7 the operator $\mathbb{L}_{1k}$ from $W_p^2((-1, 0, 1); H(A_k), H)$ into

$$\begin{aligned} &L_p((-1, 1); H) \dot{+} (H(A_1), H)_{\theta_{11}, p} \dot{+} (H(A_1), H)_{\theta_{21}, p} \dot{+} (H(A_2), H)_{\theta_{12}, p} \\ &\dot{+} (H(A_2), H)_{\theta_{22}, p} \end{aligned}$$

is compact. The proof of the theorem is completed in view of application by S.G. Krein ([16], sec., 14, Theorem 14.1). $\square$

REFERENCES

- [1] A. A. Shkalikov, *Boundary value problems for ordinary differential Equations with a parameter in boundary conditions*, Trudy Sem. Petrovsk. **9** (1983), 190-229.
- [2] A. L. Skubachevskii, *Elliptic Functional Differential Equations and Applications*, Birkhasuer Verlag, Basel, 1997.
- [3] A. L. Skubachevskii, *Nonlocal elliptic problems with a parameter*, Math. Sb. **49** (1984), 197-206.
- [4] A. V. Bitsadze and A. A. Samarskii, *On some simplest generalizations of linear elliptic problems*, Dokl. Akad. Nauk SSSR **185** (1969), 398-400.

- [5] B. A. Aliev, *Solvability of a boundary value problem for a second order elliptic-differential operator equation with spectral parameter in the equation and boundary conditions*, Transactions of NAS of Azerbaijan, **4** (2010), 3-16.
- [6] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*, Nort-Holland, Amsterdam, 1978.
- [7] M. Kandemir , O. Sh. Mukhtarov and Ya. Yakubov, *Irregular boundary value problems with discontinuous coefficients and the eigenvalue parameter*, Mediterranean Journal of Mathematics, **6** (2009), 317-338.
- [8] M. Kandemir and Ya. Yakubov, *Regular boundary value problems with a discontinuous coefficient, functional-multipoint conditions, and a linear spectral parameter*, Israel Journal of Mathematics, **180** (2010), 255-270.
- [9] M. L. Rasulov, *Methods of contour integration*, Nort Holland publishing company, Amsterdam, 1967.
- [10] M. S. Agronovich and M. I. Vishik, *Elliptic problems with a parameter parabolic problems of general type*, Russian Math. Surveys, **19** (1964), 53-161.
- [11] N. Dunford and J. T. Schwartz, *Linear Operators. Part II. Spectral Theory*, Interscience, New York-London, 1963.
- [12] O. Sh. Mukhtarov and H. Demir, *Coerciveness of the Discontinuous Initial-Boundary Value Problem for Parabolic Equation*, Israel Journal of Mathematics **114** (1999), 239-252.
- [13] O. Sh. Mukhtarov and S. Yakubov, *Problems for ordinary differential equations with transmission conditions*, Applicable Analysis, **81**(2002), 1033-1064.
- [14] P. L. Gurevich, *Elliptic problems with nonlocal boundary conditions and Feller semigroups*, Journal of Mathematical Sciences **182**(3) (2012), 256-440.
- [15] S. G. Krein, *Linear Differential Equations in Banach Spaces* , Providence, 1971.
- [16] S. G. Krein, *Linear Equations in Banach Spaces*, Birkhauser, 1982.
- [17] S. Yakubov and Ya. Yakubov, *Differential-Operator Equation Ordinary and Partial Differential Equation*, Chapman and Hall/CRC Boca Raton London New York Washington, D. C, 1999.
- [18] T. Kato, *Perturbation theory for linear operators*, Springer-Verlag, Berlin-Heidelberg, New York, 1966.
- [19] V. B. Shakhmurov, *Linear and nonlinear abstract elliptic equations with VMO coefficients and applications, Fixed point theory and applications*, **6** (2013), 1-21.
- [20] V. V. Podyapolskii, *Completeness of the root functions system of a nonlocal problem in L_p* , Mathematical Notes, **6**: 804–814. Translated from Matematicheskie Zametki, **6** (2002), 878-889.

AMASYA UNIVERSITY, EDUCATION FACULTY, DEPARTMENT OF MATHEMATICS, 05200
AMASYA /TURKEY,

E-mail address: mkandemir5@yahoo.com