Gulf Journal of Mathematics ISSN: 2309-4966
Vol 17, Issue 1 (2024) 44-57 https://doi.org/10.56947/gjom.v17i1.1723

SOLUTION FORMS OF DISCRETISED DIRAC SYSTEM WITH SUMMABLE AND UNBOUNDED POTENTIALS

BENARD OWINO[1]*, FREDRICK NYAMWALA[2] AND DAVID AMBOGO[3]

Abstract. We have investigated the form of solutions of discretised Dirac system with bounded and unbounded potentials. This we have done using asymptotic summation based on the discretised Levinson's theorem. We have shown that the asymptotic behaviour of the matrix solutions of the associated Dirac systems are determined by the dominant corrected eigenvalues depending on the spectral summable parameter in the case of bounded potential and on the dominant corrected eigenvalues depending on the unbounded potential. In both cases, the spectrum of the associated Dirac system is discrete.

1. Introduction

Consider the continuous matrix Cauchy problem for the Dirac system:

$$D'(x) + J(x)D(x) = A_\mu D(x), \qquad D(0) = I \tag{1.1}$$

where $x \in [0,1]$, $A_\mu = i\mu J_0$ and

$$J_0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad J(x) = \begin{bmatrix} 0 & \delta_1(x) \\ \delta_2(x) & 0 \end{bmatrix}, \qquad I := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$\mu \in \mathbb{C}$ is a spectral parameter, $\delta_j(x) \in L_1[0,1]$, $j = 1,2$ are complex valued functions known as potentials. In this case the potentials are integrable. The solutions of (1.1) is a matrix D with entries from the space of absolutely continuous functions on $[0,1]$, see [10] for more details. In this section, we have given some literature concerning the investigations of (1.1), formulation of the discretised version of (1.1) and the gap in knowledge especially on the discretised Cauchy Dirac Systems. The asymptotic behaviour of the fundamental solutions of Dirac system (1.1) has been investigated by the authors, see [6, 10] and some related literature in the references [4, 5, 7]. As such, a variety of asymptotic formulae for the solutions to (1.1) in different settings and under a wide range of assumptions on the potentials have been generated. However, a similar study for the case of the discretised version of (1.1) is scarce and needs to be analysed under different decay and growth conditions. Moreover, when the potentials are unbounded, the analysis of how asymptotic behaviour of solutions of the discretised Dirac system

Date: Received: Feb 6, 2024; Accepted: Apr 28, 2024.
* Corresponding author.
2010 *Mathematics Subject Classification.* Primary 39A05; Secondary 39A12.
Key words and phrases. Dirac System, Transformations, Asymptotic behaviour, Discretised Levinson's theorem.

is affected by the growth of the potentials is lacking. In this study, we have considered the discretised system of (1.1) and analysed the form of solutions when the potentials are summable and spectral parameter unbounded as well as when the potentials are unbounded and the spectral parameter bounded. We start by formulating the discretised version of (1.1) and explain the methodology that have been employed to obtain the main results. The discretised Dirac system considered is of the form:

$$\Delta D(t) + J(t)RD(t) = A_{\mu}RD(t), \qquad D(t_0) = I, \qquad t_0 \in \mathbb{N}, \tag{1.2}$$

t_0 is the left regular end point where the necessary boundary conditions are imposed. $D(t)$ is a 2×2 matrix defined by

$$D(t) = \begin{bmatrix} p_{11}(t) & p_{12}(t) \\ p_{21}(t) & p_{22}(t) \end{bmatrix}.$$

Δ is a forward difference operator defined by $\Delta f(t) = f(t+1) - f(t)$, $f(t)$ a complex valued function, $t \in \mathbb{N}$. R in this case is a partial forward shift operator whose action in $D(t)$ is defined by

$$R(D(t)) = \begin{bmatrix} p_{11}(t+1) & p_{12}(t+1) \\ p_{21}(t) & p_{22}(t) \end{bmatrix}.$$

Just like in the case of continuous version where J_0 was taken as the value of $J(x)$ at the initial left regular end point $x = 0$, here $J(t_0)$, t_0, the left regular end point, will be its equivalent and hence we have

$$J_{t_0} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad J(t) = \begin{bmatrix} 0 & \delta_1(t) \\ \delta_2(t) & 0 \end{bmatrix}, \qquad I := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$\mu \in \mathbb{C}$ is a spectral parameter. In the sequel, we shall define the inner products and the space of operation as follows:

$$\ell^2(\mathbb{N}) = \{D(t) : \{D(t)\}_{t=t_0}^{\infty} \subset \mathbb{C}^{2\times 2} : \textstyle\sum_{t=t_0}^{\infty} (RD(t))^*(RD(t)) < \infty\}$$

and

$$\langle D_1(t), D_2(t)\rangle = \sum_{t=t_0}^{\infty} \overline{D_1(t)} D_2(t), \qquad D_1(t), D_2(t) \in \ell^2(\mathbb{N}).$$

In the case of summable potentials, there exists $t_1 \in \mathbb{N}$, $t_1 > t_0$, such that the integral domain $\mathcal{I} = [t_0, t_1]$ has the property

$$\sum_{t=t_0}^{t_1-1} (RD(t))^*(RD(t)) < \infty.$$

This integral domain leads to the Hilbert space $\ell^2(\mathcal{I})$. In the first part of the analysis, we will assume that the potentials $\delta_j(t)$ are summable and spectral parameter $\mu \in \mathbb{C}$ is in the set P_d defined by

$$P_d = \{\mu = z_0 + iz_1 : z_0, z_1 \in \mathbb{R}, |z_1| < d\}$$

though we will allow $|\mu| \to \infty$ in the case $|z_0| \to \infty$. Here, $d > 0$. This will be adjusted appropriately in the case of unbounded potentials. The methodology largely applied to obtain the solutions of (1.2) in Sections 2 and 3 under different growth and decay conditions of the potentials and spectral parameter is asymptotic summation based on the discretised Levinson's theorem [3, 8, 9]. Since we are interested in the form of solutions for (1.2) that we are going to solve using asymptotic summation, it is advisable that we convert it to a form which we can easily apply the discretised Levinson's theorem. Thus reformulation of (1.2) leads to the following relations:

$$\begin{aligned} p_{11}(t+1) &= (1-i\mu)^{-1}p_{11}(t) - (1-i\mu)^{-1}\delta_1 p_{21}(t), \\ p_{12}(t+1) &= (1-i\mu)^{-1}p_{12}(t) - (1-i\mu)^{-1}\delta_1 p_{22}(t), \\ p_{21}(t+1) &= -(1-i\mu)^{-1}\delta_2 p_{11}(t) + [(1-i\mu)^{-1}\delta_1\delta_2 + (1-i\mu)]p_{21}(t), \\ p_{22}(t+1) &= -(1-i\mu)^{-1}\delta_2 p_{12}(t) + [(1-i\mu)^{-1}\delta_1\delta_2 + (1-i\mu)]p_{22}(t). \end{aligned} \tag{1.3}$$

Here, $\delta_j = \delta_j(t)$, $j = 1, 2$. Application of the equations (1.3) in (1.2) and re-writing (1.2) in its first order transfer form [2, 11]

$$D(t+1,\mu) = C(t,\mu)D(t,\mu) \tag{1.4}$$

where,

$$C(t,\mu) = \frac{1}{1-i\mu}\begin{bmatrix} 1 & -\delta_1 \\ -\delta_2 & 1+\delta_1\delta_2-\mu^2-2i\mu \end{bmatrix}.$$

Equation (1.4) is now solved based on discretised version of Levinson's theorem. In its simplest form, the theorem can be restated in its fundamental matrix form, see [2, 3, 11] for details.

Theorem 1.1. *Let*

$$D(t+1,\mu) = [\Lambda(t,\mu) + \mathfrak{R}(t,\mu)]D(t,\mu), \tag{1.5}$$

$D(t_0) = I$, *be the first order system of* (1.2) *and assume that* $\|\mathfrak{R}_{kj}(t,\mu)\| \in \ell^1$ *with* $\Lambda(t,\mu) = diag\{\lambda_1(t,\mu), \lambda_2(t,\mu)\}$ *satisfying the* $\mu-$*uniform dichotomy condition, that is*

$$\left| \frac{\lambda_k(t,\mu)}{\lambda_j(t,\mu)} \right| < 1-\gamma \ \ or \ \left| \frac{\lambda_k(t,\mu)}{\lambda_j(t,\mu)} \right| > 1+\gamma,\ k \neq j,\ \gamma > 0,\ \mu \in P_d,$$

then the solutions of this system and those of (1.2) *are of the form*

$$D(t,\mu) = E(t,\mu)\prod_{l=t_0}^{t_1-1}(\Lambda(l,\mu)).$$

Here, $E(t,\mu)$ is a normalized 2×2 matrix with its first normalised components given by

$$E_{1k}(t,\mu) = e_k(t),$$

$k = 1, 2, e_k(t)$ is a normalized eigenvector. The proof of this theorem is done using the variation of constant approach and for details, see [1, 2, 3, 11]. Behncke and Nyamwala [2], greatly simplified the analysis of uniform dichotomy condition to the extent that it is established for eigenvalues with unit magnitude since it will be satisfied for the other set of eigenvalues. Our interest in this study was the solution form and asymptotic behaviour of the solutions of (1.2) as well as (1.4) when either the spectral parameter, μ, is unbounded in some strip within the complex plane and the potentials are bounded within some integral interval or when the potentials are unbounded and the spectral parameter is within a bounded strip. The method used to transform (1.4) into its Levison's form, see [2, 3, 11] for more details, involve a chain of transformations and application of discretised Levinson's theorem. Here, because of the decay conditions imposed on the entries of $D(t)$ and the potentials $\delta_j(t)$, see (2.2), we have applied two diagonalisations and $[I + Q(t, \mu)]-$transformations, see [3] to change (1.4) into the required form (1.5).

Our results state that if the eigenvalues of $C(t, \mu)$ in (1.4) satisfy uniform dichotomy in μ, $\mu \in P_d$, then the form of solutions and asymptotic behaviour of solutions of (1.2) are determined by the magnitude of the corrected eigenvalues of $C(t, \mu)$ and the product of the transforming matrices. These largely depend on the dominant parameter. In the unbounded potential in the second case, they depend on $\delta_1(t)$. In both cases, the solutions which are square summable are uniformly square summable and similar observation is true for non-square summable solutions. The spectrum of the associated discrete operator is discrete.

In the sequel, the expansion of the eigenvalues and characteristic polynomial in this section, is done with the assumption that μ is unbounded and $\delta_j(t)$, $j = 1, 2$ are summable. The case of unbounded potentials are discussed in Section 4. Though, the analysis largely remain the same with some slight difference being in the eigenvalue expansion either in μ or in $\delta_1(t)$ which are the unbounded elements in Sections 1-3 and Section 4 respectively.

Asymptotic summation based on Theorem 1.1 requires that we find the eigenvalues of $C(t, \mu)$ which are determined from the polynomial

$$\mathcal{P}_\lambda(t, \mu) = det(C(t, \mu) - \lambda I_2).$$

Thus, we obtain the characteristic polynomial

$$\mathcal{P}_\lambda(t, \mu) = (1 - i\mu)^2\lambda^2 - (1 - i\mu)(2 + \delta_1\delta_2 - \mu^2 - 2i\mu)\lambda + (1 - \mu^2 - 2i\mu) \quad (1.6)$$

whose zeros are the eigenvalues of $C(t, \mu)$ and which when approximated to the term $O(\mu^{-5})$ results into the following roots:

$$\lambda_1(t, \mu) \approx -i\mu + i\mu^{-1}\delta_1(t)\delta_2(t) - i\mu^{-3}\delta_1(t)\delta_2(t) + O(\mu^{-5}), \quad (1.7)$$
$$\lambda_2(t, \mu) \approx i\mu^{-1} + i\mu^{-3}\delta_1(t)\delta_2(t) + O(\mu^{-5}).$$

The polynomial $\mathcal{P}_\lambda(t, \mu)$ has two distinct roots. The corresponding eigenvectors are then approximated from the relations

$$\nu_1(t, \mu) \approx \begin{bmatrix} 1 & 1 - \lambda_2(t, \mu) \end{bmatrix}^{tr} \quad (1.8)$$
$$\nu_2(t, \mu) \approx \begin{bmatrix} 1 & 1 - \lambda_1(t, \mu) \end{bmatrix}^{tr}.$$

Here $'tr'$ means transpose. Thus a matrix of the form

$$T(t, \mu) \approx \begin{bmatrix} \nu_1(t, \mu) & \nu_2(t, \mu) \end{bmatrix}$$

diagonalises the matrix $C(t, \mu)$. The matrix for second diagonalisation and the transformation, $[I + Q(t)]$, that accelerates the decay of the off-diagonal terms that are not summable after second diagonalisation, are discussed in Section 2. The analysis as carried out here can be extended to any discretised Dirac system of order n in a straight forward way. The only task shall be computation of eigenvalues and establishment of uniform dichotomy conditions. We now give preliminary results required in transforming the discretised Dirac system to achieve the desired Levinson's form.

2. Preliminaries

Here, we have given some preliminaries on the transformations that are required to convert (1.4) into the required form (1.5). The assumptions made here are that

$$\delta_j(t) \in \ell^1([t_0, t_1]), \qquad |\mu| \to \infty, \qquad \mu \in P_d. \tag{2.1}$$

$\delta_j(t)$ have second forward difference with

$$\begin{aligned} &\Delta(p_{kj}(t)), \Delta(\delta_j(t)) \in \ell^2[t_0, t_1] \\ &\Delta^2(\delta_j(t)), (\Delta(\delta_j(t)))^2, \Delta^2(p_{kj}(t)), (\Delta(p_{kj}(t)))^2 \in \ell^1[t_0, t_1]. \end{aligned} \tag{2.2}$$

In the next Lemma we have established that the roots of $\mathcal{P}_\lambda(t, \mu)$ as given in (1.8) satisfy the required $\mu-$uniform dichotomy condition.

Lemma 2.1. *Suppose (2.1)and (2.2) are satisfied, then the eigenvalues of $C(t, \mu)$ which are the zeros of $\mathcal{P}_\lambda(t, \mu)$ given in (1.7) satisfy the $\mu-$uniform dichotomy condition.*

Proof. As stated in Section 1, we need to show that the value

$$\left| \frac{\lambda_1(t,\mu)}{\lambda_2(t,\mu)} \right| < 1 - \gamma \text{ or } \left| \frac{\lambda_1(t,\mu)}{\lambda_2(t,\mu)} \right| > 1 + \gamma$$

as $|\mu| \to \infty$ for some $\gamma > 0$, however small. This should be true for all $\mu \in P_d$. Here, $\lambda_1(t, \mu)$ and $\lambda_2(t, \mu)$ are as given in (1.7). Therefore, computing $\left| \frac{\lambda_1(t,\mu)}{\lambda_2(t,\mu)} \right|$ with the assumptions in (2.1) we obtain

$$\left| \frac{\lambda_1(t,\mu)}{\lambda_2(t,\mu)} \right| \approx |\mu|^2 + c$$

where $c = O(|\delta_1\delta_2 - 2\mu^{-2}\delta_1^2\delta_2^2|)$. As $|\mu| \to \infty$ and invoking the results of [2], the required dichotomy condition is satisfied since $|\lambda_1(t, \mu)| > 1$ while $|\lambda_2(t, \mu)| < 1$ uniformly in μ. ☐

For us to reduce (1.4) into (1.5) we will require two diagonalisations because of (2.2). Therefore the required diagonalising matrix which is now used to transform (1.4) is given by

$$T(t, \mu) \approx \begin{bmatrix} 1 & 1 \\ 1 - \lambda_2(t, \mu) & 1 - \lambda_1(t, \mu) \end{bmatrix}. \tag{2.3}$$

Hence we make a transformation of the form

$$D(t, \mu) = T(t, \mu)W(t, \mu) \tag{2.4}$$

that takes (1.4) into

$$\begin{aligned} W(t+1, \mu) &= T^{-1}(t+1, \mu)C(t, \mu)T(t, \mu)W(t, \mu) \\ &= [\Lambda(t, \mu) + \mathfrak{R}(t, \mu)]W(t, \mu) \end{aligned}$$

where,

$$\mathfrak{R}(t,\mu) = -T^{-1}(t+1,\mu)\Delta T(t,\mu)\Lambda(t,\mu). \tag{2.5}$$
$$\Lambda(t,\mu) = diag\,(\lambda_k(t,\mu)),\ k=1,2.$$

Here, $\mathfrak{R}(t,\mu)$ consists of ℓ^2 and ℓ^1 terms. This suggests that the eigenvalues of $C(t,\mu)$ are responsible for the asymptotic behaviour of the solutions of (1.4) although there will be some correction terms as a result of the first transformation which are given by $\mathfrak{R}_{kk}(t,\mu)$, $k=1,2$. In (2.5), $\Delta T(t,\mu) = T(t+1,\mu) - T(t,\mu)$. Explicitly, the correction terms are of the form $O(i\mu^{-3}\Delta(\delta_1\delta_2))$. Because of the decay assumptions in (2.2) on the functions $\delta_j(t)$, $j=1,2$, one requires second diagonalisation. This is done by use of the eigenvectors of the matrix $[\Lambda(t,\mu) + \mathfrak{R}(t,\mu)]$. Using the results of Behncke [1], a matrix of the form $[I + B(t,\mu)]$ with

$$B_{kk}(t,\mu) = 0, \qquad B_{kj}(t,\mu) = (\lambda_j - \lambda_k)^{-1}\mathfrak{R}_{kj}, \tag{2.6}$$
$$k \neq j, \qquad k,j = 1,2, \qquad t \geq t_0.$$

will be required for the second diagonalisation. The second diagonalisation will result into correction terms added to the diagonals given by $(\Lambda_2)_{kk} = diag((\mathfrak{R}B)_{kk})$ which explicitly are given by $diag(O(\mu^{-7}(\Delta(\delta_1\delta_2))^2))$. The second diagonalisation is thus done using the transformation

$$W(t,\mu) = [I + B(t,\mu)]X(t,\mu) \tag{2.7}$$

and which results into a system of the form

$$X(t+1,\mu) = \{[\Lambda(t,\mu) + \Lambda_2(t,\mu)] + [I + B(t+1,\mu)]^{-1}\mathfrak{R}(t,\mu)[I + B(t,\mu)]\}X(t,\mu) \tag{2.8}$$

where $(\Lambda_2) = diag(\mathfrak{R}B)_{kk}$. One, therefore obtains the following result on the corrected eigenvalues with their expansion on μ.

Lemma 2.2. *Define Dirac system in* (1.4)*with* (2.1)– (2.2) *satisfied. The eigenvalues of the form* $\tilde{\lambda_k}(t,\mu) \approx \lambda_k(t,\mu) + O(i\mu^{-3}\Delta(\delta_1\delta_2)) + O(\mu^{-7}(\Delta(\delta_1\delta_2))^3)$ *determines the asymptotic behaviour of the solutions when two diagonalisations are performed.*

Proof. In this case we assume that after two diagonalisations, the Dirac system will be converted into the form in (1.5). One therefore, computes the eigenvalues of $C(t,\mu)$ which are the zeros of the polynomial $\mathcal{P}_\lambda(t,\mu) = det[C(t,\mu) - \lambda I_2]$ and are of the form:

$$\begin{aligned} \lambda_1(t,\mu) &\approx -i\mu + i\mu^{-1}\delta_1(t)\delta_2(t) - i\mu^{-3}\delta_1(t)\delta_2(t) + O(\mu^{-5}), \\ \lambda_2(t,\mu) &\approx i\mu^{-1} + i\mu^{-3}\delta_1(t)\delta_2(t) + O(\mu^{-5}). \end{aligned}$$

Thus the eigenvectors can be computed and approximated to the term $O(\mu^{-5})$ and then a transformation using the matrix of the eigenvectors applied. In our case, this is the matrix $T(t,\mu)$ whose determinant is $O(\mu)$ and hence $T(t,\mu)$ has inverse. Using the transformation as given in (2.4), the corrections to the diagonals are given by the diagonals of the matrix $\mathfrak{R}(t,\mu)$ and hence $(\mathfrak{R}(t,\mu))_{kk} \approx i\mu^{-3}\Delta(\delta_1\delta_2) + O(\mu^{-5}\delta_1\delta_2\Delta(\delta_1\delta_2))$. In order to perform a second diagonalisation as explained earlier, we absorb $(\mathfrak{R}(t,\mu))_{kk}$ into $\Lambda(t,\mu) = diag(\lambda_k(t,\mu))$, $k = 1,2$ to obtain

$$W(t+1,\mu) = [\tilde{\Lambda}(t,\mu) + \tilde{\mathfrak{R}}(t,\mu)]W(t,\mu) \tag{2.9}$$

where

$$\tilde{\Lambda}(t,\mu) = diag\{\lambda_k(t,\mu) + O(i\mu^{-3}\Delta(\delta_1\delta_2))\}$$

and

$$\tilde{\mathfrak{R}}(t,\mu) = \mathfrak{R}(t,\mu) - diag((\mathfrak{R}(t,\mu))_{kk}, k = 1,2.$$

A second diagonalisation is then performed using the matrix $[I + B(t,\mu)]$ with

$$(B(t,\mu))_{kk} = 0, \qquad (B(t,\mu))_{kj} = (\lambda_j - \lambda_k)^{-1}(\tilde{\mathfrak{R}}(t,\mu))_{kj}, \ \ k,j = 1,2, \ \ k \neq j.$$

The matrices $\tilde{\mathfrak{R}}(t,\mu)$ and $B(t,\mu)$ are approximately given by

$$\begin{bmatrix} 0 & -i\mu^{-3}\Delta(\delta_1\delta_2) + O(\mu^{-4}) \\ -i\mu^{-3}\Delta(\delta_1\delta_2) + O(\mu^{-4}) & 0 \end{bmatrix} \tag{2.10}$$

and

$$\begin{bmatrix} 0 & -\mu^{-4}(\Delta(\delta_1\delta_2))^2 + O(\mu^{-5}) \\ \mu^{-4}(\Delta(\delta_1\delta_2))^2 + O(\mu^{-5}) & 0 \end{bmatrix} \tag{2.11}$$

respectively. Thus by making a transformation of the form (2.7), a second diagonalisation as outlined above will now result into correction terms of the form $(\tilde{\mathfrak{R}}B)_{kk}$, $k = 1,2$ that explicitly are of the form

$$\begin{aligned} (\tilde{\mathfrak{R}}B)_{11} &\approx -i\mu^{-7}(\Delta(\delta_1\delta_2))^3, \\ (\tilde{\mathfrak{R}}B)_{22} &\approx i\mu^{-7}(\Delta(\delta_1\delta_2))^3. \end{aligned}$$

Hence, after two diagonalisations, the corrected eigenvalues are $\tilde{\lambda}_k(t,\mu)$ given by

$$\tilde{\lambda}_k(t,\mu) \approx \lambda_k(t,\mu) + O[\mu^{-3}\Delta(\delta_1\delta_2)] + O[\mu^{-7}(\Delta(\delta_1\delta_2))^3]. \tag{2.12}$$

Each summand after $\lambda_k(t,\mu)$ contributed by each diagonalisation. By assumptions in (2.1) and (2.2), the system (2.8) will be in the form of (1.5) since the off diagonal terms are absolutely summable and by application of the discretised Levinson's theorem the asymptotics of solutions of (1.4) are determined by $\tilde{\lambda}_k(t,\mu)$. □

Finally, we discuss transformations that accelerates the decay of the off-diagonal terms, that is, $[I + Q(t,\mu)]-$transformations as discussed by Behncke [1, 3]. It should be noted that after two diagonalisations, not all the off-diagonal terms may be in ℓ^1, especially when $\delta_j(t)$ are unbounded. This transformation is required to convert any ℓ^2-terms that might remain after two diagonalisations into ℓ^1-terms and accelerate the decay of ℓ^1-terms. Now re-write the system (2.8) in the form

$$X(t+1,\mu) = [\tilde{\Lambda}(t,\mu) + V(t,\mu)]X(t,\mu) \tag{2.13}$$

where,

$$\begin{aligned}\tilde{\Lambda}(t,\mu) &= \Lambda(t,\mu)+\Lambda_2(t,\mu)\\ \Lambda(t,\mu) &= diag(\lambda_k+O(\mu^{-3}\Delta(\delta_1\delta_2)))\\ \Lambda_2(t,\mu) &= diag(O(\mu^{-7}(\Delta(\delta_1\delta_2))^2))\\ V_{kk}(t,\mu) &= 0\\ V_{kj}(t,\mu) &= (\Re B)_{kj}(t,\mu)\end{aligned}$$

for $k,j=1,2$. Applying the transformation

$$X(t,\mu) = [I+Q(t,\mu)]Z(t,\mu)$$

on the system (2.13), we obtain

$$Z(t+1,\mu)=[I+Q(t+1,\mu)]^{-1}[\tilde{\Lambda}(t,\mu)+V(t,\mu)][I+Q(t,\mu)]Z(t,\mu). \tag{2.14}$$

Here,

$$Q_{12}(t,\mu)\approx -\sum_{t_0}^{t_1-1}\frac{V_{12}(s,\mu)}{\tilde{\lambda}_2(t,\mu)}\prod_{l=s}^{t}\frac{\tilde{\lambda}_2(t,\mu)}{\tilde{\lambda}_1(t,\mu)}$$

since $|\frac{\lambda_1(t,\mu)}{\lambda_2(t,\mu)}|>1+\gamma$, $\gamma>0$ and

$$Q_{21}(t,\mu)\approx \sum_{t_0}^{t_1-1}\frac{V_{21}(s,\mu)}{\tilde{\lambda}_2(s,\mu)}\prod_{l=s}^{t}\frac{\tilde{\lambda}_2(t,\mu)}{\tilde{\lambda}_1(t,\mu)}$$

because $|\frac{\lambda_2(t,\mu)}{\lambda_1(t,\mu)}|<1-\gamma$, $\gamma>0$ small, as $|\mu|\to\infty$. Explicit computation now gives

$$\begin{aligned}V_{kk}(t,\mu) &= 0\\ V_{kj}(t,\mu) &= (\Re B)_{kj}(t,\mu)\\ &\approx O(\mu^{-7}(\Delta(\delta_1\delta_2))^2)\end{aligned}$$

Therefore, applying the relations in (3.16a), (3.16b), (3.19) and (3.20) in Benzaid-Lutz paper [3] together with the eigenvalues in (1.7) and the workings in the proof of Theorem 2.1 in [3], we have $Q_{kj}(t,\mu)$ as follows:

$$\begin{aligned}Q_{12}(t,\mu) &= -Q_{21}(t,\mu)\\ &= -\sum_{t_0}^{t_1-1}\frac{V_{21}(s,\mu)}{\tilde{\lambda}_2(s,\mu)}\prod_{l=s}^{t}\frac{\tilde{\lambda}_2(t,\mu)}{\tilde{\lambda}_1(t,\mu)}\\ &\approx -\sum_{s=t_0}^{t_1-1}O\left(\mu^{-6}(\Delta\delta_1\delta_2)^2\right)(s)\prod_{l=s}^{t}[O(\mu^{-2})+O(\mu^{-4}\delta_1\delta_2)](l).\end{aligned}$$

Any subsequent $[I+Q_j(t,\mu)]-$transformations are applied with assumptions that $Q_j(t,\mu)=o(Q(t,\mu))$ for $j=1,2,\cdots$. Hence, mathematically, only the first $[I+Q(t,\mu)]-$transformation has significance in the asymptotic behaviour of solutions of (1.4). In line with the arguements in Benzaid-Lutz paper [3] and highlighted by Behncke [1], $Q_{kj}(t,\mu)$ are bounded since $|Q_{kj}(t,\mu)|$ are majorised by convergent geometric series that can be shown using Hölder's inequality. Thus in Section 3, which contains our main results we will assume the following growth conditions on $Q_k(t,\mu)$.

$$Q_k(t,\mu) = o(Q(t,\mu)), \qquad k = 1,2,3,\cdots \qquad t \in [t_0,t_1], \qquad \mu \in P_d. \tag{2.15}$$

3. Main Results

Here , we have proved main results based on different growth and decay conditions imposed on the spectral parameters and potentials.

Theorem 3.1. *Suppose all the conditions in Lemma 2.2 are satisfied in addition to (2.15) then the solutions of the Dirac system (1.2) after two diagonalisations and a chain of $[I+Q(t,\mu)]-$transformations are given by*

$$D(t,\mu) \quad = \quad T(t,\mu)[I+B(t,\mu)][I+Q(t,\mu)][E(t,\mu)+\circ(I)]\prod_{s=t_o}^{t-1}[\tilde{\Lambda}(s,\mu)]$$

where $E(t,\mu)$ is a 2×2 matrix of the normalized eigenvectors, $\tilde{\Lambda}(s,\mu) = diag[\tilde{\lambda_1}(s,\mu),\tilde{\lambda_2}(s,\mu)]$ and

$$\tilde{\lambda_k}(s,\mu) = \lambda_k(s,\mu) + O(\mu^{-3}\Delta(\delta_1\delta_2)^2) + O(\mu^{-7}\Delta(\delta_1\delta_2)^2).$$

The asymptotic behaviour of these solutions are determined by the leading power of μ.

Proof. The proof of this theorem follow closely from that of Lemma 2.2. The Dirac system in (1.2) is converted into its first order system (1.4) and the eigenvalues of the 2×2 matrix $C(t,\mu)$ are computed. The uniform dichotomy condition now follows from Lemma 2.1. The corresponding eigenvectors are computed and diagonalising matrix $T(t,\mu)$ is obtained with the columns of $T(t,\mu)$ as the eigenvectors of $C(t,\mu)$. A transformation of the form (2.4) is then applied on the system (1.4) to obtain a system with remainder terms $\mathfrak{R}(t,\mu)$ given by (2.5) and $\Lambda(t,\mu)$ basically the diagonal matrix $diag(\lambda_k(t,\mu))$, $k=1,2$. Because of the decay assumptions in (2.2), the remainder matrix can be split as a sum of two matrices $\mathfrak{R}_1(t,\mu)$ and $\mathfrak{R}_2(t,\mu)$ with elements of the former in ℓ^1 and those of the latter in ℓ^2. A second diagonalisation can then be carried out using the transformation $W(t,\mu) = [I+B(t,\mu)]X(t,\mu)$ with the matrix $B(t,\mu)$ computed as explained in Section 2. This will result into a system of the form (2.8). Any elements in the remainder matrix that is not small enough, that is, not in ℓ^1 can be made smaller using a chain of $[I+Q(t,\mu)]-$transformation based on the assumption (2.15). In this case, it is assumed that the first $[I+Q(t,\mu)]-$transformation will dominate the other $[I+Q_k(t,\mu)]-$transformation, $k=1,2,\cdots$. Hence the transformation

$$X(t,\mu) = [I+Q(t,\mu)]Z(t,\mu)$$

has an effect on the asymptotics of the solutions of the system (1.4). The asymptotics of the solutions of the Dirac system (1.4) as well as (1.2) now depend on the two diagonalisations and the first $[I+Q(t,\mu)]-$transformation. Therefore, by performimg backward transformation after obtaining the transformed system

(2.14) which is assumed to be in Levinson-Benzaid-Lutz form, the solutions of the system (1.2) as well as (1.4) are given by the product of the matrices $T(t,\mu)$, $[I+B(t,\mu)]$, $[I+Q(t,\mu)]$ applied on the product of the matrix of normalised eigenvectors with remainder elements that tend to zero as $|\mu|\to\infty$ and the direct product of corrected diagonal system. Hence, the solution of (1.2) is the matrix $D(t,\mu)$ given by

$$D(t,\mu) \;=\; T(t,\mu)[I+B(t,\mu)][I+Q(t,\mu)][E(t,\mu)+\circ(I)]\prod_{s=t_o}^{t_1-1}\tilde{\Lambda}(s,\mu)$$

with $D(t_0,\mu)=I$. The matrix $T(T,\mu)[I+B(t,\mu)][I+Q(t,\mu)][E(t,\mu)+\circ(I)]$ does not affect the asymptotic behaviour of the solution but rather the matrix $\prod_{s=t_o}^{t_1-1}\tilde{\Lambda}(s,\mu)$. The matrix $\prod_{s=t_o}^{t_1-1}\tilde{\Lambda}(s,\mu)$ has dominant unbounded diagonal and thus of the corresponding $D(t,\mu)$. The associated discretised Dirac operator has compact resolvent and the values of μ in the strip P_d are isolated points hence the spectrum is discrete at most. □

We extend the above results to the case of bounded spectral parameter and unbounded potential $\delta_1(t)$. Hence, we assume

$$\mu=\circ(\delta_k), |\delta_1(t)|\nearrow\infty, \delta_2(t)=\circ(\delta_1(t)) \qquad \textit{and} \qquad 0\le|\mu|<1 \tag{3.1}$$

for $k=1,2$ and as $t\to\infty$ and the Hilbert space which is $\ell^2(\mathbb{N})$. Note that unlike in Section 2 where $t\in[t_0,t_1]$, here we will have $t\in\mathbb{N}$. We also assume that

$$\Delta(\delta_2(t))\to 0 \qquad as \qquad t\to\infty. \tag{3.2}$$

The assumption in (3.2) is to ensure that the correction terms to the eigenvalues after first diagonalisation are controllable. The analysis carried out in this section can also be done in a similar way without alteration of results when $|\delta_2(t)|\nearrow\infty$ as $t\to\infty$ while $\delta_1(t)=\circ(\delta_2(t))$ and $\Delta(\delta_1(t))\to 0$ as $t\to\infty$. The characteristic polynomial (1.7) yields the characteristic roots approximated to $O(\delta_1^{-3})$ given by:

$$\begin{aligned}\lambda_1(t,\mu,\delta_k) &\approx (1+\delta_2\delta_1)-(1-i\mu)\delta_2^{-1}\delta_1^{-1}+(1-i\mu)(1+(1-i\mu)^2)\delta_2^{-2}\delta_1^{-2}+O(\delta_1^{-3})\\ \lambda_2(t,\mu,\delta_k) &\approx (1-i\mu)\delta_2^{-1}\delta_1^{-1}+(1-i\mu)(1+(1-i\mu)^2)\delta_2^{-2}\delta_1^{-2}+O(\delta_1^{-3}).\end{aligned} \tag{3.3}$$

After computing and normalizing the first component of the respective eigenvectors, we obtain the transformation matrix of the form:

$$T(t,\mu,\delta_k)\approx\begin{bmatrix}1 & 1\\ \delta_2(t)+O(\delta^{-1}(t)) & \delta_1^{-1}(t)+O(\delta_1^{-2}(t))\end{bmatrix} \tag{3.4}$$

with the determinant $O(\delta_2(t))$ and whose inverse is approximately

$$T^{-1}(t,\mu,\delta_k)\approx-[\delta_2(t)+O(\delta_1^{-1}(t))]^{-1}\begin{bmatrix}\delta_1^{-1}(t)+O(\delta_1^{-2}(t)) & -1\\ -\delta_2(t)+O(\delta_1^{-1}(t)) & 1\end{bmatrix}. \tag{3.5}$$

Just like in Section 2, after the first diagonalisation we obtain

$$\begin{aligned} Y(t+1,\mu,\delta_k) &= T^{-1}(t+1,\mu,\delta_k)C(t,\mu,\delta_k)T(t,\mu,\delta_k)Y(t,\mu,\delta_k) \\ &= (\Lambda+R)(t,\mu,\delta_k)Y(t,\mu,\delta_k) \end{aligned}$$

where

$$R(t,\mu,\delta_k) = -T^{-1}(t+1,\mu,\delta_k)\Delta T(t,\mu,\delta_k)\Lambda(t,\mu,\delta_k). \tag{3.6}$$

The elements of $R(t,\mu,\delta_k)$ are approximately given by

$$R(t,\mu,\delta_k) \approx \begin{bmatrix} -\delta_2^{-1}(1+\delta_2\delta_1)\Delta\delta_2 & -(1-i\mu)\delta_2^{-2}\delta_1^{-1}\Delta\delta_1^{-1} \\ \delta_2^{-1}(1+\delta_2\delta_1)\Delta\delta_2 & (1-i\mu)\delta_2^{-2}\delta_1^{-1}\Delta\delta_1^{-1} \end{bmatrix} \tag{3.7}$$

whereas $\Lambda(t,\mu,\delta_k)$ is approximately given by

$$\begin{bmatrix} (1+\delta_2\delta_1)-(1-i\mu)\delta_2^{-1}\delta_1^{-1}+O(\delta_1^{-2}) & 0 \\ 0 & (1-i\mu)\delta_2^{-1}\delta_1^{-1}+O(\delta_1^{-2}) \end{bmatrix}. \tag{3.8}$$

Explicitly, working on the off-diagonal terms of the matrix $R(t,\mu,\delta_k)$ in (3.6) we have

$$\delta_2^{-1}(t)\Delta(\delta_2(t)), \qquad \delta_1(t)\Delta(\delta_2(t)), \qquad \delta_2^{-1}(t)\delta_1^{-1}(t)\Delta(\delta_1^{-1}(t)) \in \ell^2. \tag{3.9}$$

The correction terms to the diagonals are given by

$$R_{11}(t,\mu,\delta_k) \approx -\delta_2^{-1}(1+\delta_2\delta_1)\Delta(\delta_2(t)) \text{ and}$$
$$R_{22}(t,\mu,\delta_k) \approx (1-i\mu)\delta_2^{-1}\delta_1^{-1}\Delta(\delta_1^{-1}(t)).$$

It is the term $R_{11}(t,\mu,\delta_k)$ that warranted the assumption in (3.2). This results into corrected eigenvalues of the form:

$$\begin{aligned} \tilde{\lambda}_1(t,\mu,\delta_k) &\approx \lambda_1(t,\mu,\delta_k)+R_{11}(t,\mu,\delta_k) \\ \tilde{\lambda}_2(t,\mu,\delta_k) &\approx \lambda_2(t,\mu,\delta_k)+R_{22}(t,\mu,\delta_k). \end{aligned}$$

Making transformation similar to (2.4) in Section 2, we obtain the following after first diagonalisation

$$W(t+1,\mu,\delta_k) \approx [\tilde{\Lambda}(t,\mu,\delta_k)+\tilde{R}(t,\mu,\delta_k)]W(t,\mu,\delta_k) \tag{3.10}$$

with

$$\tilde{\Lambda}(t,\mu,\delta_k) \approx diag(\tilde{\Lambda}(t,\mu,\delta_k)$$

and

$$\tilde{R}(t,\mu,\delta_k) \approx \begin{bmatrix} 0 & -(1-i\mu)\delta_2^{-2}\delta_1^{-1}\Delta(\delta_1^{-1}(t)) \\ \delta_2^{-1}(1+\delta_2\delta_1)\Delta\delta_2(t) & 0 \end{bmatrix}$$

where $\tilde{R}_{kj}(t,\mu,\delta_k) \in \ell^2$, $k \neq j$. Now as in Section 2, we proceed with second diagonalisation by computing the matrix $B(t,\mu,\delta_k)$. Thus, we use $B_{kk}(t,\mu,\delta_k) = 0$ and for $k \neq j$, $B_{kj}(t,\mu,\delta_k) = (\lambda_j-\lambda_k)R_{kj}$, which implies that

$$\begin{aligned} B_{12}(t,\mu,\delta_k) &\approx (1-i\mu)^2\delta_2^{-3}\delta_1^{-2}\Delta(\delta_1^{-1}) \\ B_{21}(t,\mu,\delta_k) &\approx (1-i\mu)\delta_2^{-1}\Delta(\delta_2). \end{aligned}$$

The correction terms to the diagonals are

$$\begin{aligned} (\tilde{R}B)_{11} &\approx O((1-i\mu)\delta_2^{-3}\delta_1^{-1}\Delta(\delta_1^{-1})\Delta(\delta_2)) \\ (\tilde{R}B)_{22} &\approx O((1-i\mu)\delta_2^{-3}\delta_1^{-1}\Delta(\delta_1^{-1})\Delta(\delta_2)). \end{aligned}$$

The second diagonalisation is then obtained through transformation similar to (2.7) so that a form similar to (2.8) can be obtained. We perform the $(I + Q(t,\mu,\delta_k))$ transformation as discussed earlier in Section 2. Hence we obtain that

$$\begin{aligned} V_{kk}(t,\mu,\delta_k) &= 0 \\ V_{kj}(t,\mu,\delta_k) &= (RB)_{kj}(t,\mu,\delta_k) \\ &= O(\delta_2^{-3}\delta_1^{-1}\Delta(\delta_2)\Delta(\delta_1^{-1})) \end{aligned}$$

while

$$\begin{aligned} Q_{12}(t,\mu,\delta_k) &= -Q_{21}(t,\mu,\delta_k) \\ &\approx -\Sigma_{s=t_0}^{t_1-1} O(\delta_2^{-2}\Delta(\delta_2)\Delta(\delta_1^{-1}))\prod_{l=s}^{t}[O(\delta_2^{-2}\delta_1^{-2})]. \end{aligned}$$

An extension of the results of Lemma 2.2 in the case of unbounded potentials and bounded spectral parameter is thus formulated.

Theorem 3.2. *Let the discretised Dirac systems be given by (1.4) and assume that (3.1)– (3.2) and (3.8) are satisfied then the corrected eigenvalues of (1.4) are given by*

$$\begin{aligned} \tilde{\lambda}_1(t,\mu,\delta_k) &\approx \lambda_1(t,\mu,\delta_k) - \delta_2^{-1}(1+\delta_2\delta_1)\Delta(\delta_2(t)) - (1-i\mu)\delta_2^{-3}\delta_1^{-1}\cdot \\ &\quad \Delta(\delta_1^{-1})\Delta(\delta_2) \\ \tilde{\lambda}_2(t,\mu,\delta_k) &\approx \lambda_{t,\mu,\delta_k}(t) + (1-i\mu)\delta_2^{-1}\delta_1^{-1}\Delta(\delta_1^{-1}(t)) + (1-i\mu)\delta_2^{-3}\delta_1^{-1}\cdot \\ &\quad \Delta(\delta_1^{-1})\Delta(\delta_2). \end{aligned}$$

Here, $\lambda_k(t,\mu,\delta_k)$ is given in (3.3).

Proof. The proof follows closely that of Lemma 2.2. The only difference is the expansion of the eigenvalues based on the assumptions in (3.1). Thus the magnitude of the eigenvalues are determined by the leading value of $\delta_1(t)$. The rest follow from Lemma 2.2. □

The next result is for the case of unbounded potential and bounded spectral parameter.

Theorem 3.3. *The asymptotic behaviour of the Dirac system (1.2) under the conditions (3.1)– (3.2) are determined by the magnitude of $\delta_1(t)$ in the corrected eigenvalues and the solutions are of the form*

$$\begin{aligned} D(t,\mu,\delta_k) &= T(t,\mu,\delta_k)(I+B(t,\mu,\delta_k))(I+Q(t,\mu,\delta_k))\cdot \\ &\quad (E(t,\mu,\delta_k)+\circ(I_2))\prod_{s=t_0}^{t-1}\tilde{\Lambda}(s,\mu,\delta_k), \end{aligned} \tag{3.11}$$

where $\tilde{\Lambda}(t, \mu, \delta_k) = diag(\tilde{\lambda}(t, \mu, \delta_k))$ *with*

$$\begin{aligned} \tilde{\lambda_1}(t, \mu, \delta_k) &= \lambda_1(t, \mu, \delta_k) + O(\delta_1 \Delta(\delta_2)) + O(\delta_2^{-3}\delta_1^{-1}\Delta(\delta_2)\Delta(\delta_1^{-1})) \\ \tilde{\lambda_2}(t, \mu, \delta_k) &= \lambda_2(t, \mu, \delta_k) + O(\delta_2^{-2}\delta_1^{-1}\Delta(\delta_1^{-1})) + O(\delta_2^{-3}\delta_1^{-1}\Delta(\delta_2)\Delta(\delta_1^{-1})). \end{aligned}$$

Proof. The proof closely follows from that of Theorem 3.1. Thus diagonalisations based on the assumption (3.2) will make off-diagonal terms to be ℓ^1 and those that are still ℓ^2- terms their decay can be accelerated using $[I + Q(t, \mu)]-$ transformations. The rest now follow from the proof of Theorem 3.1. Here, the asymptotic behaviour of the solutions are determined by the matrix $\prod_{s=t_0}^{t-1} \tilde{\Lambda}(s, \mu, \delta_k)$ whose diagonals are the eigenvalues of $C(t, \mu, \delta_k)$ and hence the leading term of $\delta_1(t)$ plays a great role. Since the diagonal of $\tilde{\Lambda}(s, \mu, \delta_k)$ has unbounded term in $\tilde{\lambda}_1(t, \mu, \delta_k)$, the resolvent of the corresponding discretised Dirac operator is compact and the spectrum is discrete as expected. □

4. Conclusion

The results obtained in this research extends those of Rzepnicki [10] to discrete situation. Though in the said article, the focus was on bounded potentials and unbounded spectral parameter, our results extend to the case of unbounded potentials. Thus using asymptotic summation based on the discretised Levinson's theorem, we have concluded that the asymptotic behaviour of the discretised Cauchy Dirac System depends on the dominant, unbounded term or parameter both in the unbounded spectral term and unbounded potentials. In all the cases, square and non-square summable solutions are uniformly square summable implying that the associated spectrum to the Dirac operator is at most discrete due to the compact resolvent. In future, we recommend this investigation to be extended to other higher order discretised Dirac systems of orders equal or greater than four.

Acknowledgement. Fredrick Nyamwala thanks the faculty at Institute for Analysis, Leibniz University of Hannover for the hospitality, DAAD and TWAS-UNESCO for funding. Benard Owino thanks DAAD for their financial support through ICIR Scholarships.

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[1] Department of Mathematics, Physics and Computing, Moi University, Eldoret, Kenya.
Email address: benardowino17@yahoo.com

[2] Department of Mathematics, Physics and Computing, Moi University, Eldoret, Kenya.
Email address: foluoch2000@yahoo.com

[3] Department of Pure and Applied Mathematics, Maseno University, Maseno, Kenya.
Email address: otivoe@yahooo.com