

OPTIMIZATION TECHNIQUES FOR MIXED FINITE ELEMENT METHODS OF AN SEMILINEAR ELLIPTIC PDES - SUPERCONVERGENCE

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ABSTRACT. In this paper, we shall investigate the superconvergence property of semilinear quadratic elliptic optimal control problems by triangular mixed finite element methods. The state and co-state are approximated by Raviart - Thomas mixed finite element spaces of order k , and the control is approximated by piecewise constant functions. We derive the superconvergence properties for both the control variable and the state variables.

1. INTRODUCTION

Optimal control problems are playing an increasingly important role in the design of modern life. They have various application backgrounds in the operation of physical, social, and economic processes. Efficient numerical methods are among the keys to successful applications of optimal control in practical areas. In many control problems, the objective functional contains the gradient of the state variables. Thus, the accuracy of the gradient is important in the numerical approximation of the state equations. In the finite element community, mixed finite element methods should be used for discretization of the state equations in such cases, both the scalar variable and its flux variable can be approximated in the same accuracy.

In computational optimal control, mixed finite element methods are not as widely used as in engineering simulations. In particular there does not seem to exist much work on theoretical analysis of mixed finite element approximation of optimal control problems in the literature. Finite element approximation of optimal control problems plays a very important role in the numerical solution of these problems. Many researchers have done a lot of works on some topics of finite element methods for optimal control problems. A systematic introduction of finite element method for PDEs and optimal control problem can be found in, [10, 11, 13, 14, 19, 22, 23]. For optimal control problems governed by semilinear elliptic state equations, a priori error estimates of finite element approximation were studied in [2, 3, 5, 9] and a posteriori error estimates in [16, 18].

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Recently, Chen et al. in [6, 7, 8] and Meyer et al. [20], the author have presented error estimates and superconvergence of mixed finite element for quadratic elliptic optimal control problems. Hou et al. in [15], the author have discussed superconvergence of RT1 mixed finite element approximations for elliptic control problems. Yang et al. in [24], the author have presented the superconvergence properties for optimal control problem of bilinear type. The purpose of this paper is to extend the superconvergence property of the semilinear control problem and also to get the superconvergence properties for the state variables.

In this paper, we consider the superconvergence of mixed finite element methods for the following semilinear quadratic optimal control problems:

$$\min_{u \in K \subset L^2(\Omega)} \frac{1}{2} \left\{ \|\mathbf{p} - \mathbf{p}_d\|^2 + \|y - y_d\|^2 + \|u\|^2 \right\} \quad (1.1)$$

subject to the state equation

$$-\operatorname{div}(A(x) \mathbf{grad} \, y) + \phi(y) = f + Bu, \quad x \in \Omega, \quad (1.2)$$

with the boundary condition

$$y = 0, \quad x \in \partial\Omega. \quad (1.3)$$

where Ω is a bounded domain in $\mathbb{R}^2$ and with smooth boundary $\partial\Omega$. $\mathbf{p}_d$ and y_d are two known functions, $\mathbf{p}$ and y are the state variables, u is the control variable. We shall assume that $f \in L^2(\Omega)$ and B is a continuous linear operator from $L^2(\Omega)$ to $L^2(\Omega)$. Furthermore, we assume that the coefficient matrix $A(x) = (a_{i,j}(x))_{2 \times 2} \in L^\infty(\Omega; \mathbb{R}^{2 \times 2})$ is a symmetric and positive definite 2×2 matrix with $a_{i,j}(x) \in W^{1,\infty}(\Omega)$ and there is a constant $c > 0$ satisfying for any vector $X \in \mathbb{R}^2$, $X^t A X \geq c \|X\|_{\mathbb{R}^2}^2$. Here, K denotes the admissible set of the control variable, define by

$$K = \left\{ v \in L^2(\Omega) : \int_{\Omega} v \geq 0 \right\}. \quad (1.4)$$

In this paper, [1] we adopt the standard notation $W^{m,p}(\Omega)$ for sobolev spaces on Ω with a norm $\|\cdot\|_{m,p}$ given by

$$\|\Phi\|_{m,p}^p = \sum_{|\alpha| \leq m} \|D^\alpha \Phi\|_{L^p(\Omega)}^p$$

and the semi-norm $|\cdot|_{m,p}$ given by

$$|\Phi|_{m,p}^p = \sum_{|\alpha|=m} \|D^\alpha \Phi\|_{L^p(\Omega)}^p$$

We set $W_0^{m,p}(\Omega) = \{\Phi \in W^{m,p}(\Omega) : \Phi|_{\partial\Omega} = 0\}$. For $p = 2$, we denote $H^m(\Omega) = W^{m,2}(\Omega)$, $H_0^m(\Omega) = W_0^{m,2}(\Omega)$, and $\|\cdot\|_m = \|\cdot\|_{m,2}$, $\|\cdot\| = \|\cdot\|_{0,2}$. In addition C or c denotes a general positive constant independent of h .

The plan of this paper is as follows. In Sect. 2, we formulate the mixed finite element approximation for the optimal control problems. In Sect. 3, we state some preliminaries and some estimates for the intermediate errors. Finally, in Sect. 4, superconvergence results for both control and state variables were derived.

