MATCHED PAIRS IN ABELIAN GROUPOIDS

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ABSTRACT. We use a canonical matched pair in an abelian groupoid to decompose it as a semidirect product. We use this decomposition to study the corresponding full $C^*$-algebra.

1. Introduction

In this paper we study $C^*$-algebras on abelian groupoids. The structure of abelian groupoids is recently studied by the authors in [9]. Our basic reference for groupoids is [12].

A groupoid Γ is a small category in which each morphism is invertible. The unit space $X = \Gamma(0)$ of Γ is the subset of elements $\gamma\gamma^{-1}$ where γ ranges over Γ. The range map $r : \Gamma \to \Gamma(0)$ and source map $s : \Gamma \to \Gamma(0)$ are defined by $r(\gamma) = \gamma\gamma^{-1}$, $s(\gamma) = \gamma^{-1}\gamma$, for $\gamma \in \Gamma$. For $u \in \Gamma(0)$, we set $\Gamma_u = s^{-1}(u)$ and $\Gamma_u = r^{-1}(u)$. The loop space $\Gamma(u) = \Gamma_u = \{\gamma \in \Gamma | r(\gamma) = s(\gamma) = u\}$ is called the isotropy group of Γ (at u).

For subgroupoids $\mathcal{H}$ and $\mathcal{K}$ of Γ, let us denote by $\Gamma/\mathcal{K}$ (resp. $\mathcal{H}\backslash\Gamma$) the set of right (resp. left) classes in Γ modulo $\mathcal{K}$ (resp. $\mathcal{H}$), that is $\{\gamma\mathcal{K}\gamma^{-1} | \gamma \in \Gamma\}$ (resp. $\{\mathcal{H}\gamma\gamma^{-1} | \gamma \in \Gamma\}$). We know that $\Gamma/\mathcal{K}$ and $\mathcal{H}\backslash\Gamma$ are fibered by $\Gamma(0)$ : for any $x$ in $\Gamma(0)$, one can define $(\Gamma/\mathcal{K})^x = \{\gamma\mathcal{K}\gamma^{-1} | r(\gamma) = x\}$ and $(\mathcal{H}\backslash\Gamma)^x = \{\mathcal{H}\gamma\gamma^{-1} | s(\gamma) = x\}$.

A subgroupoid $\mathcal{N}$ of Γ is called normal if there exist a groupoid $\mathcal{G}$ and a surjective groupoid morphism $\pi : \Gamma \to \mathcal{G}$ such that $\pi^{-1}(\mathcal{G}(0)) = \mathcal{N}$. In this case, $\Gamma/\mathcal{N}$ and $\mathcal{G}$ are isomorphic, as groupoids. The set $\Gamma(X) = \bigcup_{x \in X} \Gamma(x)$ is a closed normal subgroupoid of locally compact groupoid Γ. The subgroupoid $\Gamma(X)$ is closed. It is also a group bundle with locally compact topology [12, page 18]. This is an example of a continuous groupoid whose bundle map need not be open.

Remark 1.1. (i) [2, prop. 1.1.10] The quotient topology on $\Gamma/\mathcal{N}$ is induced by the continuous surjection $\pi : \Gamma \to \Gamma/\mathcal{N}$. For a topological groupoid Γ with subgroupoid $\mathcal{N}$ with induced topology, if Γ is second countable and locally compact

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and \( N \) is locally closed, then \( N \) is locally compact. If \( N \) is open in \( \Gamma \), then its range and source maps are open.

(ii) \([5, \text{def } 1.10]\) A normal locally compact (resp. measured) subgroupoid \( N \) of a locally compact (resp. measured) groupoid \( \Gamma \) is a continuous (resp. measurable) field \( \{ x \in X \to N(x) \subset \Gamma(x) \} \) of subgroups such that \( \gamma N(x) \gamma^{-1} = N(y) \), where \( \gamma : x \to y \).

