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(N, ε)-PSEUDOSPECTRA OF BOUNDED LINEAR OPERATORS ON ULTRAMETRIC BANACH SPACES

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Abstract. In this paper, we prove that the essential pseudospectrum of bounded linear operator pencils is invariant under perturbation of each completely continuous linear operators on ultrametric Banach spaces over a spherically complete field $\mathbb{K}$ and we establish a characterization of the essential pseudospectrum of bounded linear operator pencils by means of the spectrum of each perturbed completely continuous operators. Furthermore, we introduce and study the notion of (n, ε)-pseudospectrum of bounded linear operators and the concept of (n, ε)-pseudospectrum of bounded linear operator pencils on ultrametric Banach spaces. We establish some results about them. Finally, several examples are provided.

1. Introduction

In the classical operator theory, Hansen [10] introduced and studied the notion of (n, ε)-pseudospectra of continuous linear operators on separable Hilbert spaces. Seidel [16] extended and studied the notion of (n, ε)-pseudospectra of continuous linear operators on complex Banach spaces. For more details, we refer to [10, 11, 16].

Inspired by the ideas introduced in [10, 11, 16], we will extend and study the (n, ε)-pseudospectra of continuous linear operators on ultrametric Banach spaces. The notion of ultrametric (n, ε)-pseudospectra is useful in ultrametric spectral theory of linear operators such as the spectra and the pseudospectra of linear operators on utrametric Banach spaces.

This paper is motivated by many studies on spectral theory of linear operators in ultrametric Banach spaces. For more details, we refer to [1, 2, 4, 5, 8, 13].

The goal of this paper is to develop and study the notions of (n, ε)-pseudospectra of continuous linear operators and continuous linear operator pencils in ultrametric Banach spaces.

Throughout this paper, $\mathbb{K}$ is a complete ultrametric field with a non-trivial valuation $|\cdot|$, $\mathcal{E}$ is an ultrametric Banach space over $\mathbb{K}$, $\mathcal{B}(\mathcal{E})$ denotes the collection of each continuous linear operators on $\mathcal{E}$, $\mathcal{E}^* = \mathcal{B}(\mathcal{E}, \mathbb{K})$ is the dual space of $\mathcal{E}$ and $\mathbb{Q}_p$ is the field of p-adic numbers [4, 19]. For $S \in \mathcal{B}(\mathcal{E})$, $\rho(S), \sigma(S), R(S)$ and $N(S)$

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denote the resolvent set, the spectrum, the range and the kernel of S respectively [4].

2. Preliminaries

We begin with the following:

Definition 2.1. [4] Let $\mathcal{E}$ be a vector space over $\mathbb{K}$. A function $\|\cdot\| : \mathcal{E} \to \mathbb{R}_+$ is called an ultrametric norm if:

(i) For each $u \in \mathcal{E}$, $\|u\| = 0$ if and only if $u = 0$;
(ii) For each $u \in \mathcal{E}$ and $\lambda \in \mathbb{K}$, $\|\lambda u\| = |\lambda|\|u\|$;
(iii) For any $u, v \in \mathcal{E}$, $\|u + v\| \leq \max(\|u\|, \|v\|)$.

Definition 2.2. [4] An ultrametric normed space is a pair $(\mathcal{E}, \|\cdot\|)$ where $\mathcal{E}$ is a vector space over $\mathbb{K}$ and $\|\cdot\|$ is an ultrametric norm on $\mathcal{E}$.

Definition 2.3. [4] An ultrametric Banach space is a complete ultrametric normed space.

Example 2.4. [4] Let $\mathcal{M}$ be a compact (topological) space. Define the space of any continuous functions from $\mathcal{M}$ into $\mathbb{K}$ by

$$C(\mathcal{M}, \mathbb{K}) = \{g : \mathcal{M} \mapsto \mathbb{K}, g \text{ is continuous}\}.$$

Then $(C(\mathcal{M}, \mathbb{K}), \|\cdot\|)$ is an ultrametric Banach space over $\mathbb{K}$ where $\|g\| = \sup_{u\in\mathcal{M}} |g(u)|$ for all $g \in C(\mathcal{M}, \mathbb{K})$.

Suppose that $\mathbb{K}$ is a spherically complete, we get the following:

Theorem 2.5. [19] *For each $u \in \mathcal{E}\backslash\{0\}$, there is $u^* \in \mathcal{E}^*$ with $u^*(u) = 1$ and $\|u^*\| = \|u\|^{-1}$.*

Lemma 2.6. [13] *Let $\mathcal{E}$ be an ultrametric normed space over $\mathbb{K}$. If $f_1^*, \cdots, f_n^*$ are linearly independent vectors in $\mathcal{E}^*$, hence there are vectors $f_1, \cdots, f_n$ in $\mathcal{E}$ with*

$$f_i^*(f_k) = \delta_{i,k} = \begin{cases} 1, & \text{if } i = k; \\ 0, & \text{if } i \neq k. \end{cases} \quad 1 \leq i, k \leq n. \tag{2.1}$$

Moreover, if $f_1, \cdots, f_n$ are linearly independent vectors in $\mathcal{E}$, hence there are vectors $f_1^*, \cdots, f_n^*$ in $\mathcal{E}^*$ with (2.1) holds.

Lemma 2.7. [7] *Let $\mathcal{E}, \mathcal{F}$ be two ultrametric normed spaces over $\mathbb{K}$. If $S \in \mathcal{B}(\mathcal{E}, \mathcal{F})$, hence $R(S)^\perp = N(S^*)$.*

Definition 2.8. [14] Let $S \in \mathcal{B}(\mathcal{E})$. S is called an upper semi-Fredholm operator if

$$\alpha(S) = \dim N(S) \text{ is finite and R(S) is closed.}$$

$\Phi_+(\mathcal{E})$ is the collection of each continuous upper semi-Fredholm operators on $\mathcal{E}$.

Definition 2.9. [14] Let $S \in \mathcal{B}(\mathcal{E})$. S is said to be a lower semi-Fredholm operator if

$$\beta(S) = \dim(\mathcal{E}/R(S)) \text{ is finite.}$$

The set of each continuous lower semi-Fredholm operators on $\mathcal{E}$ is $\Phi_-(\mathcal{E})$. The collection of each continuous Fredholm operators on $\mathcal{E}$ is

$$\Phi(\mathcal{E}) = \Phi_+(\mathcal{E}) \cap \Phi_-(\mathcal{E}).$$

If $S \in \Phi(\mathcal{E})$, hence the index $ind(S)$ of S is $ind(S) = \alpha(S) - \beta(S)$.

For additional details on ultrametric continuous Fredholm operators, see [5], [13], [14] and [19].

Definition 2.10. [4] Let $S \in \mathcal{B}(\mathcal{E})$. S is said to an operator of finite rank if $\dim R(S)$ is finite.

The set of each finite rank operators on $\mathcal{E}$ is $\mathcal{F}_0(\mathcal{E})$.

Definition 2.11. [4] Let $S \in \mathcal{B}(\mathcal{E})$. S is said to be completely continuous if, there is a sequence $(S_n)_{n\in\mathbb{N}} \in \mathcal{F}_0(\mathcal{E}) : \|S_n - S\| \to 0$ as $n \to \infty$.

$\mathcal{C}_c(\mathcal{E})$ is the set of each completely continuous linear operators on $\mathcal{E}$. In the following lemmas, we suppose that $\mathbb{K}$ is a spherically complete.

Lemma 2.12. [5] *If $S \in \Phi(\mathcal{E})$ and $C \in \mathcal{C}_c(\mathcal{E})$, hence $S + C \in \Phi(\mathcal{E})$.*

Lemma 2.13. [2] *If $S \in \Phi(\mathcal{E})$, hence for each $C \in \mathcal{C}_c(\mathcal{E})$, $S + C \in \Phi(\mathcal{E})$ and $ind(S + C) = ind(S)$.*

We obtain the following definitions.