2. MIXED FINITE ELEMENT APPROXIMATION

In this section, we study the mixed finite element approximation of the problem (1.1) – (1.3). First, the problem can be rewritten in the first order system as following:

$$\min_{u \in K \subset L^2(\Omega)} \frac{1}{2} \left\{ \|\mathbf{p} - \mathbf{p}_d\|^2 + \|y - y_d\|^2 + \|u\|^2 \right\} \quad (2.1)$$

$$\operatorname{div} \mathbf{p} + \phi(y) = f + Bu, \quad x \in \Omega \quad (2.2)$$

$$\mathbf{p} = -A(x) \mathbf{grad} y, \quad x \in \Omega \quad (2.3)$$

with the boundary condition

$$y = 0, \quad x \in \partial\Omega \quad (2.4)$$

Next, we introduce the co-state elliptic equation

$$-\operatorname{div}(A(x)(\mathbf{grad} z + \mathbf{p} - \mathbf{p}_d)) + \phi'(y)z = y - y_d, \quad x \in \Omega \quad (2.5)$$

which can be expressed in the form of the first-order system

$$\operatorname{div} \mathbf{q} + \phi'(y)z = y - y_d, \quad x \in \Omega \quad (2.6)$$

$$\mathbf{q} = -A(x)(\mathbf{grad} z + \mathbf{p} - \mathbf{p}_d), \quad x \in \Omega \quad (2.7)$$

with the boundary condition

$$z = 0, \quad x \in \partial\Omega. \quad (2.8)$$

Now, we shall describe the mixed finite discretization of semilinear quadratic optimal control problems (2.1)-(2.4). Let

$$\mathbf{V} = H(\operatorname{div}; \Omega) = \{\mathbf{v} \in (L^2(\Omega))^2, \operatorname{div} \mathbf{v} \in L^2(\Omega)\}$$

endowed with the norm given by

$$\|\mathbf{v}\|_{H(\operatorname{div}; \Omega)} = (\|\mathbf{v}\|_{0, \Omega}^2 + \|\operatorname{div} \mathbf{v}\|_{0, \Omega}^2)^{1/2}, \quad W = L^2(\Omega) \text{ and } U = L^2(\Omega).$$

Then, the mixed weak formulation for the problems (2.1)-(2.4): find $(\mathbf{p}, y, u) \in \mathbf{V} \times W \times U$

$$\min_{u \in K} \frac{1}{2} \left\{ \|\mathbf{p} - \mathbf{p}_d\|^2 + \|y - y_d\|^2 + \|u\|^2 \right\}, \quad (2.9)$$

$$(A^{-1} \mathbf{p}, \mathbf{v}) - (y, \operatorname{div} \mathbf{v}) = 0, \quad \forall \mathbf{v} \in \mathbf{V}, \quad (2.10)$$

$$(\operatorname{div} \mathbf{p}, w) + (\phi(y), w) = (f + Bu, w), \quad \forall w \in W, \quad (2.11)$$

where $(\cdot)$ denotes the inner product in $L^2(\Omega)$. It is well known that the convex control problems (2.9)-(2.11) has a unique solution $(\mathbf{p}, y, u)$, and that a triple $(\mathbf{p}, y, u)$ is the solution of (2.9)-(2.11) if and only if there is a costate $(\mathbf{q}, z) \in \mathbf{V} \times W$ such that $(\mathbf{p}, y, \mathbf{q}, z, u)$ satisfies the following optimality conditions [17]:

$$(A^{-1} \mathbf{p}, \mathbf{v}) - (y, \operatorname{div} \mathbf{v}) = 0, \quad \forall \mathbf{v} \in \mathbf{V}, \quad (2.12)$$

$$(\operatorname{div} \mathbf{p}, w) + (\phi(y), w) = (f + Bu, w), \quad \forall w \in W, \quad (2.13)$$

$$(A^{-1} \mathbf{q}, \mathbf{v}) - (z, \operatorname{div} \mathbf{v}) = -(\mathbf{p} - \mathbf{p}_d, \mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{V}, \quad (2.14)$$

$$(\operatorname{div} \mathbf{q}, w) + (\phi'(y)z, w) = (y - y_d, w), \quad \forall w \in W, \quad (2.15)$$

$$(B^*z + u, \tilde{u} - u)_U \geq 0, \quad \forall \tilde{u} \in K. \quad (2.16)$$

where $(\cdot, \cdot)_U$ is the inner product of U , and B^* is the adjoint operator of B . For simplify, the product $(\cdot, \cdot)_U$ will be denoted as $(\cdot, \cdot)$. The given function y_d and p_d in the problem satisfy the following regularity: $y_d \in L^2(\Omega)$, $p_d \in (L^2(\Omega))^2$. For ease of exposition, we will assume that Ω is a polygon. Let $\mathcal{T}_h$ be regular triangulations or rectangulations of Ω . They are assumed to satisfy the angle condition, which means that there is a positive constant C such that, for all $T \in \mathcal{T}_h$, $C^{-1}h_T^2 \leq |T| \leq Ch_T^2$, where $|T|$ is the area of T and h_T is the diameter of T . Let $h = \max h_T$.

