

SOME MODULAR RELATIONS OF ORDER SIX WITH ITS APPLICATIONS TO PARTITIONS

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ABSTRACT. In the paper, we establish some modular relations involving cubic functions which are analogous to Ramanujan's forty identities. We also give new proof of some modular relations of the same nature established earlier by C. Adiga, K. R. Vasuki and N. Bhaskar [3]. Furthermore, we extract interesting partition results from some of our modular relations.

1. INTRODUCTION

Throughout the paper, we use the customary notation

$$\begin{aligned}(a; q)_0 &:= 1, \\ (a; q)_n &:= \prod_{k=0}^{n-1} (1 - aq^k), \quad n \geq 1, \\ (a; q)_\infty &:= \lim_{n \rightarrow \infty} (a; q)_n, \quad |q| < 1,\end{aligned}$$

and

$$(a_1, a_2, \dots, a_n; q)_\infty := \prod_{i=1}^n (a_i; q)_\infty.$$

The well known Rogers- Ramanujan functions [15, 16, 19] are

$$G(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q; q)_n} = \frac{1}{(q, q^4; q^5)_\infty}, \quad (1.1)$$

and

$$H(q) := \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q; q)_n} = \frac{1}{(q^2, q^3; q^5)_\infty}. \quad (1.2)$$

S. Ramanujan [18] found forty modular relation for $G(q)$ and $H(q)$, which are completely listed in 1975 by B. J. Birch [9] called Ramanujan's forty identities. The forty identities have been established in a series of papers by L. J. Rogers [20] in 1919, G. N. Watson [21] in 1933, D. Bressoud [10] in 1977, A. F. J.

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Biagioli [8] in 1989 and most recently H. Yesilyurt [22] in 2009. Although all the identities can be proved using the theory of modular forms, the method employed by Biagioli, it is more instructive to find proofs that Ramanujan might have found. Outside the theory of modular forms, Watson method, Rogers method, which generalized by Bressoud and also extended by Yesilyurt, are the general methods that has been devised for proving identities from Ramanujan's list. Recently, B. C. Berndt et al. [7] have given the alternative proof of all the identities in the spirit of Ramanujan's mathematics. S. -S. Huang [14], S. -L. Chen and S. -S. Huang [11] have established modular relations for the Ramanujan Göllnitz-Gordon functions [12, 13] employing modular forms. N. D. Baruah et al. [5] have given alternative proofs of them by using Schröter formulas and theta functions identities due to Ramanujan [17]. In the process they also found some new relations.

In this paper, we consider the following two functions $S(q)$ and $T(q)$, which are analogous to Rogers-Ramanujan functions:

$$\begin{aligned} S(q) &:= \sum_{n=0}^{\infty} \frac{(-q; q^2)_n q^{n^2+2n}}{(q^4; q^4)_n} = \frac{(-q; q^2)_{\infty} (q^6; q^6)_{\infty} (q; q^6)_{\infty} (q^5; q^6)_{\infty}}{(q^2; q^2)_{\infty}}, \\ &= \frac{f(-q, -q^5)}{\psi(-q)} \end{aligned} \quad (1.3)$$

and

$$\begin{aligned} T(q) &:= \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^2; q^2)_n} = \frac{(-q; q^2)_{\infty} (q^6; q^6)_{\infty} (q^2; q^6)_{\infty} (q^4; q^6)_{\infty}}{(q^2; q^2)_{\infty}}, \\ &= \frac{f(-q^2, -q^4)}{\psi(-q)}. \end{aligned} \quad (1.4)$$

The identities (1.3) and (1.4) can be found in [4]. C. Adiga, K. R. Vasuki and N. Bhaskar [3] have established modular relations for $S(q)$ and $T(q)$. In this paper, we establish some new modular relations for $S(q)$ and $T(q)$ and also give alternative proof of some of the modular relations established by C. Adiga et al. [3]. In the last section, we mention some applications of these new modular relations to the theory of partitions.

2. DEFINITIONS AND PRELIMINARY RESULTS

Ramanujan's general theta function is defined by

$$f(a, b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab| < 1. \quad (2.1)$$

The Jacobi triple product identity [2] in Ramanujan's notation is

$$f(a, b) = (-a; ab)_{\infty} (-b; ab)_{\infty} (ab; ab)_{\infty}. \quad (2.2)$$

The function $f(a, b)$ satisfies the following basic properties [2, 6]:

$$f(a, b) = f(b, a), \quad (2.3)$$

$$f(1, a) = 2f(a, a^3), \quad (2.4)$$

$$f(-1, a) = 0, \quad (2.5)$$

and, if n is an integer,

$$f(a, b) = a^{n(n+1)/2} b^{n(n-1)/2} f(ab^n, b(ab)^{-n}). \quad (2.6)$$

The three special cases of (2.1) are [6, 17]

$$\varphi(q) := f(q, q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q; q^2)_{\infty}^2 (q^2; q^2)_{\infty}, \quad (2.7)$$

$$\psi(q) := f(q, q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2; q^2)_{\infty}}{(q; q^2)_{\infty}}, \quad (2.8)$$

and

$$f(-q) := f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q; q)_{\infty}. \quad (2.9)$$

Also, after Ramanujan, define

$$\chi(q) := (-q; q^2)_{\infty}.$$

For convenience, we define

$$f_n := f(-q^n) = (q^n; q^n)_{\infty},$$

for positive integer n . In order to prove our modular relations, we first establish some Lemmas. From Entries 24 and 25 [6, pp.39-40] the following Lemma follows easily.

Lemma 2.1. *We have*

$$\begin{aligned} \varphi(q) &= \frac{f_2^5}{f_1^2 f_4^2}, & \psi(q) &= \frac{f_2^2}{f_1}, & f(q) &= \frac{f_2^3}{f_1 f_4}, & \chi(q) &= \frac{f_2^2}{f_1 f_4}, \\ \varphi(-q) &= \frac{f_1^2}{f_2}, & \psi(-q) &= \frac{f_1 f_4}{f_2}, & \chi(-q) &= \frac{f_1}{f_2}. \end{aligned}$$

Lemma 2.2. [6, pp. 46-49]. *We have*

$$f(a, b) f(-a, -b) = f(-a^2, -b^2) \varphi(-ab), \quad (2.10)$$

$$f(a, b) + f(-a, -b) = 2f(a^3 b, ab^3), \quad (2.11)$$

$$f(a, b) - f(-a, -b) = 2af(b/a, a^5 b^3), \quad (2.12)$$

$$\varphi(q^4) + 2q\psi(q^8) = \varphi(q). \quad (2.13)$$

Lemma 2.3. [1] *Let $m = \left\lfloor \frac{s}{s-r} \right\rfloor$, $l = m(s-r) - r$, $k = -m(s-r) + s$ and*

$h = mr - \frac{m(m-1)(s-r)}{2}$, $0 \leq r < s$. Here $[x]$ denote the largest integer less than or equal to x . Then,

(i) $f(q^{-r}, q^s) = q^{-h} f(q^l, q^k),$

(ii) $f(-q^{-r}, -q^s) = (-1)^m q^{-h} f(-q^l, -q^k).$

Following Yesilyurt [22], we define

$$f_k(a, b) = \begin{cases} f(a, b) & \text{if } k \equiv 0 \pmod{2}, \\ f(-a, -b) & \text{if } k \equiv 1 \pmod{2}. \end{cases} \quad (2.14)$$

Let m be an integer and α, β, p and λ be positive integers such that

$$\alpha m^2 + \beta = p\lambda.$$

Let δ and ε be integers. Further let l and t be real and x and y be nonzero complex numbers. Recall that the general theta functions f, f_k are defined by (2.1) and (2.14). With the parameters defined this way, we set

$$\begin{aligned} & R(\varepsilon, \delta, l, t, \alpha, \beta, m, p, \lambda, x, y) \\ &:= \sum_{\substack{k=0 \\ n=2k+t}}^{p-1} (-1)^{\varepsilon k} y^k q^{\{\lambda n^2 + p\alpha l^2 + 2\alpha n m l\}/4} f_{\delta}(x q^{(1+l)p\alpha + \alpha n m}, x^{-1} q^{(1-l)p\alpha - \alpha n m}) \\ &\quad \times f_{\varepsilon p + m\delta}(x^{-m} y^p q^{p\beta + \beta n}, x^m y^{-p} q^{p\beta - \beta n}). \end{aligned} \quad (2.15)$$

Lemma 2.4. [22, Corollary 3.2]. *We have*

$$R(\varepsilon, \delta, l, t, \alpha, \beta, m, p, \lambda, x, y) = R(\delta, \varepsilon, t, l, 1, \alpha\beta, \alpha m, \lambda, p\alpha, y, x).$$

Lemma 2.5. [6, Corollary, p.49]. *We have*

$$\varphi(q) = \varphi(q^9) + 2qf(q^3, q^{15}), \quad (2.16)$$

$$f(q, q^5) = \chi(q)\psi(-q^3), \quad (2.17)$$

$$\varphi(q)\varphi(-q) = \varphi^2(-q^2). \quad (2.18)$$

Lemma 2.6. [6]. *We have*

$$\begin{aligned} & \frac{1}{2} \{ f(Aq^{\mu+\gamma}, q^{\mu+\gamma}/A) f(Bq^{\mu-\gamma}, q^{\mu-\gamma}/B) + f(-Aq^{\mu+\gamma}, -q^{\mu+\gamma}/A) \\ & f(-Bq^{\mu-\gamma}, -q^{\mu-\gamma}/B) \} = \sum_{m=0}^{\mu-1} \left(\frac{A}{B} \right)^m q^{2\mu m^2} \\ & f\left(\frac{A^{\mu-\gamma}}{B^{\mu+\gamma}} q^{(2\mu+4m)(\mu^2-\gamma^2)}, \frac{B^{\mu+\gamma}}{A^{\mu-\gamma}} q^{(2\mu-4m)(\mu^2-\gamma^2)} \right) f\left(ABq^{2\mu+4\gamma m}, \frac{q^{2\mu-4\gamma m}}{AB} \right). \end{aligned} \quad (2.19)$$

Lemma 2.7. [6]. *We have*

$$\begin{aligned} & \frac{1}{2} \left\{ \varphi(q^{2\mu-\omega^2})\varphi(q) + \varphi(-q^{2\mu-\omega^2})\varphi(-q) \right\} + 2q^{\mu/2-(\omega^2-1)/4} \psi(q^{4\mu-2\omega^2})\psi(q^2) \\ &= \sum_{m=0}^{\mu-1} q^{4m^2} f(q^{(2\mu-\omega^2)(2\mu+4m)}, q^{(2\mu-\omega^2)(2\mu-4m)}) f(q^{\mu/2-2\omega m}, q^{\mu/2+2\omega m}). \end{aligned} \quad (2.20)$$

We now establish some modular relations for $S(q)$ and $T(q)$ using ideas similar to those of Watson [21]. In Watson's method, one expresses the left sides of the identities in terms of theta functions. After clearing fractions, we see that the right side can be expressed as a product of two theta functions, say with

summations indices m and n . One then tries to find a change of indices of the form

$$\alpha m + \beta n = sM + a \text{ and } \gamma m + \delta n = sN + b,$$

so that the product on the right side decomposes into the requisite sum of two products of theta functions on the left side.

Theorem 2.8. *We have*

$$T(q^2)T(q^{10}) + q^3 S(q^2)S(q^{10}) = \frac{f_4 f_{20}}{2q f_2 f_8 f_{10} f_{40}} \left\{ \frac{f_2^5 f_5^2}{f_1^2 f_4^2 f_{10}} - \frac{f_6^2 f_{30}^2}{f_{12} f_{60}} \right\}, \quad (2.21)$$

$$T(q^2)T(q^{10}) - q^3 S(q^2)S(q^{10}) = \frac{f_4 f_{20}}{2q f_2 f_8 f_{10} f_{40}} \left\{ \frac{f_6^2 f_{30}^2}{f_{12} f_{60}} - \frac{f_1^2 f_{10}^5}{f_2 f_5^2 f_{20}^2} \right\}, \quad (2.22)$$

$$T(q^4)T(q^8) + q^3 S(q^4)S(q^8) = \frac{1}{2q f_4 f_{32}} \left\{ \frac{f_2^5 f_8^2}{f_1^2 f_4^2 f_{16}} - \frac{f_{12}^2 f_{24}^2}{f_{48}} \right\}, \quad (2.23)$$

$$T(q^4)T(q^8) - q^3 S(q^4)S(q^8) = \frac{1}{2q f_4 f_{32}} \left\{ \frac{f_{12}^2 f_{24}^2}{f_{48}} - \frac{f_1^2 f_8^2}{f_2 f_{16}} \right\}, \quad (2.24)$$

$$T^2(q^6) + q^3 S^2(q^6) = \frac{f_3^3 f_{12} f_{36}}{f_3 f_{18}^2 f_{24}^2} \quad (2.25)$$

and

$$T^2(q^6) - q^3 S^2(q^6) = \frac{f_3 f_{12}^2 f_{18}^7}{f_6^3 f_9^3 f_{24}^2 f_{36}^2}. \quad (2.26)$$

Proof. For given integers m and n , we choose integers M and N such that

$$m - n = 6M + a \text{ and } m + 5n = 6N + b,$$

where a and b have values selected from the set $\{0, \pm 1, \pm 2, 3\}$. Then

$$m = 5M + N + (5a + b)/6 \text{ and } n = N - M + (b - a)/6.$$

It follows easily that $a = b$, and so $m = 5M + N + a$ and $n = N - M$, where $-2 \leq a \leq 3$. We have

$$\varphi(q)\varphi(-q^5) = f(q, q)f(-q^5, -q^5) = \sum_{m, n=-\infty}^{\infty} (-1)^n q^{m^2+5n^2}. \quad (2.27)$$

It follows that, values of a and b are associated as in the following table:

a	0	± 1	± 2	3
b	0	± 1	± 2	3
m	$5M + N$	$5M + N \pm 1$	$5M + N \pm 2$	$5M + N + 3$
n	$N - M$	$N - M$	$N - M$	$N - M$

When a assume the values $-2, -1, 0, 1, 2, 3$ in succession, it is easy to see that the corresponding values of $m^2 + 5n^2$ are, respectively,

$$\begin{aligned} 30M^2 - 20M + 6N^2 - 4N + 4, \\ 30M^2 - 10M + 6N^2 - 2N + 1, \\ 30M^2 + 6N^2, \\ 30M^2 + 10M + 6N^2 + 2N + 1, \end{aligned}$$

$$30M^2 + 20M + 6N^2 + 4N + 4,$$

$$30M^2 + 30M + 6N^2 + 6N + 9.$$

It is evident, from the equations connecting m and n with M and N that, there is a one-one correspondence between all pairs of integers (m, n) and all sets of integers (M, N, a) . We can write (2.27) as

$$\begin{aligned} \varphi(q)\varphi(-q^5) &= q^4 \sum_{M,N=-\infty}^{\infty} q^{30M^2-20M+6N^2-4N} \\ &\quad + q \sum_{M,N=-\infty}^{\infty} q^{30M^2-10M+6N^2-2N} + \sum_{M,N=-\infty}^{\infty} q^{30M^2+6N^2} \\ &\quad + q \sum_{M,N=-\infty}^{\infty} q^{30M^2+10M+6N^2+2N} + q^4 \sum_{M,N=-\infty}^{\infty} q^{30M^2+20M+6N^2+4N} \\ &\quad + q^9 \sum_{M,N=-\infty}^{\infty} q^{30M^2+30M+6N^2+6N} \\ &= \varphi(-q^6)\varphi(-q^{30}) + 2qf(-q^{20}, -q^{40})f(-q^4, -q^8) \\ &\quad + 2q^4f(-q^{10}, -q^{50})f(-q^2, -q^{10}). \end{aligned} \tag{2.28}$$

Using (1.3), (1.4) and Lemma 2.1 in (2.28), we get (2.21). Change q to $-q$ in (2.21) and employing Lemma 2.1, we obtain (2.22). In a similar way, we prove the identities (2.23) and (2.24).

We have

$$\varphi(q)\varphi(-q^9) = f(q, q)f(-q^9, -q^9) = \sum_{m,n=-\infty}^{\infty} (-1)^n q^{m^2+9n^2}.$$

For given integers m and n , we choose integers M and N such that

$$m - 3n = 6M + a \text{ and } m + 3n = 6N + b,$$

where a and b have values selected from the set $\{0, \pm 1, \pm 2, 3\}$. Then

$$m = 3M + 3N + (a + b)/2 \text{ and } n = N - M + (b - a)/6.$$

It follows easily that $a = b$, and so $m = 3M + 3N + a$ and $n = N - M$, where $-2 \leq a \leq 3$. Thus, there is one-to-one correspondence between the set of all pairs of integers (m, n) , $-\infty < m, n < \infty$, and triples of integers (M, N, a) ,

$-\infty < M, N < \infty$, $-2 \leq a \leq 3$. Thus

$$\begin{aligned}
\varphi(q)\varphi(-q^9) &= \sum_{m,n=-\infty}^{\infty} (-1)^n q^{m^2+9n^2} \\
&= q^4 \sum_{M,N=-\infty}^{\infty} (-1)^{M+N} q^{18M^2-12M+18N^2-12N} \\
&\quad + q \sum_{M,N=-\infty}^{\infty} (-1)^{M+N} q^{18M^2-6M+18N^2-6N} \\
&\quad + \sum_{M,N=-\infty}^{\infty} (-1)^{M+N} q^{18M^2+18N^2} + q \sum_{M,N=-\infty}^{\infty} (-1)^{M+N} q^{18M^2+6M+18N^2+6N} \\
&\quad + q^4 \sum_{M,N=-\infty}^{\infty} (-1)^{M+N} q^{18M^2+12M+18N^2+12N} \\
&\quad + q^9 \sum_{M,N=-\infty}^{\infty} (-1)^{M+N} q^{18M^2+18M+18N^2+18N} \\
&= q^4 f^2(-q^6, -q^{30}) + q f^2(-q^{12}, -q^{24}) + \varphi^2(-q^{18}) \\
&\quad + q f^2(-q^{12}, -q^{24}) + q^4 f^2(-q^6, -q^{30}).
\end{aligned}$$

Equivalently

$$\varphi(-q^9)\{\varphi(q) - \varphi(q^9)\} = 2q f^2(-q^{12}, -q^{24}) + 2q^4 f^2(-q^6, -q^{30}). \quad (2.29)$$

Employing (1.3), (1.4), (2.16), (2.17) and Lemma 2.1 in (2.29), we obtain (2.25). Changing q to $-q$ in (2.25) and then employing Lemma 2.1, we obtain (2.26). $\square$

Remark 2.9. Identities (2.25) and (2.26) are due to Adiga et al. [3] and our proofs are different from the proofs given by Adiga et al. [3].

Theorem 2.10. *We have*

$$S(q)T(q) = \frac{f(-q)\psi(q^3)}{\psi^2(-q)}, \quad (2.30)$$

$$S(q) + T(q) = \frac{1}{f_4} \left\{ \psi(q) + \frac{\psi(q^6)\varphi(q^6)}{\psi(-q^3)} \right\}. \quad (2.31)$$

Proof. For given integers m and n , we choose integers M and N such that

$$-m + n = 2M + a \text{ and } m + n = 2N + b,$$

where a and b have values selected from the set $\{0, 1\}$. Then

$$m = N - M + (b - a)/2 \text{ and } n = M + N + (a + b)/2.$$

It follows that, values of a and b are associated as in the following table:

a	0	1
b	0	1
m	$N - M$	$N - M$
n	$M + N$	$M + N + 1$

When a assume the values 0, 1 in succession, it is easy to see that the corresponding values of $3m^2 - m + 3n^2 + 3n$ are, respectively,

$$6M^2 + 6N^2 + 4M + 2N$$

and

$$6M^2 + 6N^2 + 10M + 8N + 6.$$

It is evident, from the equations connecting m and n with M and N that, there is a one-one correspondence between all pairs of integer (m, n) and all sets of integers (M, N, a) . Thus

$$\begin{aligned}
2f(-q)\psi(q^3) &= \sum_{n,m=-\infty}^{\infty} (-1)^m q^{(m(3m-1))/2} q^{(3n(n+1))/2} \\
&= \sum_{m,n=-\infty}^{\infty} (-1)^m q^{(3m^2-m+3n^2+3n)/2} \\
&= \sum_{M,N=-\infty}^{\infty} (-1)^{M+N} q^{(6M^2+6N^2+4M+2N)/2} \\
&\quad + q^3 \sum_{M,N=-\infty}^{\infty} (-1)^{M+N} q^{(6M^2+6N^2+10M+6N)/2} \\
&= f(-q, -q^5)f(-q^2, -q^4) + q^3 f(-q^8, -q^{-2})f(-q^7, -q^{-1}) \\
&\quad (\text{Using Lemma 2.3}) \\
&= f(-q, -q^5)f(-q^2, -q^4) + f(-q, -q^5)f(-q^2, -q^4) \\
&= 2f(-q, -q^5)f(-q^2, -q^4). \tag{2.32}
\end{aligned}$$

Employing (1.3) and (1.4) in (2.32), we obtain (2.30).

Putting $a = q^2$ and $b = q^4$ in (2.10), we find that

$$f(-q^2, -q^4) = \frac{f(-q^4, -q^8)}{f(q^2, q^4)} \varphi(-q^6). \tag{2.33}$$

The following three identities, from [6], will be useful.

$$f(q, q^2) = \frac{\varphi(-q^3)}{\chi(-q)}, \tag{2.34}$$

$$\psi(-q) = \chi(-q)f(-q^4), \tag{2.35}$$

$$\chi(q) = \frac{f(q)}{f(-q^2)} = \frac{f(-q^2)}{\psi(-q)}. \tag{2.36}$$

Dividing both sides of (2.33) by $\psi(-q)$ and employing (2.34) and (2.35) in resulting identity, we obtain

$$T(q) = \frac{\chi(-q^2)}{\chi(-q)}. \quad (2.37)$$

Putting $a = q$ and $b = q^5$ in (2.10), we find that

$$f(-q, -q^5) = \frac{f(-q^2, -q^{10})}{f(q, q^5)} \varphi(-q^6). \quad (2.38)$$

Employing (2.17) in (2.38) and dividing both sides by $\psi(-q)$, we get

$$S(q) = \frac{\chi(-q^2)\psi(q^6)\varphi(q^6)}{\psi(-q)\psi(-q^3)\chi(q)}. \quad (2.39)$$

Adding (2.37) and (2.39), we deduce

$$T(q) + S(q) = \frac{\chi(-q^2)}{\chi(-q)} + \frac{\chi(-q^2)\psi(q^6)\varphi(q^6)}{\psi(-q)\psi(-q^3)\chi(q)}. \quad (2.40)$$

Employing (2.36) in (2.40), we get

$$\begin{aligned} S(q) + T(q) &= \chi(-q^2) \left\{ \frac{\psi(q)}{f_2} + \frac{\psi(q^6)\varphi(q^6)}{\psi(-q^3)f_2} \right\}, \\ &= \frac{\chi(-q^2)}{f_2} \left\{ \psi(q) + \frac{\psi(q^6)\varphi(q^6)}{\psi(-q^3)} \right\}, \\ &= \frac{1}{f_4} \left\{ \psi(q) + \frac{\psi(q^6)\varphi(q^6)}{\psi(-q^3)} \right\}. \end{aligned}$$

□

Theorem 2.11. *We have*

$$\begin{aligned} T(q^{20})T(q^{28}) + q^{12}S(q^{20})S(q^{28}) \\ = \frac{f_{40}f_{56}}{4q^4f_{20}f_{28}f_{80}f_{112}} \left\{ \frac{f_1^2f_{70}^5}{f_2f_{35}^2f_{140}^2} + \frac{f_2^5f_{35}^2}{f_1^2f_4^2f_{70}} - 2\frac{f_{60}^2f_{84}^2}{f_{120}f_{168}} \right\}, \end{aligned} \quad (2.41)$$

$$\begin{aligned} T(q^4)T(q^{44}) + q^{12}S(q^4)S(q^{44}) \\ = \frac{f_8f_{88}}{4q^4f_4f_{16}f_{44}f_{176}} \left\{ \frac{f_1^2f_{22}^5}{f_2f_{11}^2f_{44}^2} + \frac{f_2^5f_{11}^2}{f_1^2f_4^2f_{22}} - 2\frac{f_{12}^2f_{132}^2}{f_{24}f_{264}} \right\} \end{aligned} \quad (2.42)$$

and

$$\begin{aligned} T(q^{12})T(q^{36}) + q^{12}S(q^{12})S(q^{36}) \\ = \frac{f_{24}f_{72}}{4q^4f_{12}f_{48}f_{36}f_{144}} \left\{ \frac{f_1^2f_{54}^5}{f_2f_{27}^2f_{108}^2} + \frac{f_2^5f_{27}^2}{f_1^2f_4^2f_{54}} - 2\frac{f_{36}^2f_{106}^2}{f_{72}f_{112}} \right\}. \end{aligned} \quad (2.43)$$

Proof. We have

$$\varphi(q)\varphi(-q^{35}) = \sum_{m,n=-\infty}^{\infty} (-1)^m q^{35m^2+n^2} \quad (2.44)$$

For given integers m and n , we choose integers M and N such that

$$7m + n = 12M + a \text{ and } -5m + n = 12N + b$$

where a and b have values selected from the set $\{0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, 6\}$. Then

$$m = M - N + (a - b)/12 \text{ and } n = 5M + 7N + (5a + 7b)/12.$$

And so $a = b$, since m , n , M and N are all integers. Therefore, $m = M - N$ and $n = 5M + 7N + a$, where $-5 \leq a \leq 6$.

When a assume the values $0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, 6$ in succession, it is easy to see that the corresponding values of $35m^2 + n^2$ are, respectively,

$$60M^2 + 84N^2$$

$$60M^2 \pm 10M + 84N^2 \pm 14N + 1,$$

$$60M^2 \pm 20M + 84N^2 \pm 28N + 4,$$

$$60M^2 \pm 30M + 84N^2 \pm 42N + 9,$$

$$60M^2 \pm 40M + 84N^2 \pm 56N + 16,$$

$$60M^2 \pm 50M + 84N^2 \pm 70N + 25,$$

$$60M^2 + 60M + 84N^2 + 84N + 36.$$

It is evident, from the equations connecting m and n with M and N that, there is a one-one correspondence between all pairs of integer (m, n) and all sets of integers (M, N, a) . Thus

$$\varphi(q)\varphi(-q^{35}) = \sum_{m,n=-\infty}^{\infty} (-1)^m q^{35m^2+n^2}$$

$$\begin{aligned}
\varphi(q)\varphi(-q^{35}) &= \sum_{M,N=-\infty}^{\infty} (-1)^{M+N} q^{60M^2+84N^2} + q \sum_{M,N=-\infty}^{\infty} (-1)^{M+N} q^{60M^2 \pm 10M + 84N^2 \pm 14N} \\
&+ q^4 \sum_{M,N=-\infty}^{\infty} (-1)^{M+N} q^{60M^2 \pm 20M + 84N^2 \pm 28N} \\
&+ q^9 \sum_{M,N=-\infty}^{\infty} (-1)^{M+N} q^{60M^2 \pm 30M + 84N^2 \pm 42N} \\
&+ q^{16} \sum_{M,N=-\infty}^{\infty} (-1)^{M+N} q^{60M^2 \pm 40M + 84N^2 \pm 56N} \\
&+ q^{25} \sum_{M,N=-\infty}^{\infty} (-1)^{M+N} q^{60M^2 \pm 50M + 84N^2 \pm 70N} \\
&+ q^{36} \sum_{M,N=-\infty}^{\infty} (-1)^{M+N} q^{60M^2 \pm 60M + 84N^2 \pm 84N + 36} \\
&= f(-q^{60}, -q^{60})f(-q^{84}, -q^{84}) + 2qf(-q^{50}, -q^{70})f(-q^{70}, -q^{98}) \\
&+ 2q^4f(-q^{40}, -q^{80})f(-q^{56}, -q^{112}) + 2q^9f(-q^{30}, -q^{90})f(-q^{42}, -q^{126}) \\
&+ 2q^{16}f(-q^{20}, -q^{100})f(-q^{28}, -q^{140}) + 2q^{25}f(-q^{10}, -q^{110})f(-q^{14}, -q^{154}).
\end{aligned} \tag{2.45}$$

Changing q to $-q$ in (2.45), we obtain

$$\begin{aligned}
\varphi(-q)\varphi(q^{35}) &= -2q^{25}f(-q^{10}, -q^{110})f(-q^{14}, -q^{154}) + 2q^{16}f(-q^{20}, -q^{100}) \\
&f(-q^{28}, -q^{140}) - 2q^9f(-q^{30}, -q^{90})f(-q^{42}, -q^{126}) + 2q^4f(-q^{40}, -q^{80}) \\
&f(-q^{56}, -q^{112}) - 2qf(-q^{50}, -q^{70})f(-q^{70}, -q^{98}) + f(-q^{60}, -q^{60})f(-q^{84}, -q^{84}).
\end{aligned} \tag{2.46}$$

Adding (2.45) and (2.46), we get

$$\begin{aligned}
\varphi(-q)\varphi(q^{35}) + \varphi(q)\varphi(-q^{35}) - 2\varphi(-q^{60})\varphi(-q^{84}) &= \\
4q^{16}f(-q^{20}, -q^{100})f(-q^{28}, -q^{140}) + 4q^4f(-q^{40}, -q^{80})f(-q^{56}, -q^{112}).
\end{aligned} \tag{2.47}$$

Employing (1.3), (1.4) and Lemma 2.1 in (2.47), we get (2.41). In a similar way, we prove the identities (2.42) and (2.43). This completes the proof of the Theorem (2.11). $\square$

Theorem 2.12. *We have*

$$\begin{aligned}
&T(q^3)T(q^{21}) + q^6S(q^3)S(q^{21}) \\
&= \frac{f_6f_{42}}{2q^2f_3f_{12}f_{21}f_{84}} \left\{ \frac{f_4^5f_{252}^5}{f_2^2f_8^2f_{126}^2f_{504}^2} + 2q^8\frac{f_2^2f_{126}^2}{f_1f_{63}} + 4q^{32}\frac{f_8^2f_{504}^2}{f_4f_{252}} - \frac{f_9^2f_{63}^2}{f_{18}f_{126}} \right\}, \tag{2.48}
\end{aligned}$$

and

$$\begin{aligned} & T(q)T(q^{23}) + q^6 S(q)S(q^{23}) \\ &= \frac{f_2 f_{46}}{2q^2 f_1 f_4 f_{23} f_{92}} \left\{ \frac{f_4^5 f_{92}^5}{f_2^2 f_8^2 f_{46}^2 f_{184}^2} + 4q^{12} \frac{f_8^2 f_{184}^2}{f_4 f_{92}} - \frac{f_3^2 f_{69}^2}{f_6 f_{138}} + 2q^3 \frac{f_2^2 f_{46}^2}{f_1 f_{23}} \right\}. \end{aligned} \quad (2.49)$$

Proof. Setting $\epsilon = 0, \delta = 1, l = t = 0, x = y = 1, \alpha = 21, \beta = 3, m = 1, p = 6$ and $\lambda = 4$ in (2.15), we find that

$$\begin{aligned} R(0, 1, 0, 0, 21, 3, 1, 6, 4, 1, 1) &:= \\ & \sum_{k=0}^5 q^{4k^2} f_1(q^{126+42k}, q^{126-42k}) f_1(q^{18+6k}, q^{18-6k}). \end{aligned} \quad (2.50)$$

Using (2.7), (2.14) and Lemma 2.3 in (2.50) and then changing q^2 to q , we deduce

$$\begin{aligned} R(0, 1, 0, 0, 21, 3, 1, 6, 4, 1, 1) &:= \varphi(-q^{63})\varphi(-q^9) \\ &+ 2q^8 f(-q^{21}, -q^{105})f(-q^3, -q^{15}) + 2q^2 f(-q^{42}, -q^{84})f(-q^6, -q^{12}). \end{aligned}$$

By Lemma 2.4, we also have

$$R(0, 1, 0, 0, 21, 3, 1, 6, 4, 1, 1) = R(1, 0, 0, 0, 1, 63, 21, 4, 126, 1, 1).$$

Thus

$$\begin{aligned} & 2q^8 f(-q^{21}, -q^{105})f(-q^3, -q^{15}) + 2q^2 f(-q^{42}, -q^{84})f(-q^6, -q^{12}) \\ &= \varphi(q^2)\varphi(q^{126}) + 2q^8 \psi(q)\psi(q^{63}) + 4q^{32} \psi(q^4)\psi(q^{252}) - \varphi(-q^9)\varphi(-q^{63}). \end{aligned} \quad (2.51)$$

Now, using (1.3), (1.4) and Lemma 2.1 in (2.51), we obtain (2.48).

In a similar way, setting $\epsilon = 0, \delta = 1, x = y = 1, \alpha = 23, \beta = 1, m = 1, p = 6$ and $\lambda = 4$ in (2.15), we prove the identity (2.49). $\square$

Theorem 2.13. *We have*

$$T(q)T(q^3) - qS(q)S(q^3) = \frac{f_6^3 f_9^2}{f_3^2 f_4 f_{12} f_{18}}, \quad (2.52)$$

$$T(q^5)T(q^7) - q^3 S(q^5)S(q^7) = \frac{f_{10} f_{14}}{2q f_5 f_7 f_{20} f_{28}} \left\{ \frac{f_{15}^2 f_{21}^2}{f_{30} f_{42}} - \frac{f_1^2 f_{35}^2}{f_2 f_{70}} \right\}, \quad (2.53)$$

$$T(q)T(q^{11}) - q^3 S(q)S(q^{11}) = \frac{f_2 f_{22}}{2q f_1 f_4 f_{11} f_{44}} \left\{ \frac{f_3^2 f_{33}^2}{f_6 f_{66}} - \frac{f_1^2 f_{11}^2}{f_2 f_{22}} \right\}, \quad (2.54)$$

$$\begin{aligned}
& T(q^2)T(q^{34}) - q^9 S(q^2)S(q^{34}) \\
&= \frac{f_4 f_{68}}{2q^3 f_2 f_8 f_{34} f_{136}} \left\{ \frac{f_6^2 f_{102}^2}{f_{12} f_{204}} - \frac{f_3^2 f_{102}^5}{f_6 f_{51}^2 f_{204}^2} \right\}, \tag{2.55}
\end{aligned}$$

$$\begin{aligned}
& T(q^2)T(q^{34}) + q^9 S(q^2)S(q^{34}) \\
&= \frac{f_4 f_{68}}{2q^3 f_2 f_8 f_{34} f_{136}} \left\{ \frac{f_6^5 f_{51}^2}{f_3^2 f_{12}^2 f_{102}} - \frac{f_6^2 f_{102}^2}{f_{12} f_{204}} \right\}, \tag{2.56}
\end{aligned}$$

$$\begin{aligned}
& T(q^5)T(q^{19}) - q^6 S(q^5)S(q^{19}) \\
&= \frac{f_{10} f_{38}}{2q^2 f_5 f_{19} f_{20} f_{76}} \left\{ \frac{f_{15}^2 f_{57}^2}{f_{30} f_{114}} + 2q^2 \frac{f_1 f_4 f_{95} f_{380}}{f_2 f_{190}} - \frac{f_2^2 f_{190}^2}{f_4 f_{380}} \right\}, \tag{2.57}
\end{aligned}$$

$$\begin{aligned}
& T(q^3)T(q^{21}) - q^6 S(q^3)S(q^{21}) \\
&= \frac{f_6 f_{42}}{2q^2 f_3 f_{12} f_{21} f_{84}} \left\{ \frac{f_9^2 f_{63}^2}{f_{18} f_{126}} - \frac{f_2^2 f_{126}^2}{f_4 f_{252}} + 2q^3 \frac{f_1 f_4 f_{63} f_{252}}{f_2 f_{126}} \right\}, \tag{2.58}
\end{aligned}$$

$$\begin{aligned}
& T(q)T(q^{23}) - q^6 S(q)S(q^{23}) \\
&= \frac{f_2 f_{46}}{2q^2 f_1 f_4 f_{23} f_{92}} \left\{ \frac{f_3^2 f_{69}^2}{f_6 f_{138}} - \frac{f_2^2 f_{46}^2}{f_4 f_{92}} \right\} + q. \tag{2.59}
\end{aligned}$$

Proof. Setting $\epsilon = \delta = 1$, $x = y = 1$, $\alpha = 9$, $\beta = 3$, $m = 1$, $l = t = 0$, $p = 6$ and $\lambda = 2$ in (2.15), we find that

$$\begin{aligned}
& R(1, 1, 0, 0, 9, 3, 1, 6, 2, 1, 1) := \\
& \sum_{k=0}^5 (-1)^k q^{2k^2} f_1(q^{54+18k}, q^{54-18k}) f_7(q^{18+6k}, q^{18-6k}). \tag{2.60}
\end{aligned}$$

Using (2.7), (2.14) and Lemma 2.3 in (2.60), we deduce

$$\begin{aligned}
& R(1, 1, 0, 0, 9, 3, 1, 6, 2, 1, 1) := \\
& \varphi(-q^{54})\varphi(-q^{18}) - 2q^2 f(-q^{72}, -q^{36})f(-q^{24}, -q^{12}) + 2q^8 f(-q^{90}, -q^{18})f(-q^{30}, -q^6).
\end{aligned}$$

By Lemma 2.4, we also have

$$R(1, 1, 0, 0, 9, 3, 1, 6, 2, 1, 1) = R(1, 1, 0, 0, 1, 27, 9, 2, 54, 1, 1).$$

Thus

$$\begin{aligned}
& 2q^8 f(-q^{90}, -q^{18})f(-q^{30}, -q^6) - 2q^2 f(-q^{72}, -q^{36})f(-q^{24}, -q^{12}) \\
&= \varphi(-q^{54})\{\varphi(-q^2) - \varphi(-q^{18})\}. \tag{2.61}
\end{aligned}$$

Now, using (1.3), (1.4), (2.16), (2.17), Lemma 2.1 in (2.61) and then changing q^6 to q , we obtain (2.52).

Setting $\epsilon = \delta = 1$, $x = y = 1$, $\alpha = 7$, $\beta = 5$, $m = 1$, $l = t = 0$, $p = 6$ and $\lambda = 2$ in (2.15), we find that

$$\begin{aligned}
& R(1, 1, 0, 0, 7, 5, 1, 6, 2, 1, 1) := \\
& \sum_{k=0}^5 (-1)^k q^{2k^2} f_1(q^{42+14k}, q^{42-14k}) f_7(q^{30+10k}, q^{30-10k}). \tag{2.62}
\end{aligned}$$

Using (2.14), Lemma 2.3, (2.7) in (2.62), we deduce

$$R(1, 1, 0, 0, 7, 5, 1, 6, 2, 1, 1) := \\ \varphi(-q^{42})\varphi(-q^{30}) - 2q^2 f(-q^{56}, -q^{28})f(-q^{20}, -q^{40}) + 2q^8 f(-q^{70}, -q^{14})f(-q^{50}, -q^{10}).$$

By Lemma 2.4, we also find that

$$R(1, 1, 0, 0, 7, 5, 1, 6, 2, 1, 1) = R(1, 1, 0, 0, 1, 35, 7, 2, 42, 1, 1).$$

Thus

$$2q^8 f(-q^{70}, -q^{14})f(-q^{50}, -q^{10}) - 2q^2 f(-q^{56}, -q^{28})f(-q^{20}, -q^{40}) \\ = \varphi(-q^2)\varphi(-q^{70}) - \varphi(-q^{42})\varphi(-q^{30}). \quad (2.63)$$

Now, using (1.3), (1.4), Lemma 2.1 in (2.63) and then changing q^2 and q , we obtain (2.53).

In a similar way, setting $\epsilon = \delta = 1$, $x = y = 1$, $l = t = 0$, $\alpha = 11, 17, 19, 21, 23$, $\beta = 1, 1, 5, 3, 1$, $m = 1$, $p = 6$ and $\lambda = 2, 3, 4, 4, 4$ in (2.15), we prove the identities (2.54), (2.55), (2.57), (2.58), (2.59). Changing q to $-q$ in (2.55), we get (2.56). $\square$

Remark 2.14. Identity (2.52) is due to Adiga et al. [3] and our proof is different from the proof given by Adiga et al [3].

Theorem 2.15. *We have*

$$T(q^4)T(q^{140}) - q^{36}S(q^4)S(q^{140}) = \\ \frac{f_8 f_{280}}{2q^{12} f_4 f_{16} f_{140} f_{560}} \left\{ \frac{f_{12}^2 f_{420}^2}{f_{24} f_{840}} - \frac{1}{2} \left\{ \frac{f_{14}^5 f_5^2}{f_7^2 f_{10} f_{28}^2} + \frac{f_7^2 f_{10}^5}{f_5^2 f_{14} f_{20}^2} \right\} \right\}, \quad (2.64)$$

$$S(-q) \left\{ \frac{T(q^{88})S(q^{44}) + q^{11}S(q^{88})T(q^{44})}{T(q^{44})S(q^{44})} \right\} \\ = \frac{f_1 f_{44} f_{176}^2 f_{528}}{q^4 f_2^2 f_{88}^2 f_{264}^2 f_{352}} \left\{ \frac{1}{2} \left\{ \frac{f_2^5 f_{22}^5}{f_1^2 f_4^2 f_{11}^2 f_{44}^2} + \frac{f_1^2 f_{11}^2}{f_2 f_{22}} \right\} + 2q^3 \frac{f_4^2 f_{44}^2}{f_2 f_{22}} - \frac{f_6^5 f_{66}^5}{f_3^2 f_{12}^2 f_{33}^2 f_{132}^2} \right\}, \quad (2.65)$$

$$S(q) \left\{ \frac{T(q^{88})S(q^{44}) - q^{11}S(q^{88})T(q^{44})}{T(q^{44})S(q^{44})} \right\} \\ = \frac{f_2 f_{44} f_{176}^2 f_{528}}{q^4 f_1 f_4 f_{88}^2 f_{264}^2 f_{352}} \left\{ \frac{1}{2} \left\{ \frac{f_2^5 f_{22}^5}{f_1^2 f_4^2 f_{11}^2 f_{44}^2} + \frac{f_1^2 f_{11}^2}{f_2 f_{22}} \right\} - 2q^3 \frac{f_4^2 f_{44}^2}{f_2 f_{22}} - \frac{f_3^2 f_{33}^2}{f_6 f_{66}} \right\}. \quad (2.66)$$

Proof. Setting $A = 1$, $B = -1$, $\mu = 6$, $\gamma = 1$ in (2.19), we can find

$$\frac{1}{2} \{ f(q^7, q^7)f(-q^5, -q^5) + f(-q^7, -q^7)f(q^5, q^5) \} \\ = \sum_{m=0}^5 (-1)^m q^{12m^2} f(-q^{420+140m}, -q^{420-140m})f(-q^{12+4m}, -q^{12-4m}). \quad (2.67)$$

Using (2.7) and Lemma 2.3 in (2.67), we deduce

$$\begin{aligned} & \frac{1}{2} \{ \varphi(q^7) \varphi(-q^5) + \varphi(-q^7) \varphi(q^5) \} - \varphi(-q^{420}) \varphi(-q^{12}) = \\ & 2q^{48} f(-q^{700}, -q^{140}) f(-q^{20}, -q^4) - 2q^{12} f(-q^{560}, -q^{280}) f(-q^{16}, -q^8). \end{aligned} \quad (2.68)$$

Now, using (1.3), (1.4) and Lemma 2.1 in (2.68), we obtain (2.64).

Setting $\mu = 6$, and $\omega = 1$ in (2.20), we can find

$$\begin{aligned} & \frac{1}{2} \{ \varphi(q^{11}) \varphi(q) + \varphi(-q^{11}) \varphi(-q) \} + 2q^3 \psi(q^{22}) \psi(q^2) \\ & = \sum_{m=0}^5 q^{4m^2} f(q^{132+44m}, q^{132-44m}) f(q^{3-2m}, q^{3+2m}). \end{aligned} \quad (2.69)$$

Using (2.7), and Lemma 2.3 in (2.69), we deduce

$$\begin{aligned} & \frac{1}{2} \{ \varphi(q^{11}) \varphi(q) + \varphi(-q^{11}) \varphi(-q) \} + 2q^3 \psi(q^{22}) \psi(q^2) \\ & = \varphi(q^3) \{ \varphi(q^{132}) + 2q^{33} \psi(q^{264}) \} + 2q^4 f(q^5, q) \{ f(q^{88}, q^{176}) + q^{11} f(q^{44}, q^{220}) \}. \end{aligned} \quad (2.70)$$

Employing (2.13), (2.10) in (2.70), we get

$$\begin{aligned} & \frac{1}{2} \{ \varphi(q^{11}) \varphi(q) + \varphi(-q^{11}) \varphi(-q) \} + 2q^3 \psi(q^{22}) \psi(q^2) - \varphi(q^3) \varphi(q^{33}) \\ & = 2q^4 \left\{ \frac{f(-q^{352}, -q^{176})}{f(-q^{176}, -q^{88})} + q^{11} \frac{f(-q^{88}, -q^{440})}{f(-q^{220}, -q^{44})} \right\} \varphi(-q^{264}) f(q, q^5). \end{aligned} \quad (2.71)$$

Now, using (1.3), (1.4) and Lemma 2.1 in (2.71), we obtain (2.65). Changing q to $-q$ in (2.65), we get (2.66). □

3. Applications to the theory of partitions

In the sequel, for simplicity, we adopt the notation

$$(a_1, a_2, \dots, a_n; q)_\infty = \prod_{j=1}^n (a_j; q)_\infty$$

and define

$$(q^{r\pm}; q^s)_\infty = (q^r, q^{s-r}; q^s)_\infty,$$

where r and s are positive integers and $r < s$.

We also need the notion of colored partitions. A positive integer n has k colors if there are k copies of n available and all of them are viewed as distinct objects. Partitions of positive integers into parts with colors are called colored partitions. For examples, if 1 is allowed to have 2 colors, say r (*red*), and g (*green*), then all colored partition of 2 are 2, $1_r + 1_r$, $1_g + 1_g$, and $1_r + 1_g$. An important fact is that

$$\frac{1}{(q^u; q^v)_\infty^k},$$

is the generating function for the number of partitions of n where all the parts are congruent to $u \pmod{v}$ and have k colors.

Theorem 3.1. *Let $p_1(n)$ denote the number of partitions of n into even parts that are not congruent $0, \pm 16 \pmod{48}$, with the parts congruent to $\pm 8, \pm 12, 24 \pmod{48}$ having two colors and with parts congruent to $\pm 2, \pm 6, \pm 10, \pm 14, \pm 18, \pm 22 \pmod{48}$ having three colors. Let $p_2(n)$ denote the number of partitions of n into even parts, that are not congruent to $0, \pm 4, \pm 20 \pmod{48}$, with the parts congruent to $\pm 8, \pm 12, \pm 16, 24 \pmod{48}$ having two colors, with parts congruent to $\pm 2, \pm 6, \pm 10, \pm 14, \pm 18, \pm 22 \pmod{48}$ having three colors. Let $p_3(n)$ denote the number of partitions of n into parts that are either odd or congruent $\pm 12 \pmod{48}$, with parts congruent to $\pm 1, \pm 3, \pm 5, \pm 7, \pm 9, \pm 11, \pm 13, \pm 15, \pm 17, \pm 19, \pm 21, \pm 23 \pmod{48}$ having two colors. Let $p_4(n)$ denote the number of partitions of n into even parts that are not congruent $0, \pm 12, \pm 24 \pmod{48}$, with the parts congruent to $\pm 16 \pmod{48}$ having two colors and with parts congruent to $\pm 2, \pm 6, \pm 8, \pm 10, \pm 14, \pm 18, \pm 22 \pmod{48}$ having three colors. Then, for any positive integer $n > 4$, we have*

$$2p_1(n-1) + 2p_2(n-4) = p_3(n) - p_4(n).$$

Proof. Employing (1.3) and (1.4) in (2.23) and simplifying, we obtain

$$\begin{aligned} & \frac{2q}{(q^{4\pm}, q^{20\pm}; q^{48})_{\infty} (q^{8\pm}, q^{12\pm}, q^{24}; q^{48})_{\infty}^2 (q^{2\pm}, q^{6\pm}, q^{10\pm}; q^{48})_{\infty}^3} \\ & \frac{1}{(q^{14\pm}, q^{18\pm}, q^{22\pm}; q^{48})_{\infty}^3} + \frac{2q^4}{(q^{8\pm}, q^{12\pm}, q^{16\pm}, q^{24}; q^{48})_{\infty}^2 (q^{2\pm}; q^{48})_{\infty}^3} \\ & \frac{1}{(q^{6\pm}, q^{10\pm}, q^{14\pm}, q^{18\pm}, q^{22\pm}; q^{48})_{\infty}^3} = \frac{1}{(q^{12\pm}; q^{48})_{\infty} (q^{1\pm}, q^{3\pm}; q^{48})_{\infty}^2} \\ & \frac{1}{(q^{5\pm}, q^{7\pm}, q^{9\pm}, q^{11\pm}, q^{13\pm}, q^{15\pm}, q^{17\pm}, q^{19\pm}, q^{21\pm}, q^{23\pm}; q^{48})_{\infty}^2} \\ & - \frac{1}{(q^{4\pm}, q^{20\pm}; q^{48})_{\infty} (q^{16\pm}; q^{48})_{\infty}^2 (q^{2\pm}, q^{6\pm}, q^{8\pm}, q^{10\pm}; q^{48})_{\infty}^3} \\ & \frac{1}{(q^{14\pm}, q^{18\pm}, q^{22\pm}; q^{48})_{\infty}^3}. \end{aligned}$$

Note that the left and right sides of the last equality represents the generating functions for $2p_1(n-1) + 2p_2(n-4)$ and $p_3(n) - p_4(n)$ respectively and this yield the desired result. $\square$

Example 3.2. The following table verifies the case $n = 5$ in Theorem 3.1 :

$p_1(4) = 7$	$p_2(1) = 0$	$p_3(5) = 14$	$p_4(5) = 0$
4		$5_g, 5_r$	
$2_g + 2_r$		$3_g + 1_r + 1_r$	
$2_g + 2_w$		$3_g + 1_g + 1_r$	
$2_r + 2_r$		$3_g + 1_g + 1_g$	
$2_r + 2_w$		$3_r + 1_r + 1_r$	
$2_w + 2_w$		$3_r + 1_r + 1_g$	
$2_g + 2_g$		$3_r + 1_g + 1_g$	
		$1_g + 1_g + 1_g + 1_g + 1_g$	
		$1_g + 1_g + 1_g + 1_g + 1_r$	
		$1_g + 1_g + 1_g + 1_r + 1_r$	
		$1_g + 1_g + 1_r + 1_r + 1_r$	
		$1_g + 1_r + 1_r + 1_r + 1_r$	
		$1_r + 1_r + 1_r + 1_r + 1_r$	

Theorem 3.3. Let $p_1(n)$ denote the number of partitions of n into even parts, that are not congruent to $0, \pm 12, \pm 24 \pmod{60}$, with parts congruent to $\pm 2, \pm 10, \pm 14, \pm 20, \pm 22, \pm 26 \pmod{60}$ having two colors, with parts congruent to $\pm 6, \pm 18 \pmod{60}$ having three colors and with parts congruent to $30 \pmod{60}$ having four colors. Let $p_2(n)$ denote the number of partitions of n into even parts congruent to $\pm 2, \pm 6, \pm 10, \pm 14, \pm 18, \pm 22, \pm 26, 30 \pmod{60}$, with parts congruent to $\pm 2, \pm 6, \pm 14, \pm 18, \pm 22, \pm 26 \pmod{60}$ having three colors and with parts congruent to $\pm 10, 30 \pmod{60}$ having four colors. Let $p_3(n)$ denote the number of partitions of n into odd parts, that are not congruent to multiples of $5 \pmod{60}$ with parts congruent to $\pm 1, \pm 3, \pm 7, \pm 9, \pm 11, \pm 13, \pm 17, \pm 19, \pm 21, \pm 23, \pm 27, \pm 29 \pmod{60}$ having two colors. Let $p_4(n)$ denote the number of partitions of n into even parts that are not congruent to $0, \pm 12, \pm 24, 30 \pmod{60}$, with parts congruent to $\pm 20 \pmod{60}$ having two colors and with parts congruent to $\pm 2, \pm 14, \pm 22, \pm 26 \pmod{60}$ having three colors and with parts congruent to $\pm 10 \pmod{60}$ having four colors. Then, for any positive integer $n > 4$, we have

$$2p_1(n-4) + 2p_2(n-1) = p_3(n) - p_4(n).$$

Proof. Employing (1.3) and (1.4) in (2.21) and simplifying, we obtain

$$\begin{aligned}
& \frac{2q^4}{(q^{4\pm}, q^{8\pm}, q^{16\pm}, q^{28\pm}; q^{60})_\infty (q^{2\pm}, q^{10\pm}, q^{14\pm}, q^{20\pm}, q^{22\pm}, q^{26\pm}; q^{60})_\infty^2} \\
& \frac{1}{(q^{6\pm}, q^{18\pm}; q^{60})_\infty^3 (q^{30}; q^{60})_\infty^4} + \frac{2q}{(q^{2\pm}, q^{6\pm}, q^{14\pm}, q^{18\pm}, q^{22\pm}, q^{26\pm}; q^{60})_\infty^3} \\
& \frac{1}{(q^{10\pm}, q^{30}; q^{60})_\infty^4} = \frac{1}{(q^{1\pm}, q^{3\pm}, q^{7\pm}, q^{9\pm}, q^{11\pm}, q^{13\pm}, q^{17\pm}, q^{19\pm}, q^{21\pm}; q^{60})_\infty^2} \\
& \frac{1}{(q^{23\pm}, q^{27\pm}, q^{29\pm}; q^{60})_\infty^2} - \frac{1}{(q^{4\pm}, q^{6\pm}, q^{8\pm}, q^{16\pm}, q^{18\pm}, q^{28\pm}; q^{60})_\infty (q^{20\pm}; q^{60})_\infty^2} \\
& \frac{1}{(q^{2\pm}, q^{14\pm}, q^{22\pm}, q^{26\pm}; q^{60})_\infty^3 (q^{10\pm}; q^{60})_\infty^4}.
\end{aligned}$$

Note that the left and right sides of the last equality represents the generating functions for $2p_1(n-4) + 2p_2(n-1)$ and $p_3(n) - p_4(n)$ respectively and this yield the desired result. $\square$

Example 3.4. The following table verifies the case $n = 5$ in Theorem 3.3 :

$p_1(1) = 0$	$p_2(4) = 6$	$p_3(5) = 12$	$p_4(5) = 0$
	$2_g + 2_g$	$3_g + 1_g + 1_g$	
	$2_r + 2_r$	$3_g + 1_g + 1_r$	
	$2_w + 2_w$	$3_g + 1_r + 1_r$	
	$2_g + 2_r$	$3_r + 1_r + 1_r$	
	$2_g + 2_w$	$3_r + 1_r + 1_g$	
	$2_r + 2_w$	$3_r + 1_g + 1_g$	
		$1_r + 1_r + 1_r + 1_r + 1_r$	
		$1_r + 1_r + 1_r + 1_r + 1_g$	
		$1_r + 1_r + 1_r + 1_g + 1_g$	
		$1_r + 1_r + 1_g + 1_g + 1_g$	
		$1_r + 1_g + 1_g + 1_g + 1_g$	
		$1_g + 1_g + 1_g + 1_g + 1_g$	

Theorem 3.5. Let $p_1(n)$ denote the number of partitions of n into parts that are not congruent to $0, \pm 6, \pm 12, \pm 18, \pm 24, \pm 30 \pmod{66}$, with parts congruent to $\pm 3, \pm 9, \pm 11, \pm 15, \pm 21, \pm 22, \pm 27 \pmod{66}$ having two colors and with parts congruent to $33 \pmod{66}$ having four colors. Let $p_2(n)$ denote the number of partitions of n into odd parts that are congruent to $\pm 1, \pm 3, \pm 5, \pm 7, \pm 9, \pm 11, \pm 13, \pm 15, \pm 17, \pm 19, \pm 21, \pm 23, \pm 25, \pm 27, \pm 29, \pm 31, 33 \pmod{66}$, with parts congruent $\pm 1, \pm 3, \pm 5, \pm 7, \pm 9, \pm 13, \pm 15, \pm 17, \pm 19, \pm 21, \pm 23, \pm 25, \pm 27, \pm 29, \pm 31 \pmod{66}$ having two colors and with part congruent to $\pm 11, 33 \pmod{66}$ having four colors. Let $p_3(n)$ denote the number of partitions of n into parts that are not congruent multiples of 3 $\pmod{66}$ with the parts congruent to $\pm 1, \pm 5, \pm 7, \pm 13, \pm 17, \pm 19, \pm 22, \pm 23, \pm 25, \pm 29, \pm 31 \pmod{66}$ having two colors and with parts congruent to $\pm 11 \pmod{66}$ having four colors. Then, for any positive integer $n > 4$, we have

$$2p_1(n-4) - 2p_2(n-1) + p_3(n) = 0.$$

Proof. Employing (1.3) and (1.4) in (2.54) and simplifying, we obtain

$$\begin{aligned}
& \frac{2q^4}{(q^{1\pm}, q^{2\pm}, q^{4\pm}, q^{5\pm}, q^{7\pm}, q^{8\pm}, q^{10\pm}, q^{13\pm}, q^{14\pm}, q^{16\pm}, q^{17\pm}, q^{19\pm}, q^{20\pm}; q^{66})_\infty} \\
& \frac{1}{(q^{23\pm}, q^{25\pm}, q^{26\pm}, q^{28\pm}, q^{29\pm}, q^{31\pm}, q^{32\pm}; q^{66})_\infty (q^{3\pm}, q^{9\pm}, q^{11\pm}; q^{66})_\infty^2} \\
& \frac{1}{(q^{15\pm}, q^{21\pm}, q^{22\pm}, q^{27\pm}; q^{66})_\infty^2 (q^{33}; q^{66})_\infty^4} - \frac{2q}{(q^{1\pm}, q^{3\pm}, q^{5\pm}, q^{7\pm}, q^{9\pm}; q^{66})_\infty^2} \\
& \frac{1}{(q^{13\pm}, q^{15\pm}, q^{17\pm}, q^{19\pm}, q^{21\pm}, q^{23\pm}, q^{25\pm}, q^{27\pm}, q^{29\pm}, q^{31\pm}; q^{66})_\infty^2} \\
& \frac{1}{(q^{11\pm}, q^{33}; q^{66})_\infty^4} = 1 - \frac{1}{(q^{2\pm}, q^{4\pm}, q^{8\pm}, q^{10\pm}, q^{14\pm}, q^{16\pm}, q^{20\pm}; q^{66})_\infty} \\
& \frac{1}{(q^{26\pm}, q^{28\pm}, q^{32\pm}; q^{66})_\infty (q^{1\pm}, q^{5\pm}, q^{7\pm}, q^{13\pm}, q^{17\pm}, q^{19\pm}, q^{22\pm}, q^{23\pm}, q^{25\pm}; q^{66})_\infty^2} \\
& \frac{1}{(q^{29\pm}, q^{31\pm}; q^{66})_\infty^2 (q^{11\pm}; q^{66})_\infty^4}
\end{aligned}$$

Note that the last equality represents the generating functions for $2p_1(n-4) - 2p_2(n-1) + p_3(n)$ and this yield the desired result. $\square$

Example 3.6. The following table verifies the case $n = 5$ in Theorem 3.5 :

$p_1(1) = 1$	$p_2(4) = 9$	$p_3(5) = 16$
1	$3_g + 1_g$	$5_g, 5_r$
	$3_g + 1_r$	$4 + 1_g, 4 + 1_r$
	$3_r + 1_r$	$2 + 2 + 1_g, 2 + 2 + 1_r$
	$3_r + 1_g$	$2 + 1_g + 1_g + 1_g$
	$1_g + 1_g + 1_g + 1_g$	$2 + 1_g + 1_g + 1_r$
	$1_g + 1_g + 1_g + 1_r$	$2 + 1_g + 1_r + 1_r$
	$1_g + 1_g + 1_r + 1_r$	$2 + 1_r + 1_r + 1_r$
	$1_g + 1_r + 1_r + 1_r$	$1_g + 1_g + 1_g + 1_g + 1_g$
	$1_r + 1_r + 1_r + 1_r$	$1_g + 1_g + 1_g + 1_g + 1_r$
		$1_g + 1_g + 1_g + 1_r + 1_r$
		$1_g + 1_g + 1_r + 1_r + 1_r$
		$1_g + 1_r + 1_r + 1_r + 1_r$
		$1_r + 1_r + 1_r + 1_r + 1_r$

Theorem 3.7. Let $p_1(n)$ denote the number of partitions of n into parts that are not congruent to $0, \pm 4, \pm 5, \pm 8, \pm 12, \pm 15, \pm 16, \pm 20, \pm 24, \pm 25, \pm 28 \pmod{60}$, with parts congruent $\pm 1, \pm 3, \pm 6, \pm 7, \pm 9, \pm 11, \pm 13, \pm 17, \pm 18, \pm 19, \pm 21, \pm 23, \pm 27, \pm 29 \pmod{60}$ having two colors and with parts congruent to $\pm 10, 30 \pmod{60}$ having four colors. Let $p_2(n)$ denote the number of partitions of n into parts that are not congruent to $0, \pm 2, \pm 5, \pm 12, \pm 14, \pm 15, \pm 22, \pm 24, \pm 25, \pm 26 \pmod{60}$, with parts congruent to $\pm 1, \pm 3, \pm 6, \pm 7, \pm 9, \pm 10, \pm 11, \pm 13, \pm 17, \pm 18, \pm 19, \pm 20, \pm 21, \pm 23, \pm 27, \pm 29 \pmod{60}$ having two colors and with parts congruent to $30 \pmod{60}$ having four colors. Let $p_3(n)$ denote the number of partitions of n into parts that are not congruent to $0, \pm 5, \pm 6, \pm 12, \pm 15, \pm 18,$

$\pm 24, \pm 25, 30 \pmod{60}$, with parts congruent to $\pm 1, \pm 3, \pm 7, \pm 9, \pm 11, \pm 13, \pm 17, \pm 19, \pm 20, \pm 21, \pm 23, \pm 27, \pm 29 \pmod{60}$ having two colors and with parts congruent $\pm 10 \pmod{60}$ having four colors. Let $p_4(n)$ denote the number of partitions of n into parts congruent to $\pm 2, \pm 4, \pm 6, \pm 8, \pm 14, \pm 16, \pm 18, \pm 22, \pm 26 \pmod{60}$. Then, for any positive integer $n > 4$, we have

$$2p_1(n-1) - 2p_2(n-4) = p_3(n) - p_4(n).$$

Proof. Employing (1.3) and (1.4) in (2.22) and simplifying, we obtain

$$\begin{aligned} & \frac{2q}{(q^{2\pm}, q^{14\pm}, q^{22\pm}, q^{26\pm}; q^{60})_{\infty} (q^{1\pm}, q^{3\pm}, q^{6\pm}, q^{7\pm}, q^{9\pm}, q^{11\pm}; q^{60})_{\infty}^2} \\ & \frac{1}{(q^{13\pm}, q^{17\pm}, q^{18\pm}, q^{19\pm}, q^{21\pm}, q^{23\pm}, q^{27\pm}, q^{29\pm}; q^{60})_{\infty}^2 (q^{10\pm}, q^{30}; q^{60})_{\infty}^4} \\ & - \frac{2q^4}{(q^{4\pm}, q^{8\pm}, q^{16\pm}, q^{28\pm}; q^{60})_{\infty} (q^{1\pm}, q^{3\pm}, q^{6\pm}, q^{7\pm}, q^{9\pm}, q^{10\pm}; q^{60})_{\infty}^2} \\ & \frac{1}{(q^{11\pm}, q^{13\pm}, q^{17\pm}, q^{18\pm}, q^{19\pm}, q^{20\pm}, q^{21\pm}, q^{23\pm}, q^{27\pm}, q^{29\pm}; q^{60})_{\infty}^2} \\ & \frac{1}{(q^{30}; q^{60})_{\infty}^4} = \frac{1}{(q^{2\pm}, q^{4\pm}, q^{8\pm}, q^{14\pm}, q^{16\pm}, q^{22\pm}, q^{26\pm}, q^{28\pm}; q^{60})_{\infty}} \\ & \frac{1}{(q^{1\pm}, q^{3\pm}, q^{7\pm}, q^{9\pm}, q^{11\pm}, q^{13\pm}, q^{17\pm}, q^{19\pm}, q^{20\pm}, q^{21\pm}, q^{23\pm}, q^{27\pm}, q^{29\pm}; q^{60})_{\infty}^2} \\ & - \frac{1}{(q^{10\pm}; q^{60})_{\infty}^4} = \frac{1}{(q^{2\pm}, q^{4\pm}, q^{6\pm}, q^{8\pm}, q^{14\pm}, q^{16\pm}, q^{18\pm}, q^{22\pm}, q^{26\pm}; q^{60})_{\infty}} \end{aligned}$$

Note that the left and right sides of the last equality represents the generating functions for $2p_1(n-1) - 2p_2(n-4)$ and $p_3(n) - p_4(n)$ respectively and this yield the desired result. $\square$

Example 3.8. The following table verifies the case $n = 5$ in Theorem 3.7 :

$p_1(4) = 13$	$p_2(1) = 2$	$p_3(5) = 22$	$p_4(5) = 0$
$3_g + 1_g, 3_g + 1_r$	$1_g, 1_r$	$4 + 1_g, 4 + 1_r$	
$3_r + 1_g, 3_r + 1_r$		$3_g + 2, 3_r + 2$	
$2 + 2, 2 + 1_g + 1_g,$		$3_g + 1_g + 1_g, 3_g + 1_g + 1_r$	
$2 + 1_g + 1_r$		$3_g + 1_r + 1_r, 3_r + 1_r + 1_r$	
$2 + 1_r + 1_r$		$3_r + 1_g + 1_r, 3_r + 1_g + 1_g$	
$1_g + 1_g + 1_g + 1_g$		$2 + 2 + 1_g, 2 + 2 + 1_r$	
$1_g + 1_g + 1_g + 1_r$		$2 + 1_g + 1_g + 1_g, 2 + 1_g + 1_g + 1_r$	
$1_g + 1_g + 1_r + 1_r$		$2 + 1_g + 1_r + 1_r, 2 + 1_r + 1_r + 1_r$	
$1_g + 1_r + 1_r + 1_r$		$1_g + 1_g + 1_g + 1_g + 1_g$	
$1_r + 1_r + 1_r + 1_r$		$1_g + 1_g + 1_g + 1_g + 1_r$	
		$1_g + 1_g + 1_g + 1_r + 1_r$	
		$1_g + 1_r + 1_r + 1_r + 1_r$	
		$1_r + 1_r + 1_r + 1_r + 1_r$	

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