If (continuous) Haar systems \( \lambda \) of \( \Gamma \) and \( \beta \) of \( \Gamma(X) \) exist, then (continuous) Haar system \( \alpha \) of \( R(\Gamma) := \{ (r(\gamma), s(\gamma)) : \gamma \in \Gamma \} \) exists as in Remark 6.35 in [4], for each \( x \in X \), there exists a Radon measure \( \sigma^x \) with full support on \( \Gamma_x/\Gamma(x) \) such that

\[
\int_{\Gamma} f(\gamma)d\lambda_x(\gamma) = \int_{\Gamma_x/\Gamma(x)} f(\gamma h)d\beta^x(h)d\sigma^x(\hat{\gamma}).
\]

We know that \( \Gamma_x/\Gamma(x) = R_x \). Also as in [7, lemma 4.2],

\[
\int_{\Gamma} f(\gamma)d\lambda_x^x(\gamma) = \int_R \int_{\Gamma(X)} f(\gamma h)d\beta^{s(\gamma)}(h)d\sigma^x(\hat{\gamma}),
\]

and the derivative of \( \alpha_x \) with respect to \( \sigma^x \) at \( \hat{\gamma} \) is \( \omega(\gamma)^{-1} \) [7, proof of lemma 4.7], that is

\[
\int_{\Gamma_x/\Gamma(x)} \omega(\gamma)^{-1} F(\hat{\gamma})d\sigma^x(\hat{\gamma}) = \int_{\Gamma_x/\Gamma(x)} F(\hat{\gamma})d\alpha_x(\hat{\gamma}),
\]

where \( \omega \) is a continuous \( \Gamma(X) \)-invariant homomorphism from \( \Gamma \) to \( \mathbb{R}^+ \) such that for all \( f \in C_c(\Gamma(X)) \),

\[
\int_{\Gamma(X)} f(g)d\beta^x(\gamma)(g) = \omega(\gamma) \int_{\Gamma(X)} f(\gamma g \gamma^{-1})d\beta^{s(\gamma)}(g).
\]

Conversely, if \( R \) is r-discrete (and therefore has continuous Haar system) the Haar systems on \( R \) bijectively corresponds to Haar systems on \( \Gamma(X) \) [9].

The subgroupoids \( \Gamma(X) \) and \( R(\Gamma) \) are measured groupoids, if \( \Gamma \) is a measured groupoid, The continuity of their Haar systems \( \beta \) and \( \alpha \) holds in some particular cases (for example see [7] and [4]), and they are locally compact groupoids in these cases.

2. MATCHED PAIRS AND SEMIDIRECT PRODUCTS IN ABELIAN GROUPOIDS

In this section we give a semidirect product decomposition for an abelian groupoid. This is done through a matched pair and leads to a \( C^* \)-diagonal (for a special case) in the next section.

Definition 2.1. An abelian groupoid is a groupoid whose isotropy groups are abelian.

In contrast to the group case, the \( C^* \)-algebra of an abelian groupoid is not necessarily commutative. Ramsay noted that if a groupoid is finite and has at least one element, whose source and range are different, then the corresponding \( C^* \)-algebra contains a \( 2 \times 2 \) matrix algebra and hence is not commutative. Something of this sort is typically going to happen for \( \acute{e} \text{tale} \) (r-discrete) groupoids, since if one element has distinct source and range, there will be a neighborhood of that
element consisting of elements whose source and range are distinct. Then we get an algebra of functions taking \(2 \times 2\) matrix values, which is not commutative. Indeed, the \(C^*\)-algebra \(C^*(\Gamma)\) of a locally compact groupoid \(\Gamma\) is commutative if and only if \(\Gamma\) is a continuous field of abelian locally compact groups [2, prop. 1.5.7].

**Theorem 2.2.** If \(\Gamma\) is a transitive abelian measured (resp. locally compact) groupoid with isotropy group \(G\), then \(G\) constitutes a uniformity for \(\Gamma\) in the sense of [18], namely there is an isomorphism \(t_x\) of \(G\) onto each isotropy \(\Gamma(x)\) such that

\[
t_{r(\gamma)}(g)\gamma = \gamma t_{s(\gamma)}(g),
\]

for each \(\gamma \in \Gamma, g \in G\), and \((g, \gamma) \mapsto t_{r(\gamma)}(g)\gamma\) is a Borel (resp. continuous) map.