Let $\mathbf{V}_h \times W_h \subset \mathbf{V} \times W$ denotes the Raviart - Thomas space [21] of the lowest order associated with the triangulations or rectangulations $\mathcal{T}_h$ of Ω . P_k denotes the space of polynomials of total degree at most k , and $Q_{m,n}$ indicates the space of polynomials of degree no more than m and n in x_1 and x_2 variables respectively, where $x = (x_1, x_2)$. If $T \in \mathcal{T}_h$ is a triangle, let

$$\mathbf{V}(T) = \mathbf{P}_K(T) \oplus \text{span}(\mathbf{x} \cdot P_K(T)), \quad W(T) = P_K(T).$$

Similarly, if T is a rectangle, let

$$\mathbf{V}(T) = Q_{K+1,k}(T) \times Q_{K,K+1}(T), \quad W(T) = P_K(T).$$

where $\mathbf{P}_K(T) = (P_K(T))^2$. Then we can define the finite dimensional space as follows

$$\begin{aligned} \mathbf{V}_h &:= \{\mathbf{v}_h \in \mathbf{V} : \forall T \in \mathcal{T}_h, \mathbf{v}_h|_T \in \mathbf{V}(T)\}, \\ W_h &:= \{w_h \in W : \forall T \in \mathcal{T}_h, w_h|_T \in W(T)\}, \\ K_h &:= \{\tilde{u}_h \in K : \forall T \in \mathcal{T}_h, \tilde{u}_h|_T = \text{constant}\}. \end{aligned}$$

By the definition of finite element subspace, the mixed finite element approximation of (2.9)-(2.11) is as follows: compute $(\mathbf{p}_h, y_h, u_h) \in V_h \times W_h \times K_h$ such that

$$\min_{u_h \in K_h} \frac{1}{2} \left\{ \|\mathbf{p}_h - \mathbf{p}_d\|^2 + \|y_h - y_d\|^2 + \|u_h\|^2 \right\}, \quad (2.17)$$

$$(A^{-1}\mathbf{p}_h, \mathbf{v}_h) - (y_h, \text{div}\mathbf{v}_h) = 0, \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \quad (2.18)$$

$$(\text{div}\mathbf{p}_h, w_h) + (\phi(y_h), w_h) = (f + Bu_h, w_h), \quad \forall w_h \in W_h. \quad (2.19)$$

It is well known that the optimal control problems (2.17)-(2.19) again has a unique solution $(\mathbf{p}_h, y_h, u_h)$, and that a triple $(\mathbf{p}_h, y_h, u_h) \in V_h \times W_h \times K_h$ is the solution of (2.17)-(2.19) if and only if there is a costate $(\mathbf{q}_h, z_h) \in \mathbf{V}_h \times W_h$ such that $(\mathbf{p}_h, y_h, \mathbf{q}_h, z_h, u_h)$ satisfies the following optimality conditions:

$$(A^{-1}\mathbf{p}_h, \mathbf{v}_h) - (y_h, \text{div}\mathbf{v}_h) = 0, \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \quad (2.20)$$

$$(\text{div}\mathbf{p}_h, w_h) + (\phi(y_h), w_h) = (f + Bu_h, w_h), \quad \forall w_h \in W_h, \quad (2.21)$$

$$(A^{-1}\mathbf{q}_h, \mathbf{v}_h) - (z_h, \text{div}\mathbf{v}_h) = -(\mathbf{p}_h - \mathbf{p}_d, \mathbf{v}_h), \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \quad (2.22)$$

$$(\text{div}\mathbf{q}_h, w_h) + (\phi'(y_h)z_h, w_h) = (y_h - y_d, w_h), \quad \forall w_h \in W_h, \quad (2.23)$$

$$(B^*z_h + u_h, \tilde{u}_h - u_h) \geq 0, \quad \forall \tilde{u}_h \in K_h. \quad (2.24)$$

3. INTERMEDIATE ERROR ESTIMATES

In this section, we shall use some intermediate variables. For any control function $\tilde{u} \in K$, we first define the state solution such that, for any $(\mathbf{v}, w) \in$

$\mathbf{V} \times W$. $(\mathbf{p}(\tilde{u}), y(\tilde{u}), \mathbf{q}(\tilde{u}), z(\tilde{u}))$ is associated with $\tilde{u}$.

$$(A^{-1}\mathbf{p}(\tilde{u}), \mathbf{v}) - (y(\tilde{u}), \operatorname{div}\mathbf{v}) = 0, \quad (3.1)$$

$$(\operatorname{div}\mathbf{p}(\tilde{u}), w) + (\phi(y(\tilde{u})), w) = (f + B(\tilde{u}), w), \quad (3.2)$$

$$(A^{-1}\mathbf{q}(\tilde{u}), \mathbf{v}) - (z(\tilde{u}), \operatorname{div}\mathbf{v}) = -(\mathbf{p}(\tilde{u}) - \mathbf{p}_d, \mathbf{v}), \quad (3.3)$$

$$(\operatorname{div}\mathbf{q}(\tilde{u}), w) + (\phi'(y(\tilde{u}))z(\tilde{u}), w) = (y(\tilde{u}) - y_d, w). \quad (3.4)$$

