

ON SOME NONLINEAR ELLIPTIC EQUATIONS WITH MEASURABLE BOUNDARY CONDITIONS IN ANISOTROPIC WEIGHTED SOBOLEV SPACES

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ABSTRACT. The novelty of this note is to establish existence result for the following anisotropic elliptic equation

$$-\operatorname{div} B(x, \vartheta, \nabla \vartheta) + H(x, \vartheta) = f - \operatorname{div} F$$

where the datum $f \in L^1(\Omega)$ and $F \in \prod_{i=1}^N L^{p'_i}(\Omega, w_i^*)$ and $H(x, \vartheta) \in L^1(\Omega)$. Furthermore only the large monotonicity conditions will be assumed on $B(x, s, \xi)$. To overcome this difficulty we will use the approach of Minty's lemma in the anisotropic weighted Sobolev spaces.

1. INTRODUCTION

The anisotropic weighted Sobolev spaces introduce a novel framework incorporating directional derivatives with varying weights. Initially, consider Ω as an open bounded subset of $\mathbb{R}^N$ ($N \geq 2$), and let p_i denote $N + 1$ exponents, where $1 < p_i < N$ for $i \in 0, \dots, N$. In this works, we aim to investigate the existence of entropy solutions for the given problem.

$$\begin{cases} -\operatorname{div}(B(x, \vartheta, \nabla \vartheta)) + H(x, \vartheta) = \mu & \text{in } \Omega \\ \vartheta = 0 & \text{on } \partial\Omega \end{cases} \quad (1.1)$$

Here, $w = \{w_i, 0 \leq i \leq N\}$ is the set of weight functions on Ω , and $w^* = \{w_i^{1-p'_i}, 0 \leq i \leq N\}$, The second term μ belongs to a measure space $L^1(\Omega) + W^{-1, p'_i}(\Omega, w^*)$.

In recent works, existence result with some qualitative properties and regularity of nonlinear anisotropic elliptic equations where the data belongs to L^1 —have been proved, for example we may refer the reader to [7] When L. Boccardo examined the problem (1.1) in Sobolev space $W_0^{1,p}(\Omega)$.

Furthermore we can mention the paper [13] when B. El haji et al have been shown the existence result of entropy solution in weighted-Orlicz spaces, other works found by Y. Akdim et al. in their paper [1] devoted to study a degenerated

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problem (1.1) via Minty's Lemma in weighted Orlicz-Sobolev space, in the similar direction faria et al (see [14]) have been treated the similar problem as (1.1) where the solution ϑ of the elliptic problem studied depend on the gradient.

The goal of this note is to examine several elliptic problems for which the classical monotone operators. (introduced by in [17], [16], [8] and in [2] do not apply. We employ a tool to address the challenge posed by the absence of strict monotonicity (which doesn't ensure the almost everywhere convergence of the gradient). This tool involves exploring certain techniques derived from Minty's lemma. The pseudo-monotonicity approach cannot be applied because of the presence of $f \in L^1(\Omega)$. To verify the almost everywhere convergence of $\nabla \vartheta_n$, the authors in [6] demonstrated the boundedness of ϑ_n in the Marcinkiewicz space. Nonetheless, in this investigation, we establish the local convergence in measure of ϑ_n (as outlined in Section 4.2.2).

The mathematical literature dealing existence of solutions to some parabolic and elliptic problems is massive; we refer the reader to [11, 12, 3, 4].

The outline of this note is as follows. After introducing our work, we recall in Section 2 the new anisotropic weighted sobolev spaces, and the section 3 is devoted to some essential assumptions which are necessary to have existence solution, finally section 4 will be devoted to give our Major Conclusions.

2. PRELIMINARIES

Here, we present the enlargement of Sobolev spaces to a novel class of anisotropic weighted Sobolev spaces. Let Ω be a bounded open set of $\mathbf{R}^N$, $p_0, p_1, \dots, p_N$ be $N+1$ exponents with $1 < p_i < \infty$ for $i = 0, 1, \dots, N$ and $w = \{w_i(x), 0 \leq i \leq N\}$ be a vector of weight functions which is measurable and strictly positive a.e. in Ω . Furthermore, we assume the following hypothesis that, there exists

$$w_i \in L^1_{\text{loc}}(\Omega), \quad (2.1)$$

$$w_i^{\frac{-1}{p_i-1}} \in L^1_{\text{loc}}(\Omega), \quad (2.2)$$

for any $0 \leq i \leq N$. The anisotropic weighted orlicz space $L^{p_i}(\Omega, \gamma)$, where γ is a weight function on Ω will be formulated by the following expression,

$$L^{p_i}(\Omega, \gamma) = \left\{ \vartheta = \vartheta(x), \vartheta \gamma^{\frac{1}{p_i}} \in L^{p_i}(\Omega) \right\}$$

and endowed by the norm

$$\|\vartheta\|_{L^{p_i}(\Omega, \gamma)} = \|\vartheta\|_{p_i, \gamma} = \left(\int_{\Omega} |\vartheta(x)|^{p_i} \gamma(x) dx \right)^{\frac{1}{p_i}}.$$