**Proof.** We may define \(t_{s(\gamma)}(g) := \gamma^{-1} t_{r(\gamma)}(g)\gamma\), where \(g \in G = \Gamma(r(\gamma))\) and \(t_{r(\gamma)}\) is the identity homomorphism on \(G\). Similarly, if \(G = \Gamma(s(\gamma))\) we set \(t_{r(\gamma)}(g) := \gamma t_{s(\gamma)}(g)\gamma^{-1}\). This is well defined, because for each \(\gamma\) and \(\gamma'\) with \(r(\gamma) = r(\gamma')\) and \(s(\gamma) = s(\gamma')\), we have

\[
\gamma^{-1} t_{r(\gamma)}(g)\gamma = \gamma'^{-1} t_{r(\gamma)}(g)\gamma',
\]

since \(\Gamma(r(\gamma))\) is abelian and \(\gamma'\gamma^{-1} t_{r(\gamma)}(g) = t_{r(\gamma)}(g)\gamma'\gamma^{-1}\). The second assertion follows from the fact that the multiplication and inversion are Borel (resp. continuous).

In general, we can find \(t_1, t_2 \in \Gamma\) with \(s(t_1) = r(t_2), r(t_1) = s(\gamma), \) and \(s(t_2) = r(\gamma)\), such that \(t_{s(t_1)} = t_{r(t_2)}\). Hence, since \(G\) is abelian, \(t_{s(t_1)}(g) = (t_2\gamma t_1)^{-1} t_{r(t_2)}(g)(t_2\gamma t_1)\). Therefore if we define \(t_{s(\gamma)}(g) = t_1 t_{s(t_1)}(g)t_1^{-1}\) and \(t_{r(\gamma)}(g) = t_2^{-1} t_{r(t_2)}(g)t_2\), then \(t_{s(\gamma)}(g) = t_{r(\gamma)}(g)\gamma\), as required. \(\square\)

**Notation 2.3.** If \(\Gamma\) is as the above, we put \(g\gamma = t_{r(\gamma)}(g)\gamma\) and \(\gamma g = t_{s(\gamma)}(g)\gamma\). Following [12, page 8], this leads to the notion of semidirect product groupoid \(\Gamma \rtimes G\).

**Definition 2.4.** [17, page 18] Let \(\Gamma\) be any groupoid and \(\mathcal{H}, \mathcal{K}\) be two subgroupoids of \(\Gamma\) such that \(\Gamma = \mathcal{H}\mathcal{K} = \{hk : h \in \mathcal{H}, k \in \mathcal{K}^{(h)}\}\) and \(\mathcal{H} \cap \mathcal{K} \subset \Gamma^{(0)}\). Such a pair \((\mathcal{H}, \mathcal{K})\) is called a matched pair of \((\mathcal{H}, \mathcal{K})\).

**Theorem 2.5.** The subgroupoids \(R(\Gamma)\) and \(\Gamma(X)\) constitute a matched pair of subgroupoids in abelian groupoid \(\Gamma\).

**Proof.** By Remark 6.38 of [4], since \(\Gamma_x\) is second countable, for each \(x \in X\), we can find a Borel cross section \(c : \Gamma_x/\Gamma(x) \rightarrow \Gamma_x\) for the quotient map. Furthermore we can define a Borel map \(\delta : \Gamma_x \rightarrow \Gamma(x)\) such that \(\gamma = c([\gamma])\delta(\gamma).\) Since \(\delta(\gamma g) = c([\gamma]^{-1}(\gamma g) = c([\gamma])^{-1}\gamma g = \delta(\gamma)g\), we have \(c([\gamma g]) = c([\gamma])\), that is, \(c([\gamma]) \in \mathcal{R}\). \(\square\)

**Proposition 2.6.** Every subgroupoid of an abelian groupoid \(\Gamma\) can be constructed by a subfield of \(\Gamma(X)\) and its related principal subgroupoid.

**Proof.** The statement follows from the fact that for a subgroupoid the above two groupoids form a matched pair. \(\square\)
Now $\Gamma = R(\Gamma)\Gamma(X)$ is a matched pair groupoid. For any $k \in R$ and $h \in \Gamma(X)^{s(k)}$, let’s denote by $k \prec h$ (resp. $k \succ h$) the unique element in $\mathcal{H}$ (resp. $R^{s(k=\gamma h)}$) such that

\[ kh = (k \succ h)(k \prec h), \]

then $\succ$ (resp. $\prec$) is a left action of the groupoid $R$ on the fibered space $\Gamma(X)$ (resp. right action of the groupoid $\Gamma(X)$ on the fibered space $R$).