Correspondingly, we define the discrete state solution $(\mathbf{p}_h(\tilde{u}), y_h(\tilde{u}), \mathbf{q}_h(\tilde{u}), z_h(\tilde{u}))$ associated with $\tilde{u} \in K$ such that, for any $(v_h, w_h) \in V_h \times W_h$,

$$(A^{-1}\mathbf{p}_h(\tilde{u}), \mathbf{v}_h) - (y_h(\tilde{u}), \operatorname{div}\mathbf{v}_h) = 0, \quad (3.5)$$

$$(\operatorname{div}\mathbf{p}_h(\tilde{u}), w_h) + (\phi(y_h(\tilde{u})), w_h) = (f + B(\tilde{u}), w_h), \quad (3.6)$$

$$(A^{-1}\mathbf{q}_h(\tilde{u}), \mathbf{v}_h) - (z_h(\tilde{u}), \operatorname{div}\mathbf{v}_h) = -(\mathbf{p}_h(\tilde{u}) - \mathbf{p}_d, \mathbf{v}_h), \quad (3.7)$$

$$(\operatorname{div}\mathbf{q}_h(\tilde{u}), w_h) + (\phi'(y_h(\tilde{u}))z_h(\tilde{u}), w_h) = (y_h(\tilde{u}) - y_d, w_h). \quad (3.8)$$

Thus, for the sake of simplicity, we write the exact solution and its approximation in the following way:

$$(\mathbf{p}, y, \mathbf{q}, z) = (\mathbf{p}(u), y(u), \mathbf{q}(u), z(u)), \quad (3.9)$$

$$(\mathbf{p}_h, y_h, \mathbf{q}_h, z_h) = (\mathbf{p}_h(u_h), y_h(u_h), \mathbf{q}_h(u_h), z_h(u_h)). \quad (3.10)$$

We introduce the standard $L^2(\Omega)$ -orthogonal projection [4], $P_h : W \rightarrow W_h$, which satisfies: for any $\phi \in W$:

$$(\phi - P_h\phi, w_h) = 0, \quad \forall w_h \in W_h. \quad (3.11)$$

Next, let us recall the Fortin projection [4], $R_h : \mathbf{V} \rightarrow \mathbf{V}_h$, which satisfies: for any $\mathbf{q} \in \mathbf{V}$:

$$(\operatorname{div}(\mathbf{q} - R_h\mathbf{q}), w_h) = 0, \quad w_h \in W_h. \quad (3.12)$$

We have the following diagram commutes:

$$\begin{array}{ccc} V & \xrightarrow{\operatorname{div}} & W \\ \Pi_h \downarrow & & \downarrow P_h \\ V_h & \xrightarrow{\operatorname{div}} & W_h \longrightarrow 0 \end{array}$$

i.e., $\operatorname{div} \circ \Pi_h = P_h \circ \operatorname{div} : \mathbf{V} \xrightarrow{\operatorname{onto}} W_h$, and $\operatorname{div}(I - \Pi_h)\mathbf{V} \perp W_h$, $\operatorname{div}R_h = P_h\operatorname{div} : \mathbf{V} \xrightarrow{\operatorname{onto}} W_h$, and the following approximation properties:

$$\|\mathbf{q} - R_h\mathbf{q}\| \leq C|\mathbf{q}|_1 h, \quad (3.13)$$

$$\|\operatorname{div}(\mathbf{q} - R_h\mathbf{q})\|_{-s} \leq C|\operatorname{div}\mathbf{q}|_1 h^{1+s}, \quad s = 0, 1, \quad (3.14)$$

$$\|\phi - P_h\phi\|_{-s} \leq C|\phi|_1 h^{1+s}, \quad s = 0, 1. \quad (3.15)$$

Furthermore, we also define the standard $L^2(\Omega_U)$ - orthogonal projection $Q_h : U \rightarrow U_h$, which satisfies: for any $\tilde{u} \in U$:

$$(\tilde{u} - Q_h\tilde{u}, \tilde{u}_h)_U = 0, \quad \forall \tilde{u}_h \in U_h. \quad (3.16)$$

The $Q_h \tilde{u}$ is in fact just the element average of $\tilde{u}$. Similar to (3.15), we have the approximation property:

$$\|\tilde{u} - Q_h \tilde{u}\|_{-s} \leq C |\tilde{u}|_1 h_U^{1+s}, \quad s = 0, 1 \text{ for } \tilde{u} \in W^{1,r}(\Omega). \quad (3.17)$$

Lemma 3.1. *Let u be the solution of (2.12)-(2.15), for h sufficiently small, there exists a positive constant C such that*

$$\|y(u) - y(Q_h u)\|_{L^2(\Omega)} + \|\mathbf{p}(u) - \mathbf{p}(Q_h u)\|_{div} \leq Ch^2, \quad (3.18)$$

$$\|z(u) - z(Q_h u)\|_{L^2(\Omega)} + \|\mathbf{q}(u) - \mathbf{q}(Q_h u)\|_{div} \leq Ch^2. \quad (3.19)$$

Proof. Choose $\tilde{u} = Q_h u$ and $\tilde{u} = u$ in (3.1)-(3.4), respectively, then we have the following error equations

$$\begin{aligned} (\operatorname{div} \mathbf{p}(Q_h u) - \operatorname{div} \mathbf{p}(u), w) + (\phi(y(Q_h u)) - \phi(y(u)), w) \\ = (B(Q_h u - u), w), \end{aligned} \quad (3.20)$$

$$\begin{aligned} (\operatorname{div} \mathbf{q}(Q_h u) - \operatorname{div} \mathbf{q}(u), w) + (\phi'(y(Q_h u))z(Q_h u) - \phi'(y(u))z(u), w) \\ = (y(Q_h u) - y(u), w), \end{aligned} \quad (3.21)$$

for any $w \in V$ and $q \in V$.