We put

$$(p_i) = (p_0, \dots, p_N), \quad D^0 \vartheta = \vartheta \quad \text{and} \quad D^i \vartheta = \frac{\partial \vartheta}{\partial x_i} \quad \text{for } i = 1, \dots, N,$$

and we consider that

$$\underline{p_i} = \min \{p_0, p_1, \dots, p_N\} \quad \text{then} \quad \underline{p_i} > 1. \quad (2.3)$$

and

$$\|\vartheta\|_{1,(p_i),w} = \|\vartheta\|_{p_0,w_0} + \sum_{i=1}^N \left\| \frac{\partial \vartheta}{\partial x_i} \right\|_{p_i,w_i}. \quad (2.4)$$

The anisotropic weighted Sobolev space $W^{1,(p_i)}(\Omega, w)$ is defined as $\vartheta \in L^{p_0}(\Omega, w_0) = \left\{ u(x), uw_0^{\frac{1}{p_0}} \in L^{p_0}(\Omega) \right\}$ such that $D^i \vartheta \in L^{p_i}(\Omega, w_i)$ for $i = 1, \dots, N$.

Remark 2.1. We use the technical lemmas introduced in [5] and in [2].

Lemma 2.2. [5] *Let $(\vartheta_n)_n$ be a bounded sequence in $W_0^{1,(p_i)}(\Omega, w)$. If $\vartheta_n \rightarrow \vartheta$ in $W_0^{1,(p_i)}(\Omega, w)$, then $T_k(\vartheta_n) \rightarrow T_k(\vartheta)$ in $W_0^{1,(p_i)}(\Omega, w)$ where $T_k(\cdot)$ is the truncation function.*

3. BASIC ASSUMPTIONS AND EXISTENCE RESULTS

We suppose that the norm :

$$\|\vartheta\| = \left(\sum_{i=1}^N \int_{\Omega} \left| \frac{\partial \vartheta}{\partial x_i} \right|^{p_i} w_i(x) dx \right)^{\frac{1}{p_i}} \quad (3.1)$$

is equivalent to (2.4), and there exists $\sigma(x)$ (a weight function) on Ω and, $1 < q_i < \infty$ such that the Hardy inequality,

$$\left(\int_{\Omega} |\vartheta(x)|^{q_i} \sigma(x) dx \right)^{\frac{1}{q_i}} \leq c \left(\sum_{i=1}^N \int_{\Omega} \left| \frac{\partial \vartheta}{\partial x_i} \right|^{p_i} w_i(x) dx \right)^{\frac{1}{p_i}} \quad (3.2)$$

is verified for every $\vartheta \in W_0^{1,(p_i)}(\Omega, w)$ with a constant $c > 0$ independent of ϑ . Furthermore, the imbedding,

$$W_0^{1,(p_i)}(\Omega, w) \hookrightarrow L^{q_i}(\Omega, \sigma) \text{ is compact.} \quad (3.3)$$

Let $\mathbb{B}$ denote a nonlinear operator mapping from $W_0^{1,(p_i)}(\Omega, w)$ to its dual space $W^{-1,p'_i}(\Omega, w^*)$. This operator is defined as:

$$\mathbb{B}(\vartheta) = -\operatorname{div}(B(x, \vartheta, \nabla \vartheta))$$

where $B(x, s, \xi) : \Omega \times \mathbf{R} \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a Carathéodory function fulfills the hypotheses listed below.

For $i = 1, \dots, N$

$$|B_i(x, s, \xi)| \leq \beta w_i^{\frac{1}{p_i}}(x) \left[k(x) + \sigma^{\frac{1}{p_i}} |s|^{\frac{q_i}{p_i}} + \sum_{j=1}^N w_j^{\frac{1}{p_j}}(x) |\xi_j|^{p_i-1} \right], \quad (3.4)$$

for a.e., $x \in \Omega$, all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$, $k(x) \in L^{p'_i}(\Omega)$ $\left(\frac{1}{p_i} + \frac{1}{p'_i} = 1 \right)$ and $\beta > 0$. Here σ and q_i are as in (3.2).

$$\text{for all } (\xi, \eta) \in \mathbb{R}^N \times \mathbb{R}^N, 0 < \langle B(x, s, \xi) - B(x, s, \eta), \xi - \eta \rangle, \quad (3.5)$$

$$\langle B(x, s, \xi), \xi \rangle \geq \alpha \sum_{i=1}^N w_i |\xi_i|^{p_i}, \quad \alpha > 0. \quad (3.6)$$

$H(x, s)$ is a Carathéodory function fulfills the hypothesis listed below

$$H(x, s)s \geq 0. \quad (3.7)$$

$$\sup_{|s| \leq n} |H(x, s)| = h_n(x) \in L^1(\Omega) \quad (3.8)$$

Let us consider, the truncation function :

$$T_k(s) = \begin{cases} s & \text{if } |s| \leq k \\ k \frac{s}{|s|} & \text{if } |s| > k \end{cases} \quad \text{for } k > 1 \text{ and } s \text{ in } \mathbb{R}.$$

Our goal is to study the problem:

$$\begin{cases} \mathbb{B}(\vartheta) + H(x, \vartheta) + \operatorname{div}(F) = f & \text{in } \Omega \\ \vartheta = 0 & \text{on } \partial\Omega \end{cases} \quad (3.9)$$

Where, $\mathbb{B}(\vartheta) = -\operatorname{div}(B(x, \vartheta, \nabla \vartheta))$, the datum f belongs to $L^1(\Omega)$ and $F \in \prod_{i=1}^N L^{p_i'}(\Omega, w_i^*)$.

Definition 3.1. A measurable function ϑ is named an entropy solution of (3.9) if $T_k(\vartheta)$ belongs to $W_0^{1,(p_i)}(\Omega, w)$ and for every $k > 0$ we have

$$\begin{aligned} \int_{\Omega} \langle B(x, \vartheta, \nabla \vartheta), \nabla T_k[\vartheta - \Psi] \rangle dx + \int_{\Omega} H(x, \vartheta) T_k[\vartheta - \Psi] dx &= \int_{\Omega} f T_k[\vartheta - \Psi] dx \\ &+ \int_{\Omega} \langle F, \nabla T_k[\vartheta - \Psi] \rangle dx \end{aligned}$$

for every $\Psi \in W_0^{1,(p_i)}(\Omega, w) \cap L^\infty(\Omega)$.

Theorem 3.2. Let the assumptions (3.1)-(3.7) holds, then ϑ is a solution of (3.9) in the sense of the definition 3.1 such that.

$$\begin{aligned} \int_{\Omega} \langle B(x, \vartheta, \nabla \vartheta), \nabla T_k[\vartheta - \Psi] \rangle dx + \int_{\Omega} H(x, \vartheta) T_k[\vartheta - \Psi] dx &= \int_{\Omega} f T_k[\vartheta - \Psi] dx \\ &+ \int_{\Omega} \langle F, \nabla T_k[\vartheta - \Psi] \rangle dx \end{aligned}$$

4. PROOF OF EXISTENCE THEOREM

4.1. Main Lemma.

Lemma 4.1. Let ϑ be a measurable function such that $T_k(\vartheta) \in W_0^{1,(p_i)}(\Omega, w)$ for every $k > 0$. Then

$$\begin{aligned} \int_{\Omega} \langle B(x, \vartheta, \nabla \vartheta), \nabla T_k[\vartheta - \Psi] \rangle dx + \int_{\Omega} H(x, \vartheta) T_k[\vartheta - \Psi] dx &\leq \int_{\Omega} f T_k[\vartheta - \Psi] dx \\ &+ \int_{\Omega} \langle F, \nabla T_k[\vartheta - \Psi] \rangle dx. \end{aligned}$$

$\Leftrightarrow$

$$\begin{aligned} \langle B(x, \vartheta, \nabla \vartheta), \nabla T_k[\vartheta - \Psi] \rangle dx + \int_{\Omega} H(x, \vartheta) T_k[\vartheta - \Psi] dx &= \int_{\Omega} f T_k[\vartheta - \Psi] dx \\ &+ \int_{\Omega} \langle F, \nabla T_k[\vartheta - \Psi] \rangle dx. \end{aligned}$$

for every Ψ in $W^{1,p_i}(\Omega, w) \cap L^{\infty}(\Omega)$, and for every $k > 0$.

Proof The proof is done in the same way as the proof in [?]

4.2. Proof of Existence theorem 3.2.

4.2.1. A Priori estimate.

Let $f_n \in L^{\infty}(\Omega) \rightarrow f$ strongly in $L^1(\Omega)$ such that $\|f_n\|_{L^1} \leq \|f\|_{L^1}$, and let ϑ_n be a solution in $W_0^{1,p_i}(\Omega, w)$ of the problem

$$\begin{cases} -\operatorname{div} B(x, \vartheta_n, \nabla \vartheta_n) + H_n(x, \vartheta_n) = f_n - \operatorname{div}(F) & \text{in } \Omega \\ \vartheta_n = 0 & \text{on } \partial\Omega \end{cases} \quad (4.1)$$

With

$$H_n(x, s) = \frac{H(x, s)}{1 + \frac{1}{n}|H(x, s)|} \theta_n(x) \text{ and } \theta_n(x) = T_{\frac{1}{n}}\left(\sigma^{\frac{1}{q_i}}(x)\right)$$

which exists thanks to [10].

Now let us taking $T_k(\vartheta_n)$ as test function in (4.1), we may obtain

$$\begin{aligned} \int_{\Omega} \langle B(x, \vartheta_n, \nabla \vartheta_n), \nabla T_k(\vartheta_n) \rangle dx + \int_{\Omega} H_n(x, \vartheta_n) T_k(\vartheta_n) dx &= \int_{\Omega} f_n T_k(\vartheta_n) dx \\ &+ \int_{\Omega} \langle F, \nabla T_k(\vartheta_n) \rangle dx \end{aligned}$$

using $\nabla T_k(\vartheta_n) = \nabla \vartheta_n \chi_{\{|\vartheta_n| \leq k\}}$ and according to hypothesis (3.6), we may get

$$\int_{\Omega} \langle B(x, \vartheta_n, \nabla \vartheta_n), \nabla T_k(\vartheta_n) \rangle dx \geq \alpha \sum_{i=1}^N \int_{\Omega} w_i \left| \frac{\partial T_k(\vartheta_n)}{\partial x_i} \right|^{p_i} dx$$

then since $H_n(x, \vartheta_n) T_k(\vartheta_n) \geq 0$ we have,

$$\begin{aligned} \alpha \sum_{i=1}^N \int_{\Omega} w_i \left| \frac{\partial T_k(\vartheta_n)}{\partial x_i} \right|^{p_i} dx &\leq k \|f\|_{L^1} + \sum_{i=1}^N \int_{\Omega} F_i \left| \frac{\partial T_k(\vartheta_n)}{\partial x_i} \right| dx \\ &\leq k \|f\|_{L^1} + \sum_{i=1}^N \int_{\Omega} F_i w_i^{\frac{-1}{p_i}} \left(\frac{\alpha}{2} \right)^{\frac{-1}{p_i}} \left| \frac{\partial T_k(\vartheta_n)}{\partial x_i} \right| w_i^{\frac{1}{p_i}} \left(\frac{\alpha}{2} \right)^{\frac{1}{p_i}} dx. \end{aligned}$$

Then Young's inequality yields

$$\alpha \sum_{i=1}^N \int_{\Omega} w_i \left| \frac{\partial T_k(\vartheta_n)}{\partial x_i} \right|^{p_i} dx \leq k \|f\|_{L^1} + \frac{c(\alpha)}{p'_i} \|F\|_{\Pi^{p'_i}(\Omega, w_i^*)} + \frac{\alpha}{2} \sum_{i=1}^N \int_{\Omega} w_i \left| \frac{\partial T_k(\vartheta_n)}{\partial x_i} \right|^{p_i} dx$$

Then,

$$\frac{\alpha}{2} \sum_{i=1}^N \int_{\Omega} w_i \left| \frac{\partial T_k(\vartheta_n)}{\partial x_i} \right|^{p_i} dx \leq k \left(\|f\|_{L^1} + \frac{c(\alpha)}{p'_i} \|F\|_{\Pi^{p'_i}(\Omega, w_i^*)} \right).$$

for $k > 1$, which implies that

$$\left(\sum_{i=1}^N \int_{\Omega} \left| \frac{\partial T_k(\vartheta_n)}{\partial x_i} \right|^{p_i} w_i(x) dx \right)^{\frac{1}{p_i}} \leq ck^{\frac{1}{p_i}} \quad \forall k > 1. \quad (4.2)$$

4.2.2. Locally convergence of ϑ_n in measure.

Let $k > 0$ large enough, by using (3.3), we may obtain

$$\begin{aligned} k \text{ meas } (\{|\vartheta_n| > k\} \cap B_R) &= \int_{\{|\vartheta_n| > k\} \cap B_R} |T_k(\vartheta_n)| dx \leq \int_{B_R} |T_k(\vartheta_n)| dx \\ &\leq \left(\int_{\Omega} |T_k(\vartheta_n)|^{p_i} w_0 dx \right)^{\frac{1}{p_i}} \cdot \left(\int_{B_R} w_0^{1-p'_i} dx \right)^{\frac{1}{q_i}} \\ &\leq c \left(\int_{\Omega} \sum_{i=1}^N \left| \frac{\partial T_k(\vartheta_n)}{\partial x_i} \right|^{p_i} w_i(x) dx \right)^{\frac{1}{p_i}} \\ &\leq c_1 k^{\frac{1}{p_i}}. \end{aligned}$$

which gives

$$\text{meas } (\{|\vartheta_n| > k\} \cap B_R) \leq \frac{c_1}{k^{1-\frac{1}{p_i}}} \quad \forall k > 1. \quad (4.3)$$

We obtain, for every $\delta > 0$,

$$\begin{aligned} \text{meas } (\{|\vartheta_n - \vartheta_m| > \delta\} \cap B_R) &\leq \text{meas } (\{|\vartheta_n| > k\} \cap B_R) + \text{meas } (\{|\vartheta_m| > k\} \cap B_R) \\ &\quad + \text{meas } \{|T_k(\vartheta_n) - T_k(\vartheta_m)| > \delta\}. \end{aligned} \quad (4.4)$$

Since $T_k(\vartheta_n)$ is bounded in $W_0^{1,(p_i)}(\Omega, w)$, there exists some $v_k \in W_0^{1,(p_i)}(\Omega, w)$, such that

$$\begin{cases} T_k(\vartheta_n) \rightharpoonup v_k & \text{weakly in } W_0^{1,(p_i)}(\Omega, w), \\ T_k(\vartheta_n) \rightarrow v_k & \text{strongly in } L^{q_i}(\Omega, \sigma) \text{ and a.e. in } \Omega. \end{cases}$$

Consequently, it can be deduced that $T_k(\vartheta_n)$ forms a Cauchy sequence in measure in Ω . Let $\varepsilon > 0$. By utilizing (4.3) and (4.4), there exists a specific $k(\varepsilon) > 0$ such that $\text{meas } (\{|\vartheta_n - \vartheta_m| > \delta\} \cap B_R) < \varepsilon$ for all $n, m \geq n_0(k(\varepsilon), \delta, R)$. This implies that (ϑ_n) is a Cauchy sequence in measure within B_R , thus converging almost everywhere to a measurable function ϑ . Then,

$$\begin{cases} T_k(\vartheta_n) \rightharpoonup T_k(\vartheta) & \text{weakly in } W_0^{1,(p_i)}(\Omega, w), \\ T_k(\vartheta_n) \rightarrow T_k(\vartheta) & \text{strongly in } L^{q_i}(\Omega, \sigma) \text{ and a.e in } \Omega. \end{cases} \quad (4.5)$$

4.2.3. Equi-integrability of $H_n(x, \vartheta_n)$.

We demonstrate that

$$H_n(x, \vartheta_n) \rightarrow H(x, \vartheta) \text{ strongly in } L^1(\Omega). \quad (4.6)$$

Establishing the equi-integrability of $H_n(x, \vartheta_n)$ is sufficient, to do this we may test (4.1) by the function $T_{l+1}(\vartheta_n) - T_l(\vartheta_n)$, we obtain

$$\begin{aligned} \int_{\Omega} \langle B(x, \vartheta_n, \nabla \vartheta_n), \nabla (T_{l+1}(\vartheta_n) - T_l(\vartheta_n)) \rangle dx + \int_{\Omega} H_n(x, \vartheta_n) (T_{l+1}(\vartheta_n) - T_l(\vartheta_n)) dx \\ = \int_{\Omega} f(T_{l+1}(\vartheta_n) - T_l(\vartheta_n)) dx \\ + \sum_{i=1}^N \int_{\Omega} F_i \nabla (T_{l+1}(\vartheta_n) - T_l(\vartheta_n)) dx \end{aligned}$$

which implies that,

$$\begin{aligned} \int_{\{l \leq |\vartheta_n| \leq l+1\}} \langle B(x, \vartheta_n, \nabla \vartheta_n), \nabla \vartheta_n \rangle dx + \int_{\{|\vartheta_n| \geq l+1\}} |H_n(x, \vartheta_n)| dx \\ \leq c \int_{\{|\vartheta_n| \geq l\}} |f| dx + \sum_{i=1}^N \int_{\{l \leq |\vartheta_n| \leq l+1\}} F_i w_i^{\frac{-1}{p_i}} \left(\frac{\alpha}{2}\right)^{\frac{-1}{p_i}} |\nabla \vartheta_n| \left(\frac{\alpha}{2}\right)^{\frac{1}{p_i}} dx \end{aligned}$$

Using Young's inequality we get

$$\begin{aligned} \int_{\{l \leq |\vartheta_n| \leq l+1\}} \langle B(x, \vartheta_n, \nabla \vartheta_n), \nabla \vartheta_n \rangle dx + \int_{\{|\vartheta_n| \geq l+1\}} |H_n(x, \vartheta_n)| dx \\ \leq c \int_{\{|\vartheta_n| \geq l\}} |f| dx + \frac{c(\alpha)}{p'_i} \sum_{i=1}^N \int_{\{|\vartheta_n| \geq l\}} |F_i|^{p'_i} w_i^{1-p'_i} dx \\ + \frac{\alpha}{2} \sum_{i=1}^N \int_{\{l \leq |\vartheta_n| \leq l+1\}} |\nabla \vartheta_n|^{p_i} w_i dx \end{aligned}$$

thus by (3.6), we have

$$\int_{\{|\vartheta_n| \geq l+1\}} |H_n(x, \vartheta_n)| dx \leq c \int_{\{|\vartheta_n| \geq l\}} |f_n| dx + \frac{c(\alpha)}{p'_i} \sum_{i=1}^N \int_{\{|\vartheta_n| \geq l\}} |F_i|^{p'_i} w_i^{1-p'_i} dx.$$

Let $\varepsilon > 0$, then there exist $l(\varepsilon) \geq 1$ such that

$$\int_{\{|\vartheta_n| > l(\varepsilon)\}} |H_n(x, \vartheta_n)| dx \leq \frac{\varepsilon}{2}. \quad (4.7)$$

For any subset $E \subset \Omega$, we get

$$\begin{aligned} \int_E |H_n(x, \vartheta_n)| dx &\leq \int_{E \cap \{|\vartheta_n| \leq l(\varepsilon)\}} |H_n(x, \vartheta_n)| dx + \int_{E \cap \{|\vartheta_n| > l(\varepsilon)\}} |H_n(x, \vartheta_n)| dx \\ &\leq \int_E |h_{l(\varepsilon)}(x)| dx + \int_{E \cap \{|\vartheta_n| > l(\varepsilon)\}} |H_n(x, \vartheta_n)| dx. \end{aligned}$$

According to (3.8) $\exists \eta(\varepsilon) > 0$ such that

$$\int_E |h_{l(\varepsilon)}(x)| dx \leq \frac{\varepsilon}{2} \quad (4.8)$$

for all E such that $\text{meas}(E) < \eta(\varepsilon)$.

Now, thanks to (4.7) and (4.8) we may easily obtain $\int_E |H_n(x, \vartheta_n)| dx \leq \varepsilon$, for all E such that $\text{meas}(E) < \eta(\varepsilon)$.

4.2.4. An intermediate Inequality.

Here, we may show for $\Psi \in W_0^{1,(p_i)}(\Omega, w) \cap L^\infty(\Omega)$, that

$$\begin{aligned} & \int_{\Omega} \langle B(x, \vartheta_n, \nabla \Psi), \nabla T_k[\vartheta_n - \Psi] \rangle dx + \int_{\Omega} H_n(x, \vartheta_n) T_k[\vartheta_n - \Psi] dx \\ & \leq \int_{\Omega} f_n T_k[\vartheta_n - \Psi] dx + \int_{\Omega} \langle F, \nabla T_k[\vartheta_n - \Psi] \rangle dx. \end{aligned} \quad (4.9)$$

Furthermore, by taking $T_k(\vartheta_n - \Psi)$ as test in (4.1), with Ψ in $W_0^{1,(p_i)}(\Omega, w) \cap L^\infty(\Omega)$, we get

$$\begin{aligned} & \int_{\Omega} \langle B(x, \vartheta_n, \nabla \vartheta_n), \nabla T_k[\vartheta_n - \Psi] \rangle dx + \int_{\Omega} H_n(x, \vartheta_n) T_k[\vartheta_n - \Psi] dx \\ & = \int_{\Omega} f_n T_k[\vartheta_n - \Psi] dx + \int_{\Omega} \langle F, \nabla T_k[\vartheta_n - \Psi] \rangle dx. \end{aligned}$$

Then,

$$\begin{aligned} & \int_{\Omega} \langle B(x, \vartheta_n, \nabla \vartheta_n), \nabla T_k[\vartheta_n - \Psi] \rangle dx + \int_{\Omega} \langle B(x, \vartheta_n, \nabla \Psi), \nabla T_k[\vartheta_n - \Psi] \rangle dx \\ & - \int_{\Omega} \langle B(x, \vartheta_n, \nabla \Psi), \nabla T_k[\vartheta_n - \Psi] \rangle dx + \int_{\Omega} H_n(x, \vartheta_n) T_k[\vartheta_n - \Psi] dx \\ & = \int_{\Omega} f_n T_k[\vartheta_n - \Psi] dx + \int_{\Omega} \langle F, \nabla T_k[\vartheta_n - \Psi] \rangle dx \end{aligned} \quad (4.10)$$

According to (3.5) and using the truncation function, we may get

$$\int_{\Omega} \langle [B(x, \vartheta_n, \nabla \vartheta_n) - B(x, \vartheta_n, \nabla \Psi)], \nabla T_k[\vartheta_n - \Psi] \rangle dx \geq 0. \quad (4.11)$$

According to (4.10) and (4.11), we get (4.9).