Definition 2.14. [4] Let $S \in \mathcal{B}(\mathcal{E})$. The resolvent set $\rho(S)$ of S on $\mathcal{E}$ is

$$\rho(S) = \{\mu \in \mathbb{K} : (S - \mu I)^{-1} \in \mathcal{B}(\mathcal{E})\}.$$

The spectrum $\sigma(S)$ of S is $\mathbb{K}\backslash\rho(S)$.

Definition 2.15. [15] Let $S \in \mathcal{B}(\mathcal{E})$. Set $\nu(S) = \inf_n \|S^n\|^{\frac{1}{n}} = \lim_n \|S^n\|^{\frac{1}{n}}$, S is called a spectral operator if

$$\sup\{|\mu| : \mu \in \sigma(S)\} = \nu(S).$$

Set

$$U_S = \{\mu \in \mathbb{K} : (I - \mu S)^{-1} \in \mathcal{B}(\mathcal{E})\}.$$

(U_S is open and $0 \in U_S$) and

$$C_S = \{\beta \in \mathbb{K} : B(0, |\gamma|) \subset U_S \text{ for some } \beta \in \mathbb{K},\ |\gamma| > |\beta|\}.$$

Definition 2.16. [1] Let $S \in \mathcal{B}(\mathcal{E})$ and $\varepsilon > 0$. The pseudospectrum $\sigma_\varepsilon(S)$ of S on $\mathcal{E}$ is

$$\sigma_\varepsilon(S) = \sigma(S) \cup \{\mu \in \mathbb{K} : \|(S - \mu I)^{-1}\| > \frac{1}{\varepsilon}\},$$

with the convention $\|(S - \mu I)^{-1}\| = \infty$ if $\mu \in \sigma(S)$.

Definition 2.17. [2] Let $S \in \mathcal{B}(\mathcal{E})$ and $\varepsilon > 0$. The condition pseudospectrum $\Lambda_\varepsilon(S)$ of S is

$$\Lambda_\varepsilon(S) = \sigma(S) \cup \{\mu \in \mathbb{K} : \|S - \mu I\|\|(S - \mu I)^{-1}\| > \frac{1}{\varepsilon}\},$$

with the convention $\|S - \mu I\|\|(S - \mu I)^{-1}\| = \infty$ if $\mu \in \sigma(S)$.

We generalize the Definition 3.1 of [1] as follows.

Definition 2.18. Let $S, B \in \mathcal{B}(\mathcal{E})$ and $\varepsilon > 0$. The pseudospectrum $\sigma_\varepsilon(S,B)$ of (S,B) is

$$\sigma_\varepsilon(S,B) = \sigma(S,B) \cup \{\mu \in \mathbb{K} : \|(S-\mu B)^{-1}\| > \frac{1}{\varepsilon}\},$$

with the convention $\|(S-\mu B)^{-1}\| = \infty$ if $\mu \in \sigma(S,B)$.

Similarly to the proof of Proposition 3.1 of [1], we obtain the following proposition.

Proposition 2.19. *If $S, B \in \mathcal{B}(\mathcal{E})$ and $\varepsilon > 0$, hence*
(i) $\sigma(S,B) = \bigcap_{\varepsilon > 0} \sigma_\varepsilon(S,B)$;
(ii) *If $0 < \varepsilon_1 < \varepsilon_2$, hence $\sigma(S,B) \subset \sigma_{\varepsilon_1}(S,B) \subset \sigma_{\varepsilon_2}(S,B)$.*

Similarly to the proof of Theorem 3.4 of [1], we get the following theorem.

Theorem 2.20. *Assume that $\mathbb{K}$ is a spherically complete and $\|\mathcal{E}\| \subseteq |\mathbb{K}|$. If $S, B \in \mathcal{B}(\mathcal{E})$ and $\varepsilon > 0$. Hence*

$$\sigma_\varepsilon(S,B) = \bigcup_{D \in \mathcal{B}(\mathcal{E}) : \|D\| < \varepsilon} \sigma(S+D, B).$$

Definition 2.21. [8] Let $S, B \in \mathcal{B}(\mathcal{E})$. The essential spectrum $\sigma_e(S,B)$ of (S,B) is

$$\sigma_e(S,B) = \{\mu \in \mathbb{K} : S - \mu B \text{ is not a Fredholm operator of index } 0\}.$$

Theorem 2.22. [8] *Suppose that $\mathbb{K}$ is a spherically complete. If $S, B \in \mathcal{B}(\mathcal{E})$ with S^*, B^* exist and $\varepsilon > 0$. Hence*

$$\sigma_e(S,B) = \bigcap_{K \in \mathcal{C}_c(\mathcal{E})} \sigma(S+K, B).$$

We obtain the following:

Definition 2.23. [8] Let $S, B \in \mathcal{B}(\mathcal{E})$ and $\varepsilon > 0$. The essential pseudospectrum $\sigma_{e,\varepsilon}(S,B)$ of (S,B) is

$$\sigma_{e,\varepsilon}(S,B) = \mathbb{K} \backslash \{\mu \in \mathbb{K} : S + D - \mu B \in \Phi_0(\mathcal{E}) \text{ for all } D \in \mathcal{B}(\mathcal{E}),\ \|D\| < \varepsilon\},$$

where $\Phi_0(\mathcal{E})$ is the collection of each continuous Fredholm operators on $\mathcal{E}$ of index 0.

Theorem 2.24. [8] *If $S, B \in \mathcal{B}(\mathcal{E})$ and $\varepsilon > 0$, hence*

$$\sigma_{e,\varepsilon}(S,B) = \bigcup_{C \in \mathcal{B}(\mathcal{E}) : \|C\| < \varepsilon} \sigma_e(S+C, B).$$

3. Main Results

We have the following results.

Theorem 3.1. *Suppose that* $\mathbb{K}$ *is a spherically complete and* $\|\mathcal{E}\| \subseteq |\mathbb{K}|$. *If* $S, B \in \mathcal{B}(\mathcal{E})$ *and* $\varepsilon > 0$. *Hence*

$$\sigma_\varepsilon(S,B) = \sigma(S,B) \cup \{\mu \in \mathbb{K} : \exists\, u \in \mathcal{E}, \|u\| = 1,\ \|(S-\mu B)u\| < \varepsilon\}.$$

Proof. If $\mu \in \sigma_\varepsilon(S,B)\backslash\sigma(S,B)$, hence $\|(S-\mu B)^{-1}\| > \varepsilon^{-1}$. Thus

$$\sup_{u\in\mathcal{E}\backslash\{0\}} \frac{\|(S-\mu B)^{-1}u\|}{\|u\|} > \frac{1}{\varepsilon}.$$

Hence there is $u \in \mathcal{E}\backslash\{0\}$ with

$$\|(S-\mu B)^{-1}u\| > \frac{\|u\|}{\varepsilon}. \tag{3.1}$$

Set $y = (S-\mu B)^{-1}u \in \mathcal{E}\backslash\{0\}$, then $(S-\mu B)y = u$. By (3.1),

$$\|(S-\mu B)y\| < \varepsilon\|y\|. \tag{3.2}$$

Since $\|\mathcal{E}\| \subseteq |\mathbb{K}|$, there is $c \in \mathbb{K}\backslash\{0\}$ with $\|y\| = |c|$, put $z = c^{-1}y$, hence $\|z\| = 1$. By (3.2), we have

$$\|(S-\mu B)z\| < \varepsilon. \tag{3.3}$$

Conversely, suppose that there is $z \in \mathcal{E}$ with $\|z\| = 1$ and

$$\|(S-\mu B)z\| < \varepsilon. \tag{3.4}$$

From Theorem 2.5, there exists $\varphi \in \mathcal{E}^*$ with $\varphi(z) = 1$ and $\|\varphi\| = \|z\|^{-1} = 1$. Set for each $y \in \mathcal{E}$, $Cy = \varphi(y)(\mu B - S)z$. Hence for any $y \in \mathcal{E}$,

$$\begin{aligned} \|Cy\| &= |\varphi(y)|\|(S-\mu B)z\| \\ &\leq \|\varphi\|\|y\|\|(S-\mu B)z\| \\ &< \varepsilon\|y\|. \end{aligned}$$