First, choose $w = y(Q_h u) - y(u)$ in (3.20), we have

$$\begin{aligned} (\operatorname{div} \mathbf{p}(Q_h u) - \operatorname{div} \mathbf{p}(u), y(Q_h u) - y(u)) + (\phi(y(Q_h u)) - \phi(y(u)), y(Q_h u) - y(u)) \\ = (B(Q_h u - u), y(Q_h u) - y(u)) \end{aligned} \quad (3.22)$$

Now, we estimate the right hand side of (3.22). Using the continuity of B and (3.17), we have

$$\begin{aligned} (B(Q_h u - u), y(Q_h u) - y(u)) &= (Q_h u - u, B^*(y(Q_h u) - y(u))) \\ &\leq C \|B^*(y(Q_h u) - y(u))\|_1 \cdot \|Q_h u - u\|_{-1} \\ &\leq Ch^2 \|u\|_1 \cdot \|y(Q_h u - y(u))\|_{H^1}. \end{aligned} \quad (3.23)$$

From (3.22), (3.23) and the assumption of A and $\phi(\cdot)$, we have

$$\begin{aligned} c \|y(Q_h u) - y(u)\|_1^2 &\leq (\operatorname{div} \mathbf{p}(Q_h u) - \operatorname{div} \mathbf{p}(u), y(Q_h u) - y(u)) \\ &\quad + (\phi(y(Q_h u)) - \phi(y(u)), y(Q_h u) - y(u)) \\ &= (B(Q_h u - u), y(Q_h u) - y(u)) \\ &\leq Ch^2 \|y(Q_h u) - y(u)\|_{L^2}. \end{aligned} \quad (3.24)$$

then, (3.18) can be obtained from (3.24). Choose $w = \mathbf{p}(Q_h u) - \mathbf{p}(u)$ in (3.21), we have

$$\begin{aligned} (\operatorname{div} \mathbf{q}(Q_h u) - \operatorname{div} \mathbf{p}(u), \mathbf{p}(Q_h u) - \mathbf{p}(u)) \\ + (\phi'(y(Q_h u))z(Q_h u) - \phi'(y(u))z(u), \mathbf{p}(Q_h u) - \mathbf{p}(u)) \\ = (y(Q_h u) - y(u), \mathbf{p}(Q_h u) - \mathbf{p}(u)) \end{aligned} \quad (3.25)$$

namely,

$$\begin{aligned}
& (\operatorname{div} \mathbf{q}(Q_h u) - \operatorname{div} \mathbf{q}(u), \mathbf{p}(Q_h u) - \mathbf{p}(u)) \\
& \quad + (\phi'(y(Q_h u))z(Q_h u) - z(u), \mathbf{p}(Q_h u) - \mathbf{p}(u)) \\
& = (y(Q_h u) - y(u), \mathbf{p}(Q_h u) - \mathbf{p}(u)) \\
& \quad + (z(u)(\phi'(y(u)) - \phi'(y(Q_h u))), \mathbf{p}(Q_h u) - \mathbf{p}(u)) \quad (3.26)
\end{aligned}$$

Notice that

$$\begin{aligned}
(y(Q_h u) - y(u), \mathbf{p}(Q_h u) - \mathbf{p}(u)) & \leq C \|y(Q_h u) - y(u)\| \cdot \|\mathbf{p}(Q_h u) - \mathbf{p}(u)\| \\
& \leq Ch^2 \|\mathbf{p}(Q_h u) - \mathbf{p}(u)\|. \quad (3.27)
\end{aligned}$$

Using the assumption of $\phi(\cdot)$ and (3.17), we have

$$\begin{aligned}
& (z(u)(\phi'(y(u)) - \phi'(y(Q_h u))), \mathbf{p}(Q_h u) - \mathbf{p}(u)) \\
& \leq C \|z(u)\|_{0,4} \|\phi'(y(u)) - \phi'(y(Q_h u))\| \cdot \|\mathbf{p}(Q_h u) - \mathbf{p}(u)\|_{0,4} \\
& \leq C \|z(u)\|_1 \|\phi\|_{W^{2,\infty}} \|(y(u)) - (y(Q_h u))\| \cdot \|\mathbf{p}(Q_h u) - \mathbf{p}(u)\|_1 \\
& \leq Ch^2 \|\mathbf{p}(Q_h u) - \mathbf{p}(u)\|_1. \quad (3.28)
\end{aligned}$$

then (3.19) can be obtained from (3.28). $\square$

Lemma 3.2. *For any $\tilde{u} \in K$, if the intermediate solution satisfies*

$$\mathbf{q}(\tilde{u}), \mathbf{p}(\tilde{u}) \in [H^2(\Omega)]^2, \quad y(\tilde{u}), z(\tilde{u}) \in H^1(\Omega),$$

and Ω is convex, then, we have

$$\|y_h(\tilde{u}) - P_h y(\tilde{u})\|_{L^2(\Omega)} + \|\mathbf{p}_h(\tilde{u}) - R_h \mathbf{p}(\tilde{u})\|_{div} \leq Ch^2, \quad (3.29)$$

$$\|z_h(\tilde{u}) - P_h z(\tilde{u})\|_{L^2(\Omega)} + \|\mathbf{q}_h(\tilde{u}) - R_h \mathbf{q}(\tilde{u})\|_{div} \leq Ch^2. \quad (3.30)$$