4.2.5. Passing to the limit.

Here we prove that for $\Psi \in W_0^{1,(p_i)}(\Omega, w) \cap L^\infty(\Omega)$, we have

$$\begin{aligned} & \int_{\Omega} \langle B(x, \vartheta, \nabla \Psi), \nabla T_k[\vartheta - \Psi] \rangle dx + \int_{\Omega} H(x, \vartheta) T_k[\vartheta - \Psi] dx \leq \int_{\Omega} f T_k[\vartheta - \Psi] dx \\ & + \int_{\Omega} \langle F, \nabla T_k[\vartheta - \Psi] \rangle dx. \end{aligned}$$

Firstly, we verify that

$$\int_{\Omega} \langle B(x, \vartheta_n, \nabla \Psi), \nabla T_k[\vartheta_n - \Psi] \rangle dx \rightarrow \int_{\Omega} \langle B(x, \vartheta(\nabla \Psi), \nabla T_k[\vartheta - \Psi]) \rangle dx \text{ as } n \rightarrow +\infty.$$

As $T_M(\vartheta_n) \rightharpoonup T_M(\vartheta)$ weakly in $W_0^{1,(p_i)}(\Omega, w)$, with $M = k + \|\Psi\|_{\infty}$, so by using Lemma 2.2, we may get

$$T_k(\vartheta_n - \Psi) \rightharpoonup T_k(\vartheta - \Psi) \text{ in } W_0^{1,(p_i)}(\Omega, w), \quad (4.12)$$

which gives

$$\frac{\partial T_k}{\partial x_i}(\vartheta_n - \Psi) \rightharpoonup \frac{\partial T_k}{\partial x_i}(\vartheta - \Psi) \text{ weakly in } L^{p_i}(\Omega, w_i) \forall i = 1, \dots, N. \quad (4.13)$$

We now show that

$$B_i(x, T_M(\vartheta_n), \nabla \Psi) \rightarrow B_i(x, T_M(\vartheta), \nabla \Psi) \text{ strongly in } L^{p'_i}(\Omega, w_i^*)$$

By the hypothesis (3.4), we may have

$$\begin{aligned} |B_i(x, T_M(\vartheta_n), \nabla \Psi)|^{p'_i} w_i^{\frac{p'_i}{p_i}} &\leq \beta \left[k(x) + |T_M(\vartheta_n)|^{\frac{q_i}{p_i}} \sigma^{\frac{1}{p_i}} + \sum_{j=1}^N \left| \frac{\partial \Psi}{\partial x_j} \right|^{p_i-1} w_i^{\frac{1}{p_i}} \right]^{p'_i} \\ &\leq \gamma \left[k(x)^{p'_i} + |T_M(\vartheta_n)|^{q_i} \sigma + \sum_{j=1}^N \left| \frac{\partial \Psi}{\partial x_j} \right|^{p_i} w_i \right], \end{aligned} \quad (4.14)$$

with $\beta, \gamma > 0$, Since $T_M(\vartheta_n) \rightharpoonup T_M(\vartheta)$ weakly in $W_0^{1,(p_i)}(\Omega, w)$ and $W_0^{1,(p_i)}(\Omega, w) \hookrightarrow L^q(\Omega, \sigma)$, then $T_M(\vartheta_n) \rightarrow T_M(\vartheta)$ strongly in $L^q(\Omega, \sigma)$ and a.e. in Ω , consequently, $|B_i(x, T_M(\vartheta_n), \nabla \Psi)|^{p'_i} w_i^* \rightarrow |B_i(x, T_M(\vartheta), \nabla \Psi)|^{p'_i} w_i^*$ a.e. in Ω . and

$$\begin{aligned} \gamma \left[k(x)^{p'_i} + |T_M(\vartheta_n)|^{q_i} \sigma + \sum_{j=1}^N \left| \frac{\partial \Psi}{\partial x_j} \right|^{p_i} w_i \right] &\rightarrow \\ \gamma \left[k(x)^{p'_i} + |T_M(\vartheta)|^{q_i} \sigma + \sum_{j=1}^N \left| \frac{\partial \Psi}{\partial x_j} \right|^{p_i} w_i \right] &\text{ a.e. in } \Omega. \end{aligned}$$

We now using Vitali's theorem in order to get

$$B_i(x, T_M(\vartheta_n), \nabla \Psi) \rightarrow B_i(x, T_M(\vartheta), \nabla \Psi) \text{ strongly in } L^{p'_i}(\Omega, w_i^*), \text{ as } n \rightarrow +\infty. \quad (4.15)$$

According to (4.12) and (4.13), we get

$$\int_{\Omega} \langle B(x, \vartheta_n, \nabla \Psi), \nabla T_k[\vartheta_n - \Psi] \rangle dx \rightarrow \int_{\Omega} \langle B(x, \eta, \nabla \Psi), \nabla T_k[\vartheta - \Psi] \rangle dx, \text{ as } n \rightarrow +\infty \quad (4.16)$$

Finally, we prove that

$$\int_{\Omega} f_n T_k [\vartheta_n - \Psi] dx \rightarrow \int_{\Omega} f T_k [\vartheta - \Psi] dx. \quad (4.17)$$

We have $f_n T_k [\vartheta_n - \Psi] \rightarrow f T_k [\vartheta - \Psi]$ a.e. in Ω and $|f_n T_k [\vartheta_n - \Psi]| \leq k |f_n|$ and $k |f_n| \rightarrow k |f|$ in $L^1(\Omega)$, then by using Vitali's theorem, we may get (4.17). Similarly according to (4.6) we can establish that

$$\int_{\Omega} H_n(x, \vartheta_n) T_k [u_n - \Psi] dx \rightarrow \int_{\Omega} H(x, \vartheta) T_k [\vartheta - \Psi] dx \text{ as } n \rightarrow \infty. \quad (4.18)$$

We now show that:

$$\int_{\Omega} \langle F, \nabla T_k [u_n - \Psi] \rangle dx \rightarrow \int_{\Omega} \langle F, \nabla T_k [\vartheta - \Psi] \rangle dx. \quad (4.19)$$

In view of (4.13) and since $F \in \prod^N L^{p'_i}(\Omega, w_i^*)$, we get (4.19).

According to (4.16), (4.17) and (4.19) we may pass to the limit in (4.9), so that $\forall \Psi \in W_0^{1,(p_i)}(\Omega, w) \cap L^\infty(\Omega)$, we deduce

$$\begin{aligned} \int_{\Omega} \langle B(x, \vartheta(\nabla \Psi), \nabla T_k [\vartheta - \Psi]) \rangle dx + \int_{\Omega} H(x, \vartheta) T_k [\vartheta - \Psi] dx &\leq \int_{\Omega} f T_k [\vartheta - \Psi] dx \\ &+ \int_{\Omega} \langle F, \nabla T_k [\vartheta - \Psi] \rangle dx. \end{aligned}$$

Based on the Lemma 4.1, we conclude that ϑ be a solution to the problem (3.9) in the sense of the definition 3.1.

5. CONCLUSION AND REMARK

In the case where $F \equiv 0$, if we assume that the second member are nonnegative, we get a nonnegative solution. Indeed, If we choose $v = T_h(\vartheta^+)$ in (P), we obtain

$$\begin{aligned} \int_{\Omega} \langle B(x, \vartheta, \nabla \vartheta), \nabla T_k (\vartheta - T_h(\vartheta^+)) \rangle dx \\ + \int_{\Omega} H(x, \vartheta) T_k (\vartheta - T_h(\vartheta^+)) dx \\ \leq \int_{\Omega} f T_k (\vartheta - T_h(\vartheta^+)) dx. \end{aligned}$$

As $H(x, \vartheta) T_k (\vartheta - T_h(\vartheta^+)) \geq 0$, we conclude

$$\int_{\Omega} \langle B(x, \vartheta, \nabla \vartheta), \nabla T_k (\vartheta - T_h(\vartheta^+)) \rangle dx \leq \int_{\Omega} f T_k (\vartheta - T_h(\vartheta^+)) dx,$$

we remark, by taking $f \geq 0$

$$\int_{\Omega} f T_k (\vartheta - T_h(\vartheta^+)) dx \leq \int_{\{\vartheta \geq h\}} f T_k (\vartheta - T_h(\vartheta)) dx.$$

Now, according to 3.6, we deduce

$$\alpha \int_{\Omega} \sum_{i=1}^N \left| \frac{\partial T_k(\vartheta^-)}{\partial x_i} \right|^p w_i dx \leq \int_{\{\vartheta \geq h\}} f T_k(\vartheta - T_h(\vartheta)) dx.$$

Let $h \rightarrow \infty$, we may conclude

$$T_k(\vartheta^-) = 0, \quad \forall k > 0,$$

this give

$$\vartheta \geq 0.$$

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