Thus $C \in \mathcal{B}(\mathcal{E})$ and $\|C\| < \varepsilon$. Moreover for $z \neq 0$, $Cz + (S-\mu B)z = 0$. Then $\mu \in \bigcup\limits_{C\in\mathcal{B}(\mathcal{E}):\|C\|<\varepsilon} \sigma(S+C,B)$. By Theorem 2.20, $\mu \in \sigma_\varepsilon(S,B)$. □

Theorem 3.2. *Assume that* $\mathbb{K}$ *is a spherically complete. Let* $S, B \in \mathcal{B}(\mathcal{E})$ *and* $\varepsilon > 0$. *Hence*

$$\sigma_{e,\varepsilon}(S,B) = \sigma_{e,\varepsilon}(S+K,B) \text{ for each } K \in \mathcal{C}_c(\mathcal{E}).$$

Proof. If $\mu \notin \sigma_{e,\varepsilon}(S,B)$, hence for each $D \in \mathcal{B}(\mathcal{E})$ with $\|D\| < \varepsilon$,

$$S + D - \mu B \in \Phi(\mathcal{E}) \text{ and } ind(S+D-\mu B) = 0.$$

From Lemma 2.12 and Lemma 2.13, for all $K \in \mathcal{C}_c(\mathcal{E})$ and $D \in \mathcal{B}(\mathcal{E})$ with $\|D\| < \varepsilon$, we obtain

$$S + D + K - \mu B \in \Phi(\mathcal{E}) \text{ and } ind(S+D+K-\mu B) = 0. \tag{3.5}$$

By (3.5), we get

$$\mu \notin \sigma_{e,\varepsilon}(S+K,B).$$

Then

$$\sigma_{e,\varepsilon}(S+K,B)\subseteq\sigma_{e,\varepsilon}(S,B).$$

Similarly, we get the opposite inclusion. □

Remark 3.3. From Theorem 3.2, the essential pseudospectrum of bounded linear operator pencils is invariant under perturbation of completely continuous linear operators on $\mathcal{E}$ over a spherically complete field $\mathbb{K}$.

Theorem 3.4. *Assume that $\mathbb{K}$ is a spherically complete and $\|\mathcal{E}\|\subseteq|\mathbb{K}|$. If $S,B\in\mathcal{B}(\mathcal{E})$ with S^*,B^* exist and $\varepsilon>0$, hence*

$$\sigma_{e,\varepsilon}(S,B)=\bigcap_{K\in\mathcal{C}_c(\mathcal{E})}\sigma_\varepsilon(S+K,B).$$

Proof. If $\mu\notin\bigcap_{K\in\mathcal{C}_c(\mathcal{E})}\sigma_\varepsilon(S+K,B)$, hence there exists $K\in\mathcal{C}_c(\mathcal{E}):\mu\notin\sigma_\varepsilon(S+K,B)$. By Theorem 2.20, for any $D\in\mathcal{B}(\mathcal{E})$ with $\|D\|<\varepsilon$, $(S+K+D-\mu B)^{-1}\in\mathcal{B}(\mathcal{E})$. Hence for any $D\in\mathcal{B}(\mathcal{E})$ with $\|D\|<\varepsilon$,

$$S+K+D-\mu B\in\Phi(\mathcal{E})\text{ and }ind(S+K+D-\mu B)=0.\tag{3.6}$$

By Lemma 2.12 and Lemma 2.13, for each $D\in\mathcal{B}(\mathcal{E}):\|D\|<\varepsilon$, we get

$$S+D-\mu B\in\Phi(\mathcal{E})\text{ and }ind(S+D-\mu B)=0.\tag{3.7}$$

Then

$$\mu\notin\sigma_{e,\varepsilon}(S,B).$$

Thus

$$\sigma_{e,\varepsilon}(S,B)\subseteq\bigcap_{K\in\mathcal{C}_c(\mathcal{E})}\sigma_\varepsilon(S+K,B).\tag{3.8}$$

Conversely, assume that $\mu\notin\sigma_{e,\varepsilon}(S,B)$. From Theorem 2.24, for each $D\in\mathcal{B}(\mathcal{E})$ with $\|D\|<\varepsilon$, $\mu\notin\sigma_e(S+D,B)$. By Theorem 2.22, there is $K\in\mathcal{C}_c(\mathcal{E}):\mu\notin\sigma(S+D+K,B)$, hence for each $D\in\mathcal{B}(\mathcal{E})$ with $\|D\|<\varepsilon$, $\mu\in\rho(S+K+D,B)$. Then

$$\mu\in\bigcap_{D\in\mathcal{B}(\mathcal{E}):\|D\|<\varepsilon}\rho(S+K+D,B).\tag{3.9}$$

By Theorem 2.20, $\mu\notin\sigma_\varepsilon(S+K,B)$. Consequently,

$$\mu\notin\bigcap_{K\in\mathcal{C}_c(\mathcal{E})}\sigma_\varepsilon(S+K,B).$$

Thus

$$\sigma_{e,\varepsilon}(S,B)=\bigcap_{K\in\mathcal{C}_c(\mathcal{E})}\sigma_\varepsilon(S+K,B).$$

□

Remark 3.5. Assume that $\mathbb{K}$ is a spherically complete and $\|\mathcal{E}\|\subseteq|\mathbb{K}|$. If $S,B\in\mathcal{B}(\mathcal{E})$ and $\varepsilon>0$, hence from Example 3.31 of [4] and Theorem 3.4, we obtain

$$\sigma_{e,\varepsilon}(S,B)=\bigcap_{F\in\mathcal{F}_0(\mathcal{E})}\sigma_\varepsilon(S+F,B).$$

Proposition 3.6. *Assume that* $\mathbb{K}$ *is a spherically complete and* $\|\mathcal{E}\| \subseteq |\mathbb{K}|$. *If* $S, B \in \mathcal{B}(\mathcal{E})$, *hence*

(i) *For any* $\varepsilon > 0, \sigma_{e,\varepsilon}(S, B) \subset \sigma_\varepsilon(S, B)$;

(ii) *For each* ε_1 *and* $\varepsilon_2 : 0 < \varepsilon_1 < \varepsilon_2$, *we get*

$$\sigma_e(S, B) \subset \sigma_{e,\varepsilon_1}(S, B) \subset \sigma_{e,\varepsilon_2}(S, B).$$

Proof.

(i) Let $\mu \in \sigma_{e,\varepsilon}(S, B)$. From Theorem 2.24, $\mu \in \bigcup_{D\in\mathcal{B}(\mathcal{E}):\|D\|<\varepsilon} \sigma_e(S + D, B)$. Since $\sigma_e(S + D, B) \subset \sigma(S + D, B)$, we obtain $\mu \in \bigcup_{D\in\mathcal{B}(\mathcal{E}):\|D\|<\varepsilon} \sigma(S + D, B)$. From Theorem 2.20, $\mu \in \sigma_\varepsilon(S, B)$.

(ii) Firstly, we show that for each $\varepsilon > 0$,

$$\sigma_e(S, B) \subset \sigma_{e,\varepsilon}(S, B).$$

If $\mu \notin \sigma_{e,\varepsilon}(S, B)$, hence for each $D \in \mathcal{B}(\mathcal{E})$ with $\|D\| < \varepsilon$, we get $S + D - \mu B \in \Phi(\mathcal{E})$ and $ind(S + D - \mu B) = 0$. As $\varepsilon \to 0$, $S - \mu B \in \Phi(\mathcal{E})$ and $ind(S - \mu B) = 0$. Hence $\mu \notin \sigma_e(S, B)$. Thus

$$\sigma_e(S, B) \subset \sigma_{e,\varepsilon}(S, B).$$

If $\mu \notin \sigma_{e,\varepsilon_2}(S, B)$, hence for each $D \in \mathcal{B}(\mathcal{E})$ with $\|D\| < \varepsilon_2$, $S + D - \mu B \in \Phi(\mathcal{E})$ and $ind(S + D - \mu B) = 0$. Since $\varepsilon_1 < \varepsilon_2$, we get for each $D \in \mathcal{B}(\mathcal{E})$ with $\|D\| < \varepsilon_1$, $S + D - \mu B \in \Phi(\mathcal{E})$ and $ind(S + D - \mu B) = 0$, hence $\mu \notin \sigma_{e,\varepsilon_1}(S, B)$. Consequently, $\sigma_{e,\varepsilon_1}(S, B) \subset \sigma_{e,\varepsilon_2}(S, B)$. $\square$

Proposition 3.7. *Assume that* $\mathbb{K}$ *is a spherically complete and* $\|\mathcal{E}\| \subseteq |\mathbb{K}|$. *If* $S, B \in \mathcal{B}(\mathcal{E}) : S^*, B^*$ *exist and* $\varepsilon > 0$. *Hence*

$$\sigma_e(S, B) = \bigcap_{\varepsilon>0} \sigma_{e,\varepsilon}(S, B).$$

Proof. Suppose that $\mu \in \bigcap_{\varepsilon>0} \sigma_{e,\varepsilon}(S, B)$ and $\|\mathcal{E}\| \subseteq |\mathbb{K}|$. By Theorem 3.4,

$$\bigcap_{\varepsilon>0} \sigma_{e,\varepsilon}(S, B) = \bigcap_{\varepsilon>0} \bigcap_{K\in\mathcal{C}_c(\mathcal{E})} \sigma_\varepsilon(S + K, B) = \bigcap_{K\in\mathcal{C}_c(\mathcal{E})} \bigcap_{\varepsilon>0} \sigma_\varepsilon(S + K, B). \tag{3.10}$$

From (i) of Proposition 2.19, $\bigcap_{\varepsilon>0} \sigma_\varepsilon(S + K, B) = \sigma(S + K, B)$. By (3.10), we get $\bigcap_{\varepsilon>0} \sigma_{e,\varepsilon}(S, B) = \bigcap_{K\in\mathcal{C}_c(\mathcal{E})} \sigma(S + K, B)$. By Theorem 2.22, $\mu \in \sigma_e(S, B)$. Conversely, assume that $\mu \in \sigma_e(S, B)$. By Theorem 2.22, $\mu \in \bigcap_{K\in\mathcal{C}_c(\mathcal{E})} \sigma(S + K, B)$. By (i) of Proposition 2.19, $\mu \in \bigcap_{\varepsilon>0} \bigcap_{K\in\mathcal{C}_c(\mathcal{E})} \sigma_\varepsilon(S+K, B)$. Since $\|\mathcal{E}\| \subseteq |\mathbb{K}|$ and from Theorem 2.20,

$$\mu \in \bigcap_{\varepsilon>0} \bigcup_{D\in\mathcal{B}(\mathcal{E}):\|D\|<\varepsilon} \bigcap_{K\in\mathcal{C}_c(\mathcal{E})} \sigma(S + D + K, B).$$

By Theorem 2.22, $\mu \in \bigcap_{\varepsilon>0} \bigcup_{D\in\mathcal{B}(\mathcal{E}):\|D\|<\varepsilon} \sigma_e(S+D,B)$. From Theorem 2.24, $\mu \in \bigcap_{\varepsilon>0} \sigma_{e,\varepsilon}(S,B)$. Consequently,

$$\sigma_e(S,B) = \bigcap_{\varepsilon>0} \sigma_{e,\varepsilon}(S,B).$$

□

We introduce the next definition.

Definition 3.8. Let $S \in \mathcal{B}(\mathcal{E}), n \in \mathbb{N}$ and $\varepsilon > 0$, the (n,ε)-pseudospectrum $\sigma_{n,\varepsilon}(S)$ of S is

$$\sigma_{n,\varepsilon}(S) = \sigma(S) \cup \{\mu \in \mathbb{K} : \|(S-\mu I)^{-2^n}\|^{\frac{1}{2^n}} > \frac{1}{\varepsilon}\},$$

by convention $\|(S-\mu I)^{-2^n}\|^{\frac{1}{2^n}} = \infty$ if $\mu \in \sigma(S)$.

From Definition 3.8, we have the following remark.

Remark 3.9. If $n = 0$, then $\sigma_{0,\varepsilon}(S) = \sigma_\varepsilon(S)$ is the pseudospectrum of S.

We continue by the following results.

Theorem 3.10. *Let $S \in \mathcal{B}(\mathcal{E}), n \in \mathbb{N}$ and $\varepsilon > 0$. Hence*
(i) *For each $\varepsilon_1, \varepsilon_2$ with $\varepsilon_1 \le \varepsilon_2$, $\sigma_{n,\varepsilon_1}(S) \subset \sigma_{n,\varepsilon_2}(S)$;*
(ii) $\sigma(S) = \bigcap_{\varepsilon>0} \sigma_{n,\varepsilon}(S)$.

Proof.
(i) If $\mu \in \sigma_{n,\varepsilon_1}(S)$, then $\|(S-\mu I)^{-2^n}\|^{\frac{1}{2^n}} > \varepsilon_1^{-1} \ge \varepsilon_2^{-1}$, thus $\mu \in \sigma_{n,\varepsilon_2}(S)$.
(ii) Since for each $\varepsilon > 0$, $\sigma(S) \subseteq \sigma_{n,\varepsilon}(S)$, hence $\sigma(S) \subseteq \bigcap_{\varepsilon>0} \sigma_{n,\varepsilon}(S)$. For the converse inclusion, if $\mu \in \bigcap_{\varepsilon>0} \sigma_{n,\varepsilon}(S)$, hence for each $\varepsilon > 0$, $\mu \in \sigma_{n,\varepsilon}(S)$. If $\mu \notin \sigma(S)$, hence $\mu \in \{\mu \in \mathbb{K} : \|(S-\mu I)^{-2^n}\|^{\frac{1}{2^n}} > \varepsilon^{-1}\}$, for $\varepsilon \to 0^+$, we obtain $\|(S-\mu I)^{-2^n}\|^{\frac{1}{2^n}} = \infty$ which is a contradiction with $(S-\mu I)^{-1}$ is bounded on $\rho(S)$. Consequently $\mu \in \sigma(S)$. □

We have the following:

Lemma 3.11. *For each $n \in \mathbb{N}$ and $\varepsilon > 0$, hence $\sigma_{n,\varepsilon}(\lambda I) = \{\lambda\} \cup B(\lambda,\varepsilon)$, where $B(\lambda,\varepsilon) = \{\mu \in \mathbb{K} : |\mu - \lambda| < \varepsilon\}$.*

Proof. It follows from Definition 3.8. □

Corollary 3.12. *Let $S \in \mathcal{B}(\mathcal{E})$ and $\varepsilon > 0$, hence for any $n \in \mathbb{N}$, $\sigma_{n+1,\varepsilon}(S) \subset \sigma_{n,\varepsilon}(S)$.*

Proof. If $\mu \in \sigma_{n+1,\varepsilon}(S)$, hence

$$\begin{aligned} \frac{1}{\varepsilon} &< \|(S-\mu I)^{-2^{n+1}}\|^{\frac{1}{2^{n+1}}} \\ &\leq \|(S-\mu I)^{-2^{n}}\|^{\frac{1}{2^{n}}}, \end{aligned}$$

thus $\mu \in \sigma_{n,\varepsilon}(S)$. □

Proposition 3.13. *Let $S \in \mathcal{B}(\mathcal{E}), n \in \mathbb{N}$ and $\varepsilon > 0$. Hence for any $\lambda \in \mathbb{K}^*, \sigma_{n,\varepsilon}(\lambda S) = \lambda \sigma_{n,\frac{\varepsilon}{|\lambda|}}(S)$.*

Proof. Let $\mu \in \sigma_{n,\varepsilon}(\lambda S)$, then $\mu \in \sigma(\lambda S)$ or $\|(\lambda S-\mu I)^{-2^n}\|^{\frac{1}{2^n}} > \frac{1}{\varepsilon}$, hence $\frac{\mu}{\lambda} \in \sigma(S)$ or $\|(S-\frac{\mu}{\lambda}I)^{-2^n}\|^{\frac{1}{2^n}} > \frac{|\lambda|}{\varepsilon}$, thus $\mu \in \lambda\sigma_{n,\frac{\varepsilon}{|\lambda|}}(S)$. Similarly, we obtain the converse inclusion. □

Proposition 3.14. *Let $S \in \mathcal{B}(\mathcal{E}), n \in \mathbb{N}$ and $\varepsilon > 0$. Then for any $\lambda \in \mathbb{K}$, $\sigma_{n,\varepsilon}(S+\lambda I) = \sigma_{n,\varepsilon}(S) + \lambda$.*

Proof. One can see that $\sigma(S+\lambda I) = \sigma(S) + \lambda$. Moreover,

$$\|(\lambda I + S - \mu I)^{-2^n}\|^{\frac{1}{2^n}} = \|(S-(\mu-\lambda)I)^{-2^n}\|^{\frac{1}{2^n}},$$

then $\mu \in \sigma_{n,\varepsilon}(S+\lambda I)$ if and only if $\mu \in \sigma_{n,\varepsilon}(S) + \lambda$. □

Proposition 3.15. *Let $S \in \mathcal{B}(\mathcal{E})$ and $\lambda \notin \sigma(S)$, then $\mu \in \sigma((S-\lambda I)^{-1})$ if and only if $\lambda + \frac{1}{\mu} \in \sigma(S)$.*

Proof. For each $\lambda \notin \sigma(S)$,

$$(S-\lambda I)^{-1} - \mu I = (S-\lambda I)^{-1}[I - \mu(S-\lambda I)] = (S-\lambda I)^{-1}\mu[(\lambda + \frac{1}{\mu})I - S].$$

Then $\mu \in \sigma((S-\lambda I)^{-1})$ if and only if $\lambda + \frac{1}{\mu} \in \sigma(S)$. □

We introduce the following definition.

Definition 3.16. Let $S \in \mathcal{B}(\mathcal{E})$, S is said to be a G_1-operator if for all $\mu \notin \sigma(S)$, $\|(S-\mu I)^{-1}\| = \frac{1}{d(\mu,\sigma(S))}$.

We give an example of G_1-operators.

Example 3.17. If $a \in \mathbb{K}^*$ and $S = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \in \mathcal{M}_2(\mathbb{K})$, then S is a G_1-operator.

Proposition 3.18. *Let $S \in \mathcal{B}(\mathcal{E})$ be a G_1-operator and $\varepsilon > 0$, then $\sigma_\varepsilon(S) = \sigma(S) \cup \{\mu \in \mathbb{K} : d(\mu, \sigma(S)) < \varepsilon\}$.*

Proof. Since S is a G_1-operator, we have for all $\mu \notin \sigma(S)$, $\|(S-\mu I)^{-1}\| = \frac{1}{d(\mu,\sigma(S))}$. Let $\mu \in \sigma(S)$ or $d(\mu,\sigma(S)) = \|(S-\mu I)^{-1}\|^{-1} < \varepsilon$, then $\sigma_\varepsilon(S) = \sigma(S) \cup \{\mu \in \mathbb{K} : d(\mu,\sigma(S)) < \varepsilon\}$. □

Proposition 3.19. *Let $S \in \mathcal{B}(\mathcal{E})$. Then*

(i) *If $S = \lambda I$ for some $\lambda \in \mathbb{K}$, then S is a G_1-operator and $\sigma(S) = \{\lambda\}$;*
(ii) *If S is a G_1-operator, hence $\lambda S + \alpha I$ is a G_1-operator for any $\lambda, \alpha \in \mathbb{K}$.*

Proof.
(i) If $S=\lambda I$, hence $\sigma(S)=\{\lambda\}$ and for all $\mu\notin\sigma(S)$, $\|(S-\mu I)^{-1}\|=\frac{1}{|\mu-\lambda|}=\frac{1}{d(\mu,\sigma(S))}$. Thus S is a G_1-operator.
(ii) Since S is a G_1-operator, we have for all $\mu\notin\sigma(S)$, $\|(S-\mu I)^{-1}\|=\frac{1}{d(\mu,\sigma(S))}$. If $\lambda=0$, it follows from (i). If $\lambda\neq 0$, hence $\mu\notin\sigma(\lambda S+\alpha I)$ if and only if $\frac{\mu-\alpha}{\lambda}\notin\sigma(S)$. Put $\beta=\frac{\mu-\alpha}{\lambda}$, then

$$\begin{aligned}\|((\lambda S+\alpha I)-\mu I)^{-1}\| &= \|(\lambda S-(\mu-\alpha)I)^{-1}\|\\ &= \frac{\|(S-\frac{\mu-\alpha}{\lambda}I)^{-1}\|}{|\lambda|}\\ &= \frac{1}{|\lambda|d(\frac{\mu-\alpha}{\lambda},\sigma(S))}\\ &= \frac{1}{d(\mu,\sigma(\lambda S+\alpha I))}.\end{aligned}$$

□

Proposition 3.20. *Let $S\in\mathcal{B}(\mathcal{E})$ be a spectral operator, hence for any $\varepsilon>0,\sigma_\varepsilon(S)\subset B(0,\|S\|+\varepsilon)$.*

Proof. Since S is a spectral, then $\sup\limits_{\mu\in\sigma(S)}|\mu|\leq\|S\|$, hence for each $\mu\in\sigma(S),|\mu|\leq\|S\|$, thus $\sigma(S)\subset\overline{B}(0,\|S\|)$. Now let $\varepsilon>0$ and let $\mu\notin B(0,\|S\|+\varepsilon)$, hence $|\mu|\geq\|S\|+\varepsilon>\|S\|$, then

$$\|(S-\mu I)^{-1}\|\leq\sup_{i\in\mathbb{N}}\frac{\|S\|^i}{|\mu|^{i+1}}=\frac{1}{|\mu|}\leq\frac{1}{\varepsilon+\|S\|},$$

thus $\mu\notin\sigma_{\varepsilon+\|S\|}(S)$. Consequently, $\sigma_{\varepsilon+\|S\|}(S)\subset B(0,\varepsilon+\|S\|)$. Since $\varepsilon\leq\varepsilon+\|S\|$ and from Proposition 2.19, $\sigma_\varepsilon(S)\subset\sigma_{\varepsilon+\|S\|}(S)$, hence for any $\varepsilon>0,\sigma_\varepsilon(S)\subset B(0,\varepsilon+\|S\|)$. □

In the next proposition we assume that for any $\mu\in\mathbb{K},S\neq\mu I$.

Proposition 3.21. *Let $S\in\mathcal{B}(\mathcal{E})$ be not a scalar multiple of identity. Set $M=\inf\{\|S-zI\|:z\in\mathbb{K}\}$, hence for any $\varepsilon>0$, $\sigma_\varepsilon(S)\subset\Lambda_{\frac{\varepsilon}{M}}(S)$.*

Proof. If $\mu\in\sigma_\varepsilon(S)$, hence $\mu\in\sigma(S)$ or $\|(S-\mu I)^{-1}\|>\frac{1}{\varepsilon}$. Moreover $\|S-\mu I\|\geq M>0$, hence $\|S-\mu I\|\|(S-\mu I)^{-1}\|>\frac{M}{\varepsilon}$. Thus $\mu\in\Lambda_{\frac{\varepsilon}{M}}(S)$. □

The following definition is introduced.

Definition 3.22. Let $S,B\in\mathcal{B}(\mathcal{E}),n\in\mathbb{N}$ and $\varepsilon>0$, the (n,ε)-pseudospectrum $\sigma_{n,\varepsilon}(S,B)$ of the operator pencil of the form $S-\mu B$ is

$$\sigma_{n,\varepsilon}(S,B)=\sigma(S,B)\cup\{\mu\in\mathbb{K}:\|(S-\mu B)^{-2^n}\|^{\frac{1}{2^n}}>\frac{1}{\varepsilon}\},$$

with the convention $\|(S-\mu B)^{-2^n}\|=\infty$, if $\mu\in\sigma(S,B)$.

By Definition 3.22, we get the following remark.

Remark 3.23. If $n = 0$, then $\sigma_{0,\varepsilon}(S,B) = \sigma_\varepsilon(S,B)$ is the pseudospectrum of the operator pencil (S,B).

We get the following results.

Theorem 3.24. *If $S, B \in \mathcal{B}(\mathcal{E}), n \in \mathbb{N}$ and $\varepsilon > 0$, hence*
(i) *For each $\varepsilon_1, \varepsilon_2$ with $\varepsilon_1 \le \varepsilon_2$, $\sigma_{n,\varepsilon_1}(S,B) \subset \sigma_{n,\varepsilon_2}(S,B)$;*
(ii) $\sigma(S,B) = \bigcap\limits_{\varepsilon>0} \sigma_{n,\varepsilon}(S,B)$.

Proof.
(i) If $\mu \in \sigma_{n,\varepsilon_1}(S,B)$, hence $\|(S-\mu B)^{-2^n}\|^{\frac{1}{2^n}} > \varepsilon_1^{-1} \ge \varepsilon_2^{-1}$, thus $\mu \in \sigma_{n,\varepsilon_2}(S,B)$.
(ii) Since for any $\varepsilon > 0$, $\sigma(S,B) \subseteq \sigma_{n,\varepsilon}(S,B)$, hence $\sigma(S,B) \subseteq \bigcap\limits_{\varepsilon>0} \sigma_{n,\varepsilon}(S,B)$. For the converse inclusion, if $\mu \in \bigcap\limits_{\varepsilon>0} \sigma_{n,\varepsilon}(S,B)$, hence for any $\varepsilon > 0$, $\mu \in \sigma_{n,\varepsilon}(S,B)$. If $\mu \notin \sigma(S,B)$, thus $\mu \in \{\mu \in \mathbb{K} : \|(S-\mu B)^{-2^n}\|^{\frac{1}{2^n}} > \varepsilon^{-1}\}$, for $\varepsilon \to 0^+$, we obtain $\|(S-\mu B)^{-2^n}\|^{\frac{1}{2^n}} = \infty$ which is a contradiction with $\mu \notin \sigma(S,B)$. Consequently $\mu \in \sigma(S,B)$. □

Corollary 3.25. *If $S, B \in \mathcal{B}(\mathcal{E})$ and $\varepsilon > 0$, then for any $n \in \mathbb{N}$, $\sigma_{n+1,\varepsilon}(S,B) \subset \sigma_{n,\varepsilon}(S,B)$.*

Proof. If $\mu \in \sigma_{n+1,\varepsilon}(S,B)$, hence

$$\begin{aligned} \frac{1}{\varepsilon} &< \|(S-\mu B)^{-2^{n+1}}\|^{\frac{1}{2^{n+1}}} && (3.11)\\ &\le \|(S-\mu B)^{-2^n}\|^{\frac{1}{2^n}}, && (3.12)\end{aligned}$$

thus $\mu \in \sigma_{n,\varepsilon}(S,B)$. □

Proposition 3.26. *If $S, B \in \mathcal{B}(\mathcal{E}), n \in \mathbb{N}$ and $\varepsilon > 0$. Then for each $\lambda \in \mathbb{K}^*$ and for each $\alpha \in \mathbb{K}, \sigma_{n,\varepsilon}(\lambda S + \alpha B, B) = \alpha + \lambda \sigma_{n,\frac{\varepsilon}{|\lambda|}}(S,B)$.*

Proof. If $\mu \notin \sigma(\lambda S + \alpha B, B)$, then

$$\begin{aligned} (\lambda S + \alpha B - \mu B)^{-1} \in \mathcal{B}(\mathcal{E}) &\Leftrightarrow \lambda^{-1}(S - \frac{\mu-\alpha}{\lambda}B)^{-1} \in \mathcal{B}(\mathcal{E})\\ &\Leftrightarrow (S - \frac{\mu-\alpha}{\lambda}B)^{-1} \in \mathcal{B}(\mathcal{E}),\end{aligned}$$

hence $\sigma(\lambda S + \alpha B, B) = \alpha + \lambda\sigma(S,B)$. Let $\mu \in \sigma_\varepsilon(\lambda S + \alpha B, B)$, then

$$\begin{aligned} \frac{1}{\varepsilon} &< \|(\lambda S + \alpha B - \mu B)^{-2^n}\|^{\frac{1}{2^n}} && (3.13)\\ &\le |\lambda|^{-1}\|(S - \frac{\mu-\alpha}{\lambda}B)^{-2^n}\|^{\frac{1}{2^n}}, && (3.14)\end{aligned}$$

hence $\frac{\mu-\alpha}{\lambda} \in \sigma_{n,\frac{\varepsilon}{|\lambda|}}(S,B)$, thus $\mu \in \alpha + \lambda\sigma_{n,\frac{\varepsilon}{|\lambda|}}(S,B)$. Similarly, we obtain the converse inclusion. □

Proposition 3.27. *Let $S, B \in \mathcal{B}(\mathcal{E})$ with B is invertible, $SB = BS$, $n \in \mathbb{N}$ and $\varepsilon > 0$. Then*
$$\sigma_{n,\frac{\varepsilon}{\|B\|}}(B^{-1}S) \subseteq \sigma_{n,\varepsilon}(S,B) \subseteq \sigma_{n,\varepsilon\|B^{-1}\|}(B^{-1}S).$$

Proof. One can see that $\sigma(B^{-1}S) = \sigma(S,B)$. If $\mu \in \sigma_{n,\frac{\varepsilon}{\|B\|}}(B^{-1}S)$, then
$$\begin{aligned}\frac{\|B\|}{\varepsilon} &< \|(B^{-1}S-\mu I)^{-2^n}\|^{\frac{1}{2^n}}\\ &= \|(B^{-1}(S-\mu B))^{-2^n}\|^{\frac{1}{2^n}}\\ &\leq \|B\|\|(S-\mu B)^{-2^n}\|^{\frac{1}{2^n}}.\end{aligned}$$
Since B is invertible, then $\|B\| > 0$, thus $\frac{1}{\varepsilon} < \|(S-\mu B)^{-2^n}\|^{\frac{1}{2^n}}$, hence $\mu \in \sigma_{n,\varepsilon}(S,B)$. If $\mu \in \sigma_{n,\varepsilon}(S,B)$, then
$$\begin{aligned}\frac{1}{\varepsilon} &< \|(S-\mu B)^{-2^n}\|^{\frac{1}{2^n}}\\ &= \|(B(B^{-1}S-\mu I))^{-2^n}\|^{\frac{1}{2^n}}\\ &\leq \|B^{-1}\|\|(B^{-1}S-\mu I)^{-2^n}\|^{\frac{1}{2^n}},\end{aligned}$$
thus $\frac{1}{\varepsilon\|B^{-1}\|} < \|(B^{-1}S-\mu I)^{-2^n}\|^{\frac{1}{2^n}}$, hence $\mu \in \sigma_{n,\varepsilon\|B^{-1}\|}(B^{-1}S)$. □

Proposition 3.28. *Let $S, B \in \mathcal{B}(\mathcal{E})$ and $\lambda \notin \sigma(S,B)$, hence $\mu \in \sigma((S-\lambda B)^{-1}, B)$ if and only if $\frac{1}{\mu} \in \sigma((S-\lambda B)B)$.*

Proof. For each $\lambda \notin \sigma(S,B)$,
$$(S-\lambda B)^{-1} - \mu B = (S-\lambda B)^{-1}[I - \mu(S-\lambda B)B].$$
Hence $\mu \in \sigma((S-\lambda B)^{-1}, B)$ if and only if $\frac{1}{\mu} \in \sigma((S-\lambda B)B)$. □

Proposition 3.29. *Let $S, B \in \mathcal{B}(\mathcal{E})$ with S is invertible and $\|S^{-1}B\| \neq 0$, then for all $\mu \in B(0, \frac{1}{\|S^{-1}B\|})$, $\|(S-\mu B)^{-1}\| = \|S^{-1}\|$.*

Proof. Since $|\mu|\|S^{-1}B\| < 1$, then $(I-\mu S^{-1}B)^{-1} = \displaystyle\sum_{k=0}^{\infty}\mu^k(S^{-1}B)^k$, hence
$$(S-\mu B)^{-1} = (S(I-\mu S^{-1}B))^{-1} = (I-\mu S^{-1}B)^{-1}S^{-1} = \sum_{k=0}^{\infty}\mu^k(S^{-1}B)^kS^{-1}.$$
Consequently, for any $\mu \in B(0, \frac{1}{\|S^{-1}B\|})$,
$$\|(S-\mu B)^{-1}\| = \|\sum_{k=0}^{\infty}\mu^k(S^{-1}B)^kS^{-1}\| \leq \sup_{k\in\mathbb{N}}\|\mu^k(S^{-1}B)^k\|\|S^{-1}\| \leq \|S^{-1}\|.$$
On the other hand, $S^{-1} = (I-\mu S^{-1}B)(S-\mu B)^{-1}$. Then
$$\|S^{-1}\| \leq \|(I-\mu S^{-1}B)\|\|(S-\mu B)^{-1}\| \leq \|(S-\mu B)^{-1}\|.$$
Thus for all $\mu \in B(0, \frac{1}{\|S^{-1}B\|})$, $\|(S-\mu B)^{-1}\| = \|S^{-1}\|$. □

For $B = I$, we get.

Lemma 3.30. *If $S \in \mathcal{B}(\mathcal{E})$ is an invertible operator, then for all $\mu \in B(0, \frac{1}{\|S^{-1}\|})$, $(S-\mu I)^{-1} = \sum_{k=0}^{\infty} \mu^k S^{-(k+1)}$ and $\|(S-\mu I)^{-1}\| = \|S^{-1}\|$.*

We extend the Theorem 2.1 of [3] to ultrametric Banach spaces as follows:

Theorem 3.31. [3] *Let $S, B \in \mathcal{B}(\mathcal{E})$. If S is invertible such that $\|S-B\| < \|S^{-1}\|^{-1}$, then B is invertible and*

$$B^{-1} = S^{-1} \sum_{k=0}^{\infty} [(S-B)S^{-1}]^k,$$

and

$$\|B^{-1} - S^{-1}\| \leq \frac{\|S^{-1}\|^2 \|S-B\|}{1 - \|S-B\| \|S^{-1}\|}.$$

For $\varepsilon > 0$, set $\sigma^*_\varepsilon(S) = \sigma(S) \cup \{\mu \in \mathbb{K} : \|(S-\mu I)^{-1}\| \geq \frac{1}{\varepsilon}\}$.

Proposition 3.32. *If $S, B \in \mathcal{B}(\mathcal{E})$ and $\varepsilon > 0$, hence $\sigma^*_\varepsilon(S+B) \subset \sigma^*_{\varepsilon+\|B\|}(S)$.*

Proof. If $B = 0$, there is nothing to prove. If $B \neq 0$. Let $\mu \notin \sigma^*_{\varepsilon+\|B\|}(S)$, hence $\|(S-\mu I)^{-1}\| < \frac{1}{\varepsilon+\|B\|}$, then

$$\|(S+B-\mu I) - (S-\mu I)\| = \|B\| < \|B\| + \varepsilon < \|(S-\mu I)^{-1}\|^{-1}.$$

By Theorem 3.31, $S+B-\mu I$ is invertible and

$$\begin{aligned}
(S+B-\mu I)^{-1} &= (S-\mu I)^{-1} \sum_{k=0}^{\infty} [((S-\mu I) - (S+B-\mu I))(S-\mu I)^{-1}]^k \\
&= (S-\mu I)^{-1} \sum_{k=0}^{\infty} [-B(S-\mu I)^{-1}]^k,
\end{aligned}$$

hence $\|(S+B-\mu I)^{-1}\| \leq \|(S-\mu I)^{-1}\|$, thus

$$\|(S+B-\mu I)^{-1}\| \leq \|(S-\mu I)^{-1}\| < \frac{1}{\varepsilon + \|B\|} < \frac{1}{\varepsilon}.$$

Consequently, $\mu \notin \sigma^*_\varepsilon(S+B)$. □

Proposition 3.33. *If $S, B \in \mathcal{B}(\mathcal{E})$ and $\varepsilon > 0$, hence $\sigma_\varepsilon(S+B) \subset \sigma_{\varepsilon+\|B\|}(S)$.*

Proof. If $\mu \notin \sigma_{\varepsilon+\|B\|}(S)$, from the proof of Proposition 3.32, we have $S+B-\mu I$ is invertible. Put $g(\mu) = (S+B-\mu I)^{-1} - (S-\mu I)^{-1}$, hence

$$\begin{aligned}
\|g(\mu)\| &= \|(S+B-\mu I)^{-1}((S-\mu I) - (S+B-\mu I))(S-\mu I)^{-1}\| \\
&\leq \frac{\|B\|}{\varepsilon + \|B\|} \|(S+B-\mu I)^{-1}\|.
\end{aligned}$$

Then

$$\begin{aligned}
\|(S+B-\mu I)^{-1}\| &\leq \|g(\mu)\| + \|(S-\mu I)^{-1}\| \\
&\leq \frac{\|B\|}{\varepsilon + \|B\|} \|(S+B-\mu I)^{-1}\| + \frac{1}{\|B\| + \varepsilon},
\end{aligned}$$

thus $\|(S+B-\mu I)^{-1}\| \leq \frac{1}{\varepsilon}$. Consequently, $\sigma_\varepsilon(S+B) \subset \sigma_{\varepsilon+\|B\|}(S)$. $\square$

We have the following examples.

Example 3.34. If $S = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \in \mathcal{M}_2(\mathbb{Q}_p)$, then $\sigma(S) = \{0\}$ and for all $\mu \in \rho(S), (S-\mu I)^{-1} = -(\frac{I}{\mu} + \frac{S}{\mu^2})$. From $S^2 = 0_{\mathcal{M}_2(\mathbb{Q}_p)}$, then for any $m \in \mathbb{N}$, $(S-\mu I)^{-m} = (-1)^m(\frac{I}{\mu^m} + \frac{mS}{\mu^{m+1}})$. In particular for $m = 2^n$, we have for each $\varepsilon > 0$ and $n \in \mathbb{N}$,

$$\sigma_{n,\varepsilon}(S) = \{0\} \cup \{\mu \in \mathbb{Q}_p : \max\{\frac{1}{|\mu|}, \|\frac{2^n S}{\mu^{2^n+1}}\|^{\frac{1}{2^n}}\} > \frac{1}{\varepsilon}\}.$$

Example 3.35. If $S = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \in \mathcal{M}_2(\mathbb{Q}_p)$, then $\sigma(S) = \{1\}$ and for all $\mu \in \rho(S), (S-\mu I)^{-1} = \frac{I}{1-\mu} - \frac{N}{(1-\mu)^2}$ where $N = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$. Since $N^2 = 0_{\mathcal{M}_2(\mathbb{Q}_p)}$, hence for any $m \in \mathbb{N}$, $(S-\mu I)^{-m} = \frac{I}{(1-\mu)^m} - \frac{mN}{(1-\mu)^{m+1}}$. In particular for $m = 2^n$, we have for each $\varepsilon > 0$ and $n \in \mathbb{N}$,

$$\sigma_{n,\varepsilon}(S) = \{1\} \cup \{\mu \in \mathbb{Q}_p : \max\{\frac{1}{|(1-\mu)|}, \|\frac{2^n N}{(1-\mu)^{2^n+1}}\|^{\frac{1}{2^n}}\} > \frac{1}{\varepsilon}\}.$$

Example 3.36. If $S = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix} \in \mathcal{M}_2(\mathbb{Q}_p)$, then $\sigma(S) = \{1,5\}$ and for all $\mu \in \rho(S)$, hence for any $m \in \mathbb{N}$, $(S-\mu I)^{-m} = \begin{pmatrix} (1-\mu)^{-m} & 0 \\ 0 & (5-\mu)^{-m} \end{pmatrix}$. Thus for each $\varepsilon > 0$ and $n \in \mathbb{N}$,

$$\sigma_{n,\varepsilon}(S) = \{1,5\} \cup \{\mu \in \mathbb{Q}_p : \max\{\frac{1}{|(1-\mu)|}, \frac{1}{|5-\mu|} > \frac{1}{\varepsilon}\}.$$

We obtain the following proposition.

Proposition 3.37. *Let $S \in \mathcal{B}(\mathcal{E})$ be an invertible operator, hence for each $\mu \in B(0, \frac{1}{\|S^{-1}\|})$ and for any $m \in \mathbb{N}$, $\|(S-\mu I)^{-m}\| = \|S^{-m}\|$. In particular, for $m = 2^n$, we have for each $\mu \in B(0, \frac{1}{\|S^{-1}\|})$ and for all $n \in \mathbb{N}$, $\|(S-\mu I)^{-2^n}\| = \|S^{-2^n}\|$.*

Proof. From Lemma 3.30, we have for all $\mu \in B(0, \frac{1}{\|S^{-1}\|})$,

$$(S-\mu I)^{-1} = \sum_{k=0}^{\infty} \mu^k S^{-(k+1)} = S^{-1}(\sum_{k=0}^{\infty} \mu^k S^{-k}) = S^{-1}(I + \sum_{k=1}^{\infty} \mu^k S^{-k}).$$

Set for all $\mu \in B(0, \frac{1}{\|S^{-1}\|})$, $R(\mu) = \sum_{k=1}^{\infty} \mu^k S^{-k}$. Hence for all $\mu \in B(0, \frac{1}{\|S^{-1}\|})$, $(S-\mu I)^{-1} = S^{-1}(I + R(\mu))$. Then for each $\mu \in B(0, \frac{1}{\|S^{-1}\|})$ and for each $m \in \mathbb{N}$, $(S-\mu I)^{-m} = S^{-m}(I+R(\mu))^m$. Consequently,

$$\text{for all } \mu \in B(0, \frac{1}{\|S^{-1}\|}) \text{ and for each } m \in \mathbb{N}, \|(S-\mu I)^{-m}\| \leq \|S^{-m}\|.$$

On the other hand, $S^{-1} = (I - \mu S^{-1})(S - \mu I)^{-1}$. Then for each $\mu \in B(0, \frac{1}{\|S^{-1}\|})$ and for each $m \in \mathbb{N}, S^{-m} = (I - \mu S^{-1})^m (S - \mu I)^{-m}$. Thus,

$$\text{for all } \mu \in B(0, \frac{1}{\|S^{-1}\|}) \text{ and for each } m \in \mathbb{N}, \|S^{-m}\| \leq \|(S - \mu I)^{-m}\|.$$

Consequently,

$$\text{for all } \mu \in B(0, \frac{1}{\|S^{-1}\|}) \text{ and for each } m \in \mathbb{N}, \|(S - \mu I)^{-m}\| = \|S^{-m}\|.$$

In particular, for $m = 2^n$, we have

$$\text{for each } \mu \in B(0, \frac{1}{\|S^{-1}\|}) \text{ and for all } n \in \mathbb{N}, \|(S - \mu I)^{-2^n}\| = \|S^{-2^n}\|.$$

□

We finish with some applications of the ε-pseudospectrum of continuous linear operators on ultrametric Banach spaces.

Example 3.38. Setting $\mathbb{Z}_p = \{\mu \in \mathbb{Q}_p : |\mu| \leq 1\}$. Let $g \in C(\mathbb{Z}_p, \mathbb{Q}_p)$, hence

$$\sigma(g) = \{g(u) : u \in \mathbb{Z}_p\}$$

and for any $\mu \notin \sigma(g)$, we get

$$\|\mu e - g\| = \max\{|\mu - g(u)| : u \in \mathbb{Z}_p\}$$

and

$$\|(\mu e - g)^{-1}\| = \max\{\frac{1}{|\mu - g(u)|} : u \in \mathbb{Z}_p\},$$

where $e : \mathbb{Z}_p \to \mathbb{Q}_p$ is the identity function i.e. $e(u) = 1$ for any $u \in \mathbb{Z}_p$. Consequently,

$$\sigma_\varepsilon(g) = \{g(u) : u \in \mathbb{Z}_p\} \cup \{\mu \in \mathbb{Q}_p : \inf\{|\mu - g(u)| : u \in \mathbb{Z}_p\} < \varepsilon\}.$$

Example 3.39. Let $g \in C(\mathbb{Z}_p)$ be defined by for each $u \in \mathbb{Z}_p, g(u) = u$. Hence $\sigma(g) = \mathbb{Z}_p$. Consequently,

$$\sigma_\varepsilon(g) = \mathbb{Z}_p \cup \{\mu \in \mathbb{Q}_p : \inf\{|\mu - u| : u \in \mathbb{Z}_p\} < \varepsilon\}.$$

Problem 3.40. Consider the generalized eigenvalue problem

$$(S - \mu B)u = 0 \tag{3.15}$$

where $S, B \in \mathcal{B}(\mathcal{E})$ and $u \in \mathcal{E}$. If $\mu \notin \sigma(S, B)$, hence the equation (3.15) has one solution $u = 0$.

4. Conclusion

In this article, we demonstrated some results on the ultrametric essential pseudospectrum of bounded linear operator pencils. On the other hand, we introduced and studied the (n, ε)-pseudospectrum of bounded linear operators and the (n, ε)-pseudospectrum of bounded linear operator pencils on ultrametric Banach spaces. We proved some results about them. In addition, we provided some illustrative examples.

In the future, we shall study the (n, ε)-pseudospectrum of closed, densely defined linear operators and the (n, ε)-pseudospectrum of closed, densely defined linear operator pencils on ultrametric Banach spaces.

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