Let $(\mathbf{p}(u), y(u))$ and $(\mathbf{p}_h(u_h), y_h(u_h))$ be the solution of (2.9)-(2.11) and (2.17)-(2.19) respectively. Let $J(\cdot) : U \rightarrow R$ be a G - differential convex functional which satisfies the following form:

$$J(u) = \frac{1}{2} \{ \|\mathbf{p} - \mathbf{p}_d\|^2 + \|y - y_d\|^2 + \|u\|^2 \} \quad (3.31)$$

We assume that we have a sequence of convex functional $J_h : U \rightarrow R$:

$$J_h(u) = \frac{1}{2} \{ \|\mathbf{p}_h(u) - \mathbf{p}_d\|^2 + \|y_h(u) - y_d\|^2 + \|u\|^2 \} \quad (3.32)$$

$$J_h(u_h) = \frac{1}{2} \{ \|\mathbf{p}_h(u_h) - \mathbf{p}_d\|^2 + \|y_h(u_h) - y_d\|^2 + \|u_h\|^2 \} \quad (3.33)$$

Then, the reduced problems of (2.9) and (2.17) read as

$$\min_{u \in K} \{ J(u) \} \quad (3.34)$$

and

$$\min_{u_h \in K_h} \{ J_h(u_h) \}. \quad (3.35)$$

respectively. It can be shown that

$$\begin{aligned} (J'(u), v) &= (B^*z + u, v), \quad (J'(u_h), v) = (B^*z(u_h) + u_h, v), \\ (J'(Q_h u), v) &= (Q_h u + B^*z(Q_h u), v), \quad (J'_h(u_h), v) = (B^*z_h + u_h, v). \end{aligned}$$

In many applications, $J(\cdot)$ is uniform convex near the solution u . The convexity of $J(\cdot)$ is closely related to the second order sufficient conditions of the control problem, which are assumed in many studies on numerical methods of the problem. If $J(\cdot)$ is uniformly convex, then there is a $c > 0$, such that

$$(J'(Q_h u) - J'(u_h), Q_h u - u_h) \geq c \|Q_h u - u_h\|^2. \quad (3.36)$$

4. SUPERCONVERGENCE FOR OPTIMAL CONTROL PROBLEMS

In this section, the superconvergence for both the control variable and the state variables are approximated by using mixed finite element methods. Let π^c be the average operator defined in [19].

- (1) The control and the state have same domain and mesh, namely, Ω and $\mathcal{T}_h$.
- (2) The continue linear operator B can be expressed as $B\tilde{u} = \alpha(x)\tilde{u}$, with $\tilde{u} \in W^{1,\infty}(\Omega)$.
- (3) Let $\Omega^+ = \{U_\tau : \tau \subset \Omega, u|_\tau > 0\}$, $\Omega^0 = \{U_\tau : \tau \subset \Omega, u|_\tau = 0\}$, $\Omega^b = \Omega \setminus (\Omega^+ \cup \Omega^0)$.

We assume that u and $\mathcal{T}_h$ are regular such that $meas(\Omega^b) \leq Ch$.

- (4) Let $j(\cdot)$ is a smooth and convex function such that $j''(u) \in W^{1,\infty}$ and $j'''(\cdot) \in L^\infty(\mathbb{R})$.

Theorem 4.1. *Let $(\mathbf{p}, y, \mathbf{q}, z, u) \in (\mathbf{V} \times W)^2 \times K$ and $((\mathbf{p}_h, y_h, \mathbf{q}_h, z_h, u_h) \in (\mathbf{V}_h \times W_h)^2 \times K_h$ be the solution of (2.12)-(2.16) and (2.20)-(2.24) respectively. If $\mathbf{p}, \mathbf{q} \in [H^2(\Omega)]^2$, $y, z \in H^1(\Omega)$, and $u, z \in W^{1,\infty}(\Omega)$, $y \in W^{1,4}(\Omega)$, Then, we have*

$$\|Q_h u - u_h\| \leq Ch^{3/2}. \quad (4.1)$$

Proof. Note that $u_h - u = u_h - Q_h u + Q_h u - u$. Adding the two inequalities (2.16) and (2.24), we have

$$(u_h + B^*z_h - u - B^*z, Q_h u - u_h) + (u + B^*z, Q_h u - u) \geq 0. \quad (4.2)$$

Hence, the functional $j(\cdot)$ is convex (see (3.36)), we can see that

$$\begin{aligned} c\|Q_h u - u_h\|^2 &\leq (Q_h u - u, Q_h u - u) + (B^*z_h - B^*z, Q_h u - u_h) \\ &\quad + (u + B^*z, Q_h u - u). \end{aligned} \quad (4.3)$$

For the first term of (4.3). By using Taylor's expansion of the function $j(\cdot)$, we have

$$\begin{aligned}
(Q_h u - u, Q_h u - u) &= j''(u)(Q_h u - u) \\
&\quad + \frac{1}{2} j'''(u + \theta(Q_h u - u))(Q_h u - u)^2, Q_h u - u) \\
&= (j''(u) - \pi^c(j''(u)) \cdot (Q_h u - u), Q_h u - u) \\
&\quad + \frac{1}{2} (j'''(u + \theta(Q_h u - u))(Q_h u - u)^2, Q_h u - u) \\
&\leq Ch \|j''(u)\|_{0,\infty} \cdot \|Q_h u - u\| \cdot \|Q_h u - u_h\| \\
&\quad + \|j''(\cdot)\|_{0,\infty} \cdot \|Q_h u - u\|_{0,4}^2 \cdot \|Q_h u - u_h\| \\
&\leq Ch^2 \cdot \|Q_h u - u\|.
\end{aligned} \tag{4.4}$$