The map $h \mapsto hR^{s(h)}$ (resp. $k \mapsto \Gamma(X)_{r(k)}k$) is a natural bijection between $\Gamma(X)$ and $\Gamma/R$ (resp. $R$ and $\Gamma(X)\backslash \Gamma$). Also $R$ (resp. $\Gamma(X)$) has a left action on $\Gamma/R$ (resp. $\Gamma(X)\backslash \Gamma$) by multiplication: for any $h \in \Gamma(X)$, $k \in R$, $\gamma$ in $\Gamma_{r(h)}$ and $\gamma'$ in $\Gamma^{s(h)}$, one can define actions $\delta_{\gamma}(\Gamma(X)_{r(\gamma')}) = \Gamma(X)_{r(\gamma')}\gamma h$ and $\Delta_{\gamma}(\gamma' R^{s(\gamma')}) = k\gamma' R^{s(\gamma')}$. By a slight abuse of notation, one can see that the right action of $\Gamma(X)$ on $R$ (resp. left action of $R$ on $\Gamma(X)$) is exactly the one coming from the above paragraph.

Next we consider the extension problem for (abelian) groupoids. Some of the upcoming results are motivated by [14].

**Definition 2.7.** [1, def. 5.3.7] Let $c : \Gamma \to \mathcal{K}$ be a homomorphism from a groupoid $\Gamma$ to a groupoid $\mathcal{K}$. We write $p := c^{(0)} : \Gamma^{(0)} \to \mathcal{K}^{(0)}$, and say that $c$ is strongly surjective if $p$ is surjective and for every $x \in \Gamma^{(0)}$, $c(\Gamma^x) = \mathcal{K}^{p(x)}$. We denote by $\mathcal{H} = \ker(c)$ the kernel $c^{-1}(\mathcal{K}^{(0)})$ of $c$. In this situation, we say that $\Gamma$ is an *extension* of $\mathcal{K}$ by $\mathcal{H}$.

Let us first give some examples to motivate the construction.

**Example 2.8.** (i) [15] (central extension of a groupoid $R$ by an abelian group $G$) Let $p : R^{(0)} \times G \to R^{(0)}$ denote the first factor projection. By taking $s = r = p$ and identifying $R^{(0)}$ with $R^{(0)} \times \{e\}$ for the unit element $e \in G$, $R^{(0)} \times G$ could be regarded as a groupoid on $R^{(0)}$. A central extension $\Gamma$ of the groupoid $R$ by the abelian group $G$ is a sequence

\[ R^{(0)} \times G \to i \Gamma \to \pi R, \]

where $i$ and $\pi$ are injective and surjective groupoid morphisms over the identifying map $R^{(0)} \to \cong \Gamma^{(0)}$ and its inverse respectively, satisfying the conditions $\text{im}(i) = \ker(\pi)$ and $(i(r \circ \pi(\gamma), g))\gamma = \gamma(i(s \circ \pi(\gamma), g))$ for any $\gamma \in \Gamma$ and $g \in G$. The above equation is abbreviated as $g\gamma = \gamma g$. Note that $\pi|_{\Gamma^{(0)}}$ is injective. We choose a section $\xi$ of $\pi$ such that $\xi|_{R^{(0)}}$ coincides with the identifying map $R^{(0)} \to \cong \Gamma^{(0)}$.

(ii) [13, prop. 3.2] Let $\Gamma$ be an $r$-discrete groupoid over $X$ and let $\alpha$ be the canonical action of the inverse semigroup of its open bisections $\mathcal{S}$ on $X$. Let $\mathcal{K}$ be the groupoid of germs of the pseudogroup $\alpha(\mathcal{S})$. Then we have a short exact sequence of $r$-discrete groupoids

\[ \text{int}(\Gamma(X)) \to \Gamma \to \mathcal{K}, \]

where $\text{int}(\Gamma(X))$ is the interior of the isotropy bundle.

**Proposition 2.9.** $\Gamma$ is an extension of $R$ by $\Gamma(X)$.

**Proof.** The map $\theta : \Gamma \to R$ is a continuous homomorphism from a groupoid $\Gamma$ to the groