

TOPLINE GRAPH OF A TREE

VENKANAGOUDA M GOUDAR¹

ABSTRACT. In this communications, the concept of topline graph of a tree is introduced. We present characterization of graphs whose topline graph of a tree is planar, maximal outerplanar and minimally nonouterplanar. Also we establish a characterization of graphs whose topline graph of a tree is Eulerian and Hamiltonian.

1. INTRODUCTION AND PRELIMINARIES

The concept of pathos of a graph G was introduced by Harary [3], as a collection of minimum number of line disjoint open paths whose union is G . The path number of a graph G is the number of paths in pathos. The path number of a tree T is equal to k where $2k$ is the number of odd degree vertices of T . Also the end-vertices of each path of any pathos of a tree are of odd degree.

The edgedegree of an edge uv of a tree T is the sum of the degrees of u and v . The pathoslength is the number of edges which lie on a particular path P_i of pathos of T . A pendent pathos is a path p_i of pathos having unit length which corresponds to a pendent edge in T . A pathosvertex is a vertex in $T_L(T)$ corresponding to the path p_i of pathos in T . A graph is said to be minimally nonouterplanar if the inner vertex number $i(G) = 1$.

All graphs considered here are finite, undirected and simple. We refer [3] for unexplained terminology and notation.

The following will be useful in the proof of our results.

Theorem 1.1. [3] *If G is a (p, q) graph whose vertices have degree d_i then $L(G)$ has q vertices and q_L edges, where $q_L = -q + \frac{1}{2} \sum d_i^2$ edges.*

Theorem 1.2. [3] *A graph is planar if and only if it has no subgraph homeomorphic to K_5 or $K_{3,3}$.*

Theorem 1.3. [3] *A graph is outerplanar if and only if it has no subgraph homeomorphic to K_4 or $K_{2,3}$.*

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Theorem 1.4. [5] *A graph G is nonempty path if and only if it is a connected graph with $p \geq 2$ vertices and $\sum d_i^2 - 4p + 6 = 0$.*

Theorem 1.5. [5] *The line graph $L(G)$ of a graph is planar if and only if G is planar, $\Delta(G) \leq 4$ and if $\deg v = 4$, for a vertex v of G , then v is a cutvertex.*

Theorem 1.6. [3] *The line graph $L(G)$ of a graph G is outerplanar if and only if degree of each vertex of G is at most three and every vertex of degree three is a cutvertex.*

Theorem 1.7. [5] *Every maximal outerplanar graph G with p vertices has $(2p-3)$ edges.*

2. MAIN RESULTS

Definition 2.1. The topline graph of a tree T denoted by $T_L(T)$ is defined as the graph whose vertex set is the union of the set of edges and set of paths of paths of T in which two vertices are adjacent if and only if the corresponding edges of T are adjacent, the edge lies on the corresponding path p_i of paths and two paths have a common vertex. Since the system of path of paths for a tree T is not unique, the corresponding topline graph of a tree T is also not unique.

Proposition 2.2. *The edgedegree of an edge uv in a tree is odd if the degree of one vertex is of odd degree.*

Proposition 2.3. *If the edge degree of an edge uv in a tree T is even (or odd) and u and v are cutvertices then the corresponding vertex in $T_L(T)$ is of odd (or even) degree.*

In the following theorem we obtain the number of vertices and edges in $T_L(T)$.

Theorem 2.4. *If T is a tree with p vertices and q edges whose vertices have degree d_i then the topline graph of a tree $T_L(T)$ has $(q+k)$ vertices and $q_L = \frac{1}{2} \sum d_i^2 + [n_i(n_i - 1)/2]$ edges where n_i be the degree of the star v_i and k be the path number.*

Proof. By definition of $T_L(T)$, the number of vertices is the sum of edges and the number of pathos of T . Hence $T_L(T)$ has $(p+k)$ vertices. The number of edges in $T_L(T)$ is the sum of the edges in $L(T)$, the number of edges which lie on the paths p_i of pathos of T which is q and the number of vertices n_i which lies on the pathos p_i which is $[n_i(n_i - 1)/2]$. Hence the number of edges in $T_L(T)$ is $q_L = -q + \frac{1}{2} \sum d_i^2 + q + [n_i(n_i - 1)/2] = q_L = \frac{1}{2} \sum d_i^2 + [n_i(n_i - 1)/2]$. $\square$

Theorem 2.5. *For any edge in a tree T with edgedegree n , the degree of the corresponding vertex in $T_L(T)$ is $(n - 1)$.*

Proof. Consider an edge uv of a tree T with edgedegree n , then it is adjacent to $(n - 2)$ edges. Since the edge itself contributes one degree to each end vertex u and v in T , $(n - 2)$ degree are contributed for the corresponding vertex in $T_L(T)$. Also by definition of $T_L(T)$, every edge in T lies on only one path p_i of pathos. This contributes one more degree to the corresponding vertex in $T_L(T)$. Hence the degree of the vertex in $T_L(T)$ corresponding to an edge in T with edgedegree n is $n - 2 + 1 = n - 1$. $\square$

Theorem 2.6. *For any tree $T \neq K_2$, $T_L(T)$ is non separable.*

Proof. Suppose $T_L(T)$ is non separable. If T has $m \geq 3$ cut vertices and n pendent pathos, we have the following cases.

Case 1. For $m = 1, T = K_{1,n}$. We have the following the subcases of case1.

subcase 1.1 If p is even then $n = 0$, the number of vertices in $T_L(T) = 0$.

Subcase 1.2 If p is odd then $n = 0$ the pendent pathos is adjacent to the remaining pathos. Clearly the number of cut vertices in $T_L(T)=0$.

Case 2. Let $m > 1$, with n pendent pathos. In $L(T)$ each block is a complete subgraph $\langle K_p \rangle$ of $L(T)$ and $T_L(T)$ and this does not increase the number of cut vertex. Since at least two pathos have a vertex in common, so $T_L(T)$ does not contain any pendent vertex. Hence it is nonseparable. $\square$

Theorem 2.7. *For any tree T , $T_L(T)$ is planar if and only if $\Delta(T) \leq 3$.*

Proof. Suppose $T_L(T)$ is planar. Assume that $\Delta(T) \geq 4$. If there exists a vertex v of degree 4 in T , then by Theorem 1.5, $L(T)$ has K_4 as a subgraph. Also by the definition of $T_L(T)$, two edges lie on one path of paths and both paths have a vertex v in common. Clearly $T_L(T)$ has $\langle K_5 \rangle$ as a subgraph which is nonplanar, a contradiction.

For sufficiency, suppose every vertex of T lies on at most three edges. By Theorem 1.5, $L(T)$ is planar. Each block of $L(T)$ is either K_2 or K_3 . The edges joining these blocks from the pathos vertices are adjacent to at most two vertices of each block of $L(T)$. Also the adjacency between the pathos does not lose the planarity. This gives planar $T_L(T)$. $\square$

Theorem 2.8. *For any tree T , $T_L(T)$ is maximal outerplanar if and only if T is a path.*

Proof. Suppose $T_L(T)$ is maximal outerplanar. Then by the Theorem 2.6, $T_L(T)$ is non-separable and hence T is connected.

If $T_L(T)$ is K_2 , then T is so. Let T be any connected tree with $p \geq 2$ vertices, q edges and path number k . Clearly $T_L(T)$ has $(q + k)$ vertices and $q_L = \frac{1}{2} \sum d_i^2 + [n_i(n_i - 1)/2]$ edges. Since $T_L(T)$ is maximal outerplanar, by Theorem 2.7, $T_L(T)$ has $2(q + k) - 3$ edges. Hence $\frac{1}{2} \sum d_i^2 + [n_i(n_i - 1)/2] = 2(q + k) - 3$. For any path T , $k = 1$ and $n_i = 0$. We have $\frac{1}{2} \sum d_i^2 + 0 = 2(q + k) - 3 \Rightarrow \frac{1}{2} \sum d_i^2 = 2q + 2k - 3 = 2q - 1 \Rightarrow \frac{1}{2} \sum d_i^2 = 2(p - 1) - 1 = 2p - 3 \Rightarrow \sum d_i^2 = 4p - 6$ is true. $\Rightarrow \sum d_i^2 - 4p + 6 = 0$. By the Theorem 1.4, it follows that T is a non-empty path. Hence necessity is proved. For sufficiency, suppose T is a path. We have considered two cases.

Case(1). Suppose T is K_2 . Then $T_L(T)$ is K_2 . Hence it is maximal outerplanar.

Case(2). Suppose T is a non-empty path. We prove that $T_L(T)$ is maximal outerplanar by an induction on the number of vertices $n \geq 3$ of T . Clearly $T_L(P_3)$, which is maximal outerplanar.

As the inductive hypothesis, let $T_L(T)$ of a non-empty path T with n vertices be maximal outerplanar. We now show that $T_L(T)$ of a path T' with $(n + 1)$ vertices is maximal outerplanar. Let $T' = v_1, e_1, v_2, e_2, \dots, v_{n-1}, e_n, v_{n+1}$. Without loss

of generality, $T' - v_{n+1} = T$. By inductive hypothesis, $T_L(T)$ has $(e_{n-1}, e_n), (e_n, p_i)$ as the additional edge and these form an induced subgraph and is not K_4 . Hence $T_L(T)$ is outerplanar. Since $T_L(T)$ is maximal outerplanar and it has $2n - 3$ edges. The outerplanar graph $T_L(T)$ has $2n - 3 + 2 = 2(n + 1) - 3$ edges. By the Theorem 1.7, $T_L(T')$ is maximal outerplanar. $\square$

Theorem 2.9. *For any tree T , $T_L(T)$ is minimally nonouterplanar if and only if i). T is a tree with c_i cut vertices and exactly one cut vertex v_c of degree 3 and exactly one cut vertex of degree 2 and ii). T contains one of the path of pathos of length either 2 or 3 and it has a cut vertex v_c as internal vertex.*

Proof. Suppose $T_L(T)$ is minimally nonouterplanar. We have the following cases.

Case 1. Assume that T is a tree with c_i cut vertices and more than one cut vertex is of degree 3. Then T has subgraph homeomorphic to $K_{1,3}$. Now $K_{1,3}$ has two path of pathos and it form $\langle K_3 \rangle$ as induced subgraph in $T_L(T)$. On embedding $T_L(T)$ in any plane, $i[T_L(T)] \geq 2$. Hence it is not minimally nonouterplanar, a contradiction.

Case 2. Assume that T contains one of the path of pathos of length more than 3, it contains a cut vertex which is an internal vertex, then T has a subgraph homeomorphic to $K_{1,3}$. Now $K_{1,3}$ has two pathos vertices and each of these are adjacent to at least two vertex of K_3 of $L(T)$. On embedding $T_L(T)$ in any plane, $i[T_L(T)] \geq 2$. Hence it is not minimally nonouterplanar, a contradiction.

Conversely, suppose T satisfies both conditions. Then T has a subgraph homeomorphic to $K_{1,3}$. Now $K_{1,3}$ has two pathosvertices and one of these vertices p_i adjacent to two vertices of K_3 . Clearly degree of the pathosvertices is 3. By definition of $T_L(T)$, it has a subgraph homeomorphic to K_4 . Hence $i[T_L(T)] = 1$. This is minimally nonouterplanar. $\square$

Theorem 2.10. *For any tree T , $T_L(T)$ is always non-Eulerian.*

Proof. We have the following cases.

Case 1. Suppose every edge of a tree T is of edgedegree odd. By the Proposition 2.2, T contains alternative even and odd degree vertices. Since every path of

pathos starts and ends at odd degree vertices, the path must pass through at least one vertex of even degree. By definition of $T_L(T)$, degree of each vertex in $T_L(T)$, except pathos vertex is even. Since the corresponding pathos are adjacent. Hence $T_L(T)$ is non-Eulerian.

Case 2. Suppose at least one edge of even degree. Then $T_L(T)$ contains at least one vertex of odd degree. Hence $T_L(T)$ is non-Eulerian. $\square$

Theorem 2.11. *For any tree T , $T_L(T)$ is always Hamiltonian.*

Proof. We have the following cases.

Case 1. If T is a path then it has exactly one path of pathos. Let $V[L(T)] = e_1, e_2, \dots, e_n$. In $T_L(T)$, the pathos vertex w is adjacent to $e_1, e_2, \dots, e_n$. Hence $V[T_L(T)] = e_1, e_2, \dots, e_n \cup w$ form a closed path $w, e_1, e_2, \dots, e_n, w$ containing all the vertices of $T_L(T)$, clearly $T_L(T)$ is Hamiltonian.

