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NORM-PEAK MULTILINEAR FORMS ON ℓ_1

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ABSTRACT. Let $n \in \mathbb{N}, n \geq 2$. A continuous n-linear form T on a Banach space E is called *norm-peak* if there is unique $(x_1, \ldots, x_n) \in E^n$ such that $\|x_1\| = \cdots = \|x_n\| = 1$ and T attains its norm only at $(\pm x_1, \ldots, \pm x_n)$. In this paper, we characterize the norm-peak multilinear forms on ℓ_1.

1. Introduction

In 1961, Bishop and Phelps [3] showed that the set of norm-attaining functionals on a Banach space is dense in the dual space. Shortly after, attention was paid to possible extensions of this result to more general settings, specially bounded linear operators between Banach spaces. Lindenstrauss [13] studied norm-attaining operators. The problem of denseness of norm-attaining functions has moved to other types of mappings like multilinear forms or polynomials. The first result about norm-attaining multilinear forms appeared in a joint work of Aron, Finet and Werner [2], where they showed that the Radon-Nikodym Property is sufficient for the denseness of norm-attaining multilinear forms. Choi and Kim [4] showed that the Radon-Nikodym Property is also sufficient for the denseness of norm-attaining polynomials. Acosta [1] studied norm attaining multilinear mappings. Jiménez-Sevilla and Payá [7] studied the denseness of norm-attaining multilinear forms and polynomials on preduals of Lorentz sequence spaces. Payá and Saleh [14] presented new sufficient conditions for the denseness of norm-attaining multilinear forms. Note that the norm denseness problem of the set of norm-attaining forms in the spaces of all continuous multilinear forms is closely related to sets with the Radon-Nikodym Property. It is also linked to the broader topic of optimization on infinite dimensional normed spaces and variational principles (see Stegall [16], Finet and Georgiev [6]).

Let $n \in \mathbb{N}, n \geq 2$. We write S_E for the unit sphere of a Banach space E. We denote by $\mathcal{L}(^nE)$ the Banach space of all continuous n-linear forms on E endowed with the norm $\|T\| = \sup_{(x_1,\cdots,x_n)\in S_E\times\cdots\times S_E} |T(x_1, \cdots, x_n)|$. $\mathcal{L}_s(^nE)$ denote the closed subspace of all continuous symmetric n-linear forms on E. An element $(x_1, \ldots, x_n) \in E^n$ is called a *norming point* of T if $\|x_1\| = \cdots = \|x_n\| = 1$ and $|T(x_1, \ldots, x_n)| = \|T\|$.

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For $T \in \mathcal{L}({}^nE)$, we define

$$\text{Norm}(T) = \Big\{(x_1, \ldots, x_n) \in E^n : (x_1, \ldots, x_n) \text{ is a norming point of } T\Big\}.$$

Norm(T) is called the *norming set* of T. Notice that $(x_1, \ldots, x_n) \in \text{Norm}(T)$ if and only if $(\epsilon_1 x_1, \ldots, \epsilon_n x_n) \in \text{Norm}(T)$ for some $\epsilon_k = \pm 1$ $(k = 1, \ldots, n)$. Indeed, if $(x_1, \ldots, x_n) \in \text{Norm}(T)$, then

$$|T(\epsilon_1 x_1, \ldots, \epsilon_n x_n)| = |\epsilon_1 \cdots \epsilon_n T(x_1, \ldots, x_n)| = |T(x_1, \ldots, x_n)| = \|T\|,$$

which shows that $(\epsilon_1 x_1, \ldots, \epsilon_n x_n) \in \text{Norm}(T)$. If $(\epsilon_1 x_1, \ldots, \epsilon_n x_n) \in \text{Norm}(T)$ for some $\epsilon_k = \pm 1$ $(k = 1, \ldots, n)$, then

$$(x_1, \ldots, x_n) = \Big(\epsilon_1(\epsilon_1 x_1), \ldots, \epsilon_n(\epsilon_n x_n)\Big) \in \text{Norm}(T).$$

The following examples show that the norming sets can be empty, finite or infinite.

Example 1.1. (a) Let $(a_k)_{k\in\mathbb{N}}$ be a real sequence such that $a_k > 0$ for all k and $\sum_{k=1}^{\infty} a_k = 1$. Let $n \geq 2$ and

$$T\Big((x_i^{(1)})_{i\in\mathbb{N}}, \ldots, (x_i^{(n)})_{i\in\mathbb{N}}\Big) = \sum_{i=1}^{\infty} a_i \; x_i^{(1)} \cdots x_i^{(n)} \in \mathcal{L}({}^n c_0).$$

We claim that Norm$(T) = \emptyset$. Obviously, $\|T\| = 1$. Assume that Norm$(T) \neq \emptyset$. Let $\Big((x_i^{(1)})_{i\in\mathbb{N}}, \ldots, (x_i^{(n)})_{i\in\mathbb{N}}\Big) \in \text{Norm}(T)$. Then,

$$1 = \Big|T\Big((x_i^{(1)})_{i\in\mathbb{N}}, \ldots, (x_i^{(n)})_{i\in\mathbb{N}}\Big)\Big| \leq \sum_{i=1}^{\infty} a_i \; |x_i^{(1)}| \cdots |x_i^{(n)}| \leq \sum_{i=1}^{\infty} a_i = 1,$$

which shows that $|x_i^{(1)}| = \cdots = |x_i^{(n)}| = 1$ for all $i \in \mathbb{N}$. Hence, $(x_i^{(1)})_{i\in\mathbb{N}}, \ldots, (x_i^{(n)})_{i\in\mathbb{N}} \notin c_0$. This is a contradiction. Therefore, Norm$(T) = \emptyset$.

(b) Let

$$T\Big((x_i^{(1)})_{i\in\mathbb{N}}, \ldots, (x_i^{(n)})_{i\in\mathbb{N}}\Big) = x_1^{(1)} \cdots x_1^{(n)} \in \mathcal{L}({}^n c_0).$$

Then, $\|T\| = 1$ and

$$\begin{aligned} \text{Norm}(T) &= \Big\{\Big((\pm 1, x_2^{(1)}, x_3^{(1)}, \ldots), \ldots, (\pm 1, x_2^{(n)}, x_3^{(n)}, \ldots)\Big) \in (c_0)^n : \\ &\qquad |x_j^{(k)}| \leq 1, \text{ for } k = 1, \ldots, n, j \geq 2\Big\}. \end{aligned}$$

(c) Let

$$T\Big((x_i^{(1)})_{i\in\mathbb{N}}, \ldots, (x_i^{(n)})_{i\in\mathbb{N}}\Big) = x_1^{(1)} \cdots x_1^{(n)} \in \mathcal{L}({}^n \ell_1).$$

Then, $\|T\| = 1$ and Norm$(T) = \{(\pm e_1, \ldots, \pm e_1)\}$, where $e_1 = (1, 0, 0, 0, \ldots)$.

(d) Let $T \in \mathcal{L}({}^nE)$ and $(x_1, \ldots, x_n) \in \text{Norm}(T)$. By Hahn-Banach theorem there are $f_j \in E^*$ such that

$$f_j(x_j) = \|f_j\| = 1 \text{ for every } j = 1, \ldots, n.$$

Let ϕ be a permutation on $\{1, \ldots, n\}$. We define $T_{\phi, f_1, \ldots, f_n} \in \mathcal{L}(^{2n}E)$ by

$$T_{\phi, f_1, \ldots, f_n}(y_1, \ldots, y_{2n}) := \prod_{j=1}^{n} f_{\phi(j)}(y_{\phi(j)})\ T(y_{n+1}, \ldots, y_{2n}).$$

Then

$$(x_{\phi(1)}, \ldots, x_{\phi(n)}, x_1, \ldots, x_n) \in \text{Norm}(T_{\phi, f_1, \ldots, f_n}).$$

If $\text{Norm}(T) \neq \emptyset$, $T \in \mathcal{L}(^nE)$ is called a *norm attaining* n-linear form (see [4]). $T \in \mathcal{L}(^nE)$ is called a *norm-peak n-linear form* on E if $\text{Norm}(T) = \{(\pm x_1, \ldots, \pm x_n)\}$ for some $(x_1, \ldots, x_n) \in (S_E)^n$.

For more details about the theory of multilinear mappings on a Banach space, we refer to [5].

For $m \in \mathbb{N}$, let $\ell_1^m := \mathbb{R}^m$ with the the ℓ_1-norm and $\ell_\infty^2 = \mathbb{R}^2$ with the supremum norm. Notice that if $E = \ell_1^m$ or ℓ_∞^2 and $T \in \mathcal{L}(^nE)$, $\text{Norm}(T) \neq \emptyset$ since S_E is compact. Kim [8–11] classified $\text{Norm}(T)$ for every $T \in \mathcal{L}_s(^2\ell_\infty^2), \mathcal{L}(^2\ell_\infty^2), \mathcal{L}(^2\ell_1^2), \mathcal{L}_s(^2\ell_1^3)$ or $\mathcal{L}_s(^3\ell_1^2)$. Kim [12] classified $\text{Norm}(T)$ for every $T \in \mathcal{L}(^2\mathbb{R}^2_{h(w)})$, where $\mathbb{R}^2_{h(w)}$ denotes the plane with the hexagonal norm with weight $0 < w < 1$ $\|(x, y)\|_{h(w)} = \max\Big\{|y|, |x| + (1-w)|y|\Big\}$.

Saleh [15] showed that the norm-attaining continuous bilinear forms on $L_1(\mu)$ are norm dense in all continuous bilinear forms if and only if μ is completely atomic. This motivates the study of the norming sets of $T \in \mathcal{L}(^2\ell_1)$, since one can expect many continuous bilinear forms with interesting norming sets in the ℓ_1-setting.

In this paper, we characterize the norm-peak multilinear forms on ℓ_1.

2. Main results

The following presents an explicit formula for the norm of $T \in \mathcal{L}(^n\ell_1)$.

Proposition 2.1. *Let $n \geq 2$. Let $T \in \mathcal{L}(^n\ell_1)$ with*

$$T\Big((x_i^{(1)})_{i\in\mathbb{N}}, \ldots, (x_i^{(n)})_{i\in\mathbb{N}}\Big) = \sum_{(i_1, \ldots, i_n)\in\mathbb{N}^n} a_{i_1\cdots i_n}\ x_{i_1}^{(1)} \cdots x_{i_n}^{(n)}$$

for some $a_{i_1\cdots i_n} \in \mathbb{R}$. Then

$$\|T\| = \sup\Big\{|a_{i_1\cdots i_n}| : (i_1, \ldots, i_n) \in \mathbb{N}^n\Big\}.$$

Proof. Let $(x_i^{(k)})_{i\in\mathbb{N}} \in S_{\ell_1}$ for $k = 1, \ldots, n$. It follows that

$$\begin{aligned} & \left|T\Big((x_i^{(1)})_{i\in\mathbb{N}}, \ldots, (x_i^{(n)})_{i\in\mathbb{N}}\Big)\right| \\ \leq & \sum_{(i_1,\ldots,i_n)\in\mathbb{N}^n} |a_{i_1\cdots i_n}|\ |x_{i_1}^{(1)}|\cdots|x_{i_n}^{(n)}| \\ \leq & \sup\Big\{|a_{i_1\cdots i_n}| : (i_1,\ldots,i_n)\in\mathbb{N}^n\Big\}\ \Big(\sum_{i_1\in\mathbb{N}}|x_{i_1}^{(1)}|\Big)\cdots\Big(\sum_{i_n\in\mathbb{N}}|x_{i_n}^{(1)}|\Big) \\ = & \sup\Big\{|a_{i_1\cdots i_n}| : (i_1,\ldots,i_n)\in\mathbb{N}^n\Big\} \\ = & \sup\Big\{|T(e_{i_1},\ldots,e_{i_n})| : (i_1,\ldots,i_n)\in\mathbb{N}^n\Big\} \leq \|T\|, \end{aligned}$$

where $(e_j)_{j\in\mathbb{N}}$ denotes the canonical basis for ℓ_1.

Thus,

$$\|T\| = \sup\Big\{|a_{i_1\cdots i_n}| : (i_1,\ldots,i_n)\in\mathbb{N}^n\Big\}.$$

□

For simplicity we write $T = (a_{i_1\cdots i_n})_{(i_1,\ldots,i_n)\in\mathbb{N}^n}$. We call $a_{i_1\cdots i_n}{}'$s the *coefficients* of T. Notice that if $\|T\| = 1$, then $|a_{i_1\cdots i_n}| \leq 1$ for all $(i_1,\ldots,i_n)\in\mathbb{N}^n$.

Lemma 2.2. *Let $n \geq 2$. Let $T = (a_{i_1\cdots i_n})_{(i_1,\ldots,i_n)\in\mathbb{N}^n} \in \mathcal{L}(^n\ell_1)$. Suppose that*

$$\Big((t_i^{(1)})_{i\in\mathbb{N}}, \ldots, (t_i^{(n)})_{i\in\mathbb{N}}\Big) \in \mathrm{Norm}(T).$$

If $t_{i_1'}^{(1)}\cdots t_{i_n'}^{(n)} \neq 0$ for some $(i_1',\ldots,i_n')\in\mathbb{N}^n$, then $|T(e_{i_1'},\ldots,e_{i_n'})| = |a_{i_1'\cdots i_n'}| = \|T\|$ and $(\pm e_{i_1'},\ldots,\pm e_{i_n'}) \in \mathrm{Norm}(T)$.

Proof. It suffices to show the following claim:

Claim. If $|a_{i_1'\cdots i_n'}| < \|T\|$ for some $(i_1',\ldots,i_n')\in\mathbb{N}^n$, then $t_{i_1'}^{(1)}\cdots t_{i_n'}^{(n)} = 0$.

Suppose not. It follows that

$$\begin{aligned} \|T\| &= \left|T\Big((t_i^{(1)})_{i\in\mathbb{N}}, \ldots, (t_i^{(n)})_{i\in\mathbb{N}}\Big)\right| \\ &\leq \sum_{(i_1,\ldots,i_n)\in\mathbb{N}^n} |a_{i_1\cdots i_n}|\ |t_{i_1}^{(1)}|\cdots|t_{i_n}^{(n)}| \\ &= |a_{i_1'\cdots i_n'}|\ |t_{i_1'}^{(1)}|\cdots|t_{i_n'}^{(n)}| + \sum_{(i_1,\ldots,i_n)\in\mathbb{N}^n\setminus\{(i_1',\ldots,i_n')\}} |a_{i_1\cdots i_n}|\ |t_{i_1}^{(1)}|\cdots|t_{i_n}^{(n)}| \\ &< \|T\|\ |t_{i_1'}^{(1)}|\cdots|t_{i_n'}^{(n)}| + \sum_{(i_1,\ldots,i_n)\in\mathbb{N}^n\setminus\{(i_1',\ldots,i_n')\}} |a_{i_1\cdots i_n}|\ |t_{i_1}^{(1)}|\cdots|t_{i_n}^{(n)}| \\ &\leq \|T\|\ |t_{i_1'}^{(1)}|\cdots|t_{i_n'}^{(n)}| + \sum_{(i_1,\ldots,i_n)\in\mathbb{N}^n\setminus\{(i_1',\ldots,i_n')\}} \|T\|\ |t_{i_1}^{(1)}|\cdots|t_{i_n}^{(n)}| \\ &= \|T\|\ \Big(\sum_{i_1\in\mathbb{N}}|t_{i_1}^{(1)}|\Big)\cdots\Big(\sum_{i_1\in\mathbb{N}}|t_{i_1}^{(1)}|\Big) = \|T\|, \end{aligned}$$

which is a contradiction. Thus, the claim holds. This completes the proof. □

We characterize the norm attaining n-linear forms on ℓ_1.

Theorem 2.3. *Let $n \geq 2$. Let $T = (a_{i_1\cdots i_n})_{(i_1,\ldots,i_n)\in\mathbb{N}^n} \in \mathcal{L}(^n\ell_1)$ with $\|T\| = 1$. Then, T is norm attaining if and only if there is $(j_1, \ldots, j_n) \in \mathbb{N}^n$ such that $|a_{j_1\cdots j_n}| = 1$.*

Proof. ($\Rightarrow$). Suppose not. Let $(x_1, \ldots, x_n) \in \mathrm{Norm}(T)$. Write $x_k = (x_j^{(k)})_{j\in\mathbb{N}}$ for $k = 1, \ldots, n$. Obviously, there is $(i_1, \ldots, i_n) \in \mathbb{N}^n$ such that $|a_{i_1\cdots i_n}| \neq 0$ and $x_{i_1}^{(1)} \cdots x_{i_n}^{(n)} \neq 0$. It follows that

$$\begin{aligned}
1 &= |T(x_1, \ldots, x_n)| \\
&\leq \sum_{(k_1,\ldots,k_n)\in\mathbb{N}^n} |a_{k_1\cdots k_n}|\ |x_{k_1}^{(1)}| \cdots |x_{k_n}^{(n)}| \\
&= |a_{i_1\cdots i_n}|\ |x_{i_1}^{(1)}| \cdots |x_{i_n}^{(n)}| + \sum_{(k_1,\ldots,k_n)\in\mathbb{N}^n\setminus\{(i_1,\ldots,i_n)\}} |a_{k_1\cdots k_n}|\ |x_{k_1}^{(1)}| \cdots |x_{k_n}^{(n)}| \\
&< |x_{i_1}^{(1)}| \cdots |x_{i_n}^{(n)}| + \sum_{(k_1,\ldots,k_n)\in\mathbb{N}^n\setminus\{(i_1,\ldots,i_n)\}} |a_{k_1\cdots k_n}|\ |x_{k_1}^{(1)}| \cdots |x_{k_n}^{(n)}| \\
&\leq |x_{i_1}^{(1)}| \cdots |x_{i_n}^{(n)}| + \sum_{(k_1,\ldots,k_n)\in\mathbb{N}^n\setminus\{(i_1,\ldots,i_n)\}} |x_{k_1}^{(1)}| \cdots |x_{k_n}^{(n)}| \\
&= (\sum_{k_1\in\mathbb{N}} |x_{k_1}^{(1)}|) \cdots (\sum_{k_n\in\mathbb{N}} |x_{k_n}^{(n)}|) = 1,
\end{aligned}$$

a contradiction.

($\Leftarrow$). Note that

$$|T(e_{j_1}, \ldots, e_{j_n})| = |a_{j_1\cdots j_n}| = 1.$$

Thus, T is norm attaining. $\square$

Corollary 2.4. *Let $n, m \geq 2$. Let $T = (a_{i_1\cdots i_n})_{1\leq i_1,\ldots,i_n\leq m} \in \mathcal{L}(^n\ell_1^m)$ with $\|T\| = 1$. Then, T is norm attaining if and only if there is $(j_1, \ldots, j_n) \in \{1, \ldots, m\}^n$ such that $|a_{j_1\cdots j_n}| = 1$.*

Theorem 2.5. *Let $n \geq 2$. Let $T = (a_{i_1\cdots i_n})_{(i_1,\ldots,i_n)\in\mathbb{N}^n} \in \mathcal{L}(^n\ell_1)$ with $\|T\| = 1$. Suppose that T is norm attaining. Let $\delta_{i_1\cdots i_n} = 1$ if $|a_{i_1\cdots i_n}| = 1$ and $\delta_{i_1\cdots i_n} = 0$ if $|a_{i_1\cdots i_n}| < 1$. We define*

$$T_\delta = \Big(a_{i_1\cdots i_n}\delta_{i_1\cdots i_n}\Big)_{(i_1,\ldots,i_n)\in\mathbb{N}^n} \in \mathcal{L}(^n\ell_1).$$

Then, $\mathrm{Norm}(T) = \mathrm{Norm}(T_\delta)$.

Proof. By Proposition 2.1, $\|T\| = \|T_\delta\| = 1$.

($\subseteq$). Let $\Big((t_i^{(1)})_{i\in\mathbb{N}}, \ldots, (t_i^{(n)})_{i\in\mathbb{N}}\Big) \in \mathrm{Norm}(T)$.

Then

$$\begin{aligned} 1 &= \Big|T\Big((t_i^{(1)})_{i\in\mathbb{N}},\ldots,(t_i^{(n)})_{i\in\mathbb{N}}\Big)\Big| \\ &= \Big|\sum_{|a_{i_1'\cdots i_n'}|<1} a_{i_1'\cdots i_n'}t_{i_1'}^{(1)}\cdots t_{i_n'}^{(n)} + \sum_{|a_{i_1\cdots i_n}|=1} a_{i_1\cdots i_n}t_{i_1}^{(1)}\cdots t_{i_n}^{(n)}\Big| \\ &= \Big|\sum_{|a_{i_1\cdots i_n}|=1} a_{i_1\cdots i_n}t_{i_1}^{(1)}\cdots t_{i_n}^{(n)}\Big| \text{ (by Lemma 2.2)} \\ &= \Big|\sum_{|a_{i_1'\cdots i_n'}|<1} \Big(a_{i_1'\cdots i_n'}\delta_{i_1'\cdots i_n'}\Big)t_{i_1'}^{(1)}\cdots t_{i_n'}^{(n)} + \sum_{|a_{i_1\cdots i_n}|=1} \Big(a_{i_1\cdots i_n}\delta_{i_1\cdots i_n}\Big)t_{i_1}^{(1)}\cdots t_{i_n}^{(n)}\Big| \\ &= \Big|T_\delta\Big((t_i^{(1)})_{i\in\mathbb{N}},\ldots,(t_i^{(n)})_{i\in\mathbb{N}}\Big)\Big|. \end{aligned}$$

Thus, $\Big((t_i^{(1)})_{i\in\mathbb{N}},\ldots,(t_i^{(n)})_{i\in\mathbb{N}}\Big) \in \mathrm{Norm}(T_\delta)$.

($\supseteq$). Let $\Big((t_i^{(1)})_{i\in\mathbb{N}},\ldots,(t_i^{(n)})_{i\in\mathbb{N}}\Big) \in \mathrm{Norm}(T_\delta)$.
Write

$$\begin{aligned} & T\Big((x_i^{(1)})_{i\in\mathbb{N}},\ldots,(x_i^{(n)})_{i\in\mathbb{N}}\Big) \\ = \ & T_\delta\Big((x_i^{(1)})_{i\in\mathbb{N}},\ldots,(x_i^{(n)})_{i\in\mathbb{N}}\Big) + \sum_{|a_{i_1\cdots i_n}|<1} a_{i_1\cdots i_n}\ x_{i_1}^{(1)}\cdots x_{i_n}^{(n)}. \end{aligned}$$

Let $T_- \in \mathcal{L}(^n\ell_1)$ be such that

$$\begin{aligned} & T_-\Big((x_i^{(1)})_{i\in\mathbb{N}},\ldots,(x_i^{(n)})_{i\in\mathbb{N}}\Big) \\ = \ & T_\delta\Big((x_i^{(1)})_{i\in\mathbb{N}},\ldots,(x_i^{(n)})_{i\in\mathbb{N}}\Big) - \sum_{|a_{i_1\cdots i_n}|<1} a_{i_1\cdots i_n}\ x_{i_1}^{(1)}\cdots x_{i_n}^{(n)}. \end{aligned}$$

By Theorem A, $\|T_-\| = 1$. It follows that

$$\begin{aligned} 1 &\geq \Big|T\Big((t_i^{(1)})_{i\in\mathbb{N}},\ldots,(t_i^{(n)})_{i\in\mathbb{N}}\Big)\Big| \\ &= \Big|T_\delta\Big((t_i^{(1)})_{i\in\mathbb{N}},\ldots,(t_i^{(n)})_{i\in\mathbb{N}}\Big) + \sum_{|a_{i_1\cdots i_n}|<1} a_{i_1\cdots i_n}\ t_{i_1}^{(1)}\cdots t_{i_n}^{(n)}\Big|, \\ &= \Big|T_\delta\Big((t_i^{(1)})_{i\in\mathbb{N}},\ldots,(t_i^{(n)})_{i\in\mathbb{N}}\Big) - \sum_{|a_{i_1\cdots i_n}|<1} a_{i_1\cdots i_n}\ t_{i_1}^{(1)}\cdots t_{i_n}^{(n)}\Big|, \end{aligned}$$

which implies that

$$\begin{aligned} 1 &\geq \Big|T_\delta\Big((t_i^{(1)})_{i\in\mathbb{N}},\ldots,(t_i^{(n)})_{i\in\mathbb{N}}\Big)\Big| + \Big|\sum_{|a_{i_1\cdots i_n}|<1} a_{i_1\cdots i_n}\ t_{i_1}^{(1)}\cdots t_{i_n}^{(n)}\Big| \\ &= 1 + \Big|\sum_{|a_{i_1\cdots i_n}|<1} a_{i_1\cdots i_n}\ t_{i_1}^{(1)}\cdots t_{i_n}^{(n)}\Big|. \end{aligned}$$

Thus,

$$\Big|\sum_{|a_{i_1\cdots i_n}|<1} a_{i_1\cdots i_n}\ t^{(1)}_{i_1}\cdots t^{(n)}_{i_n}\Big| = 0$$

and so

$$\Big|T\Big((t^{(1)}_i)_{i\in\mathbb{N}},\ldots,(t^{(n)}_i)_{i\in\mathbb{N}}\Big)\Big| = \Big|T_\delta\Big((t^{(1)}_i)_{i\in\mathbb{N}},\ldots,(t^{(n)}_i)_{i\in\mathbb{N}}\Big)\Big| = 1.$$

Thus, $\Big((t^{(1)}_i)_{i\in\mathbb{N}},\ldots,(t^{(n)}_i)_{i\in\mathbb{N}}\Big) \in \mathrm{Norm}(T)$. This completes the proof. □

We characterize the norm-peak n-linear forms on ℓ_1.

Theorem 2.6. *Let $T = (a_{i_1\cdots i_n})_{(i_1,\ldots,i_n)\in\mathbb{N}^n} \in \mathcal{L}(^n\ell_1)$ with $\|T\| = 1$ for some $a_{k_1\cdots k_n}\in\mathbb{R}$.*

Then, T is a norm-peak n-linear form on ℓ_1 if and only if there is $(j_1,\ldots,j_n)\in\mathbb{N}^n$ such that $|a_{j_1\cdots j_n}| = 1$ and $|a_{k_1\cdots k_n}| < 1$ for every $(k_1,\ldots,k_n)\in\mathbb{N}^n\backslash\{(j_1,\ldots,j_n)\}$.

Proof. $(\Rightarrow)$. Note that T is norm attaining. By Theorem 2.3, there is $(j_1,\ldots,j_n)\in\mathbb{N}^n$ such that $|a_{j_1\cdots j_n}| = 1$.

Claim. $|a_{k_1\cdots k_n}| < 1$ for every $(k_1,\ldots,k_n)\in\mathbb{N}^n\backslash\{(j_1,\ldots,j_n)\}$.

Suppose not. There is $(k_1',\ldots,k_n')\in\mathbb{N}^n$ such that $(k_1',\ldots,k_n')\neq(j_1,\ldots,j_n)$ and $|a_{k_1'\cdots k_n'}| = 1$. Note that

$$(\pm e_{j_1},\ldots,\pm e_{j_n}) \neq (\pm e_{k_1'},\ldots,\pm e_{k_n'}).$$

Since $|T(e_{j_1},\ldots,e_{j_n})| = |a_{j_1\cdots j_n}| = 1 = |T(e_{k_1},\ldots,e_{k_n})| = |a_{k_1\cdots k_n}|$,

$$\{(\pm e_{j_1},\ldots,\pm e_{j_n}),\ (\pm e_{k_1'},\ldots,\pm e_{k_n'})\} \subseteq \mathrm{Norm}(T).$$

Thus, T is not a norm-peak n-linear forms on ℓ_1.

$(\Leftarrow)$. Let

$$T_\delta((x^{(1)}_j)_{j\in\mathbb{N}},\ldots,(x^{(n)}_j)_{j\in\mathbb{N}}) = a_{j_1\cdots j_n}x^{(1)}_{j_1}\cdots x^{(n)}_{j_n} \in \mathcal{L}(^n\ell_1).$$

By Theorem 2.5,

$$\mathrm{Norm}(T) = \mathrm{Norm}(T_\delta) = \{(\pm e_{j_1},\ldots,\pm e_{j_n})\}.$$

Thus, T is a norm-peak n-linear form on ℓ_1. □

Corollary 2.7. *Let $n,m\geq 2$. Let $T = (a_{i_1\cdots i_n})_{1\leq i_1,\ldots,i_n\leq m}\in\mathcal{L}(^n\ell_1^m)$ with $\|T\| = 1$.*

Then T is a norm-peak n-linear form on ℓ_1^m if and only if there is $1\leq j_1,\ldots,j_n\leq m$ such that

$$|a_{j_1\cdots j_n}| = 1 \text{ and } |a_{k_1\cdots k_n}| < 1$$

for every $(k_1,\ldots,k_n)\in\{1,\ldots,m\}^n$ with $(k_1,\ldots,k_n)\neq(j_1,\ldots,j_n)$.

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