According to the equation (4.3), we have

$$\begin{aligned}
(B^* z_h - B^* z, Q_h u - u_h) &= (B^* z_h(u) - B^* P_h z(u), Q_h u - u_h) \\
&\quad + (B^* P_h z(u) - B^* z(u), Q_h u - u_h), \\
&= \sum_{i=1}^2 G_i.
\end{aligned} \tag{4.5}$$

Now, By applying the proof of Theorems (4.1), (5.1) in [12], we have

$$\begin{aligned}
G_1 &\leq C \|z_h(u) - P_h z(u)\| \cdot \|Q_h u - u_h\|, \\
&\leq Ch^{3/2} \cdot \|Q_h u - u_h\|.
\end{aligned} \tag{4.6}$$

Obviously,

$$\begin{aligned}
G_2 &= ((\alpha(x) - \pi^c(\alpha(x))) \cdot (P_h z - z), Q_h u - u_h), \\
&\leq Ch^2 \cdot \|Q_h u - u_h\|.
\end{aligned} \tag{4.7}$$

Inserting (4.6), (4.7) with (4.5), we can deduce that

$$(B^* z_h - B^* z, Q_h u - u_h) \leq Ch^{3/2} \cdot \|Q_h u - u_h\|. \tag{4.8}$$

For the third term of (4.3), we have

$$(u + B^* z, Q_h u - u) = \int_{\Omega} + \int_{\Omega^0} + \int_{\Omega^b} (u + B^* z, Q_h u - u) dx.$$

Obviously, $(Q_h u - u)|_{\Omega^0} = 0$. From (2.19), we have pointwise a.e. $(u + B^* z) \geq 0$. we choose $\tilde{u}|_{\Omega^+} = 0$ and $\tilde{u}|_{\Omega \setminus \Omega^+} = u$, so that $(u + B^* z, u)|_{\Omega^+} \leq 0$. Hence, $(u + B^* z)|_{\Omega^+} = 0$. Then

$$\begin{aligned}
(u + B^* z, Q_h u - u) &= (u + B^* z, Q_h u - u)_{\Omega^b} \\
&\leq (u + B^* z - \pi^c(u + B^* z), Q_h u - u)_{\Omega^b} \\
&\leq Ch^2 \|u + B^* z\|_{1,\Omega^b} \|u\|_{1,\Omega^b} \\
&\leq Ch^2 \|u + B^* z\|_{1,\infty} \|u\|_{1,\infty} \cdot meas(\Omega^b) \\
&\leq Ch^3.
\end{aligned} \tag{4.9}$$

Then, combining Lemma (3.1), (4.4), (4.8), and (4.9) with (4.3), we have

$$\|Q_h u - u_h\| \leq Ch^{3/2}. \quad (4.10)$$

We can prove the result of Theorem . $\square$

Theorem 4.2. *Let $(\mathbf{p}, y, \mathbf{q}, z, u) \in (\mathbf{V} \times W)^2 \times K$ and $(\mathbf{p}_h, y_h, \mathbf{q}_h, z_h, u_h) \in (\mathbf{V}_h \times W_h)^2 \times K_h$ be the solution of (2.9)-(2.11) and (2.17)-(2.19) respectively. If $u, u + B^*z \in W^{1,\infty}(\Omega)$, $y \in W^{1,4}(\Omega)$ and $\mathbf{q}(u), \mathbf{p}(u) \in [H^2(\Omega)]^2$, $y(u), z(u) \in H^1(\Omega)$. Then, we have*

$$\|y_h - P_h y\|_{L^2(\Omega)} + \|\mathbf{p}_h - R_h \mathbf{p}\|_{div} \leq Ch^{3/2}, \quad (4.11)$$

$$\|z_h - P_h z\|_{L^2(\Omega)} + \|\mathbf{q}_h - R_h \mathbf{q}\|_{div} \leq Ch^{3/2}. \quad (4.12)$$

Proof. From (2.9)-(2.11) and (2.17)-(2.19), we have the following error equations

$$\begin{aligned} (A^{-1}(\mathbf{p} - \mathbf{p}_h), \mathbf{v}_h) - (y - y_h, \operatorname{div} \mathbf{v}_h) &= 0, \\ (\operatorname{div} \mathbf{p} - \operatorname{div} \mathbf{p}_h, w_h) + (\phi(y) - \phi(y_h), w_h) &= (B(u - u_h), w_h), \\ (A^{-1}(\mathbf{q} - \mathbf{q}_h), \mathbf{v}_h) - (z - z_h, \operatorname{div} \mathbf{v}_h) &= -(\mathbf{p} - \mathbf{p}_h, \mathbf{v}_h), \\ (\operatorname{div} \mathbf{q} - \operatorname{div} \mathbf{q}_h, w_h) + (\phi'(y)z - \phi'(y_h)z_h, w_h) &= (y - y_h, w_h). \end{aligned} \quad (4.13)$$

for all $\mathbf{v}_h \in \mathbf{V}_h$ and $w_h \in W_h$. By using the definition (3.11), (3.12), and (3.16) of projections P_h, R_h and Q_h . We recast the error equations (4.13) as follows:

$$\begin{aligned} (A^{-1}(R_h \mathbf{p} - \mathbf{p}_h), \mathbf{v}_h) - (P_h y - y_h, \operatorname{div} \mathbf{v}_h) &= \phi_1(\mathbf{v}_h), \\ (\operatorname{div} R_h \mathbf{p} - \operatorname{div} \mathbf{p}_h, w_h) + (\phi(P_h y) - \phi(y_h), w_h) &= \psi_1(w_h), \\ (A^{-1}(R_h \mathbf{q} - \mathbf{q}_h), \mathbf{v}_h) - (P_h z - z_h, \operatorname{div} \mathbf{v}_h) &= \phi_2(\mathbf{v}_h), \\ (\operatorname{div} R_h \mathbf{q} - \operatorname{div} \mathbf{q}_h, w_h) + (\phi'(P_h y)P_h z - \phi'(y_h)z_h, w_h) &= \psi_2(w_h). \end{aligned} \quad (4.14)$$

for all $\mathbf{v}_h \in \mathbf{V}_h$ and $w_h \in W_h$, where

$$\begin{aligned} \phi_1(\mathbf{v}_h) &= -(A^{-1}(\mathbf{p} - R_h \mathbf{p}), \mathbf{v}_h), \\ \psi_1(w_h) &= (B(u - Q_h u), w_h) + (B(Q_h u - u_h), w_h), \\ \phi_2(\mathbf{v}_h) &= -(A^{-1}(\mathbf{q} - R_h \mathbf{q}), \mathbf{v}_h) - (\mathbf{p} - \mathbf{p}_h, \mathbf{v}_h), \\ \psi_2(w_h) &= (y - y_h, w_h). \end{aligned} \quad (4.15)$$

Similar to the proof of Theorem 3.1 in [13], using the standard stability argument, there exists a constant C such that

$$\|y_h - P_h y\|_{L^2(\Omega)} + \|\mathbf{p}_h - R_h \mathbf{p}\|_{div} \leq C \left(\sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{|\phi_1(\mathbf{v}_h)|}{\|\mathbf{v}_h\|_{div}} + \sup_{w_h \in W_h} \frac{|\psi_1(w_h)|}{\|w_h\|} \right), \quad (4.16)$$

$$\|z_h - P_h z\|_{L^2(\Omega)} + \|\mathbf{q}_h - R_h \mathbf{q}\|_{div} \leq C \left(\sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{|\phi_2(\mathbf{v}_h)|}{\|\mathbf{v}_h\|_{div}} + \sup_{w_h \in W_h} \frac{|\psi_2(w_h)|}{\|w_h\|} \right). \quad (4.17)$$

Next, we bound the terms of the right side of (4.14) one by one. By using Theorem 5.1 in [13], we have

$$|(A^{-1}(\mathbf{p} - R_h \mathbf{p}), \mathbf{v}_h)| \leq Ch^{3/2}(\|\mathbf{p}\|_2 \|\mathbf{v}_h\| + \|\mathbf{p}\|_1 \|\mathbf{grad} \cdot \mathbf{v}_h\|). \quad (4.18)$$

$$|(A^{-1}(\mathbf{q} - R_h \mathbf{q}), \mathbf{v}_h)| \leq Ch^{3/2}(\|\mathbf{q}\|_2 \|\mathbf{v}_h\| + \|\mathbf{q}\|_1 \|\mathbf{grad} \cdot \mathbf{v}_h\|). \quad (4.19)$$

Obviously, from the equation (4.13), we have

$$(B(u - Q_h u), w_h) = ((\alpha(x) - \pi^c \alpha(x))(u - Q_h u), w_h) \leq Ch^2 \|u\|_1 \|w_h\|. \quad (4.20)$$

From Theorem 4.1, it is easy to see that

$$(B(Q_h u - u_h), w_h) = C \|Q_h u - u_h\| \cdot \|w_h\| \leq Ch^{3/2}. \quad (4.21)$$

Hence, Finally (4.18), (4.20) and (4.21) in (4.16), we can obtain (4.11).

Finally, we note that

$$\begin{aligned} (\mathbf{p} - \mathbf{p}_h, \mathbf{v}_h) &= (\mathbf{p} - R_h \mathbf{p}, \mathbf{v}_h) + (R_h \mathbf{p} - \mathbf{p}_h, \mathbf{v}_h), \\ (y - y_h, w_h) &= (y - P_h y, w_h) + (P_h y - y_h, w_h). \end{aligned} \quad (4.22)$$

Similar to the proof in (4.4), using the assumption for functional y and the definition of $\mathbf{p}_h$, we can prove that

$$(y - P_h y, w_h) \leq Ch^2 \cdot \|w_h\|. \quad (4.23)$$

Similarly, using the assumption of $\phi(\cdot)$ and Theorem 5.1 in [13], it can be proved that

$$(\mathbf{p} - R_h \mathbf{p}, \mathbf{v}_h) \leq Ch^{3/2} \cdot \|\mathbf{v}_h\|. \quad (4.24)$$

By the Lipschitz continuity of y and $\mathbf{p}$ and (4.11), we have

$$\begin{aligned} (R_h \mathbf{p} - \mathbf{p}_h, \mathbf{v}_h) &\leq Ch^{3/2} \cdot \|\mathbf{v}_h\|. \\ (P_h y - y_h, w_h) &\leq Ch^{3/2} \cdot \|w_h\|. \end{aligned} \quad (4.25)$$

Thus, inserting (4.19) and (4.22)-(4.25) in (4.17), we can obtain (4.12). $\square$

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