Case 2. If T has cut vertex of even degree and is not a path. Again we consider the following sub cases.

Subcase 2.1. If T has exactly one vertex of even degree and is a star with n vertices. Then $L(T) = K_n$, which contains a Hamiltonian cycle $e_1, e_2, \dots, e_n, e_1$. For K_n it has $n/2$ paths of pathos with pathos vertex as $p_1, p_2, \dots, p_{n/2}$. By definition of $T_L(T)$, each pathos vertex is adjacent to exactly two vertices of $L(T)$. Clearly, $V[T_L(T)] = e_1, e_2, \dots, e_n \cup p_1, p_2, \dots, p_{n/2}$. Then there exists a cycle containing all the vertices of $T_L(T)$ as $e_1, p_1, e_2, p_2, \dots, e_{n-1}, p_{n/2}, e_n, e_1$ and is a Hamiltonian cycle. Hence $T_L(T)$ is Hamiltonian.

Subcase 2.2. If T has more than one cut vertex of even degree. Then in $L(T)$, each block is complete and every cut vertex lies on exactly two blocks of $L(T)$. Let $V[L(T)] = e_1, e_2, \dots, e_n$. Since T has p_i pathos vertices, $i > 1$, then $V[L(T)]$ contains $e_1, e_2, \dots, e_n, \cup p_1, p_2, p_{n/2}$ vertices. But each p_i is adjacent to exactly two vertices of e_n and it forms a cycle $p_1, e_1, e_2, e_3, e_4, p_2, e_{n-1}, e_n, p_1$ containing all vertices of $T_L(T)$. Hence $T_L(T)$ is Hamiltonian.

Case 3. Suppose T has at least one odd degree vertex. If T has exactly one cut vertex which is of odd degree, then $G = K_{1,n}$ and $L(T) = K_n$ and number of paths of pathos are $\frac{n+1}{2}$ in which there exist at least one path of pathos p_i ,

$1 \leq i \leq \frac{n+1}{2}$ incident to v and an end vertex of T . In $T_L(T)$, each pathosvertex is adjacent to exactly two vertices of K_n . Clearly there exist one vertex of K_n joined by a pathosvertex $p_j, j < i$ and the paths p_i, p_j are adjacent if both have a vertex v in common. Clearly $e_1, e_2, e_3, \dots, p_i, p_j, e_1$, form a cycle contains all vertices of $T_L(T)$. Hence $T_L(T)$ is Hamiltonian.

Case 4. Suppose T has more than one cut vertex of odd degree. Since every path of pathos starts and ends at odd degree vertices. In drawing different path of pathos there exist at least one path of pathos p_i which starts from $\Delta(T)$ and ends at an end vertex of T . Then in $T_L(T)$, $p_i \in V[T_L(T)]$, let $e_1, e_2, \dots, e_i$ be the number of edges incident with vertices of $\Delta(T)$. Then $e_1, e_2, \dots, e_n$ form a complete subgraph. In $T_L(T)$, p_i is adjacent with exactly one vertex of e_i and the vertex p_j which is a common vertex with p_i . Hence $e_1, e_2, \dots, e_n, p_1, p_2, p_i$ form a cycle contains all vertices of $T_L(T)$. Hence $T_L(T)$ is Hamiltonian. $\square$

REFERENCES

1. Chartrand G, Geller, D and Hedetniemi, S. Graphs with forbidden subgraphs J. Combinatorial Theory (1971)10:12.
2. Gudagudi, B R., Doctorial Thesis, Karnataka University, Dharwad, (1975), p. 17.
3. Harary F., Graph theory, Addition - Weseley Reading .Mass. (1969), pp 34 and 107.
4. Harary, F., Annals of New York Academy of Science, (1975), 175 :198.
5. Kulli V R ., On minimally non- outer planar graphs, Proceeding of the Indian National Science Academy, Vol.41, Part A, No.3 (1975), pp 275 -280.

SRI GAUTHAMA RESEARCH CENTRE, (AFFILIATED TO KUVEMPU UNIVERSITY) DEPARTMENT OF MATHEMATICS,, SRI SIDDHARTHA INSTITUTE OF TECHNOLOGY,, TUMKUR-572 102, KARNATAKA, INDIA.

E-mail address: vmgouda@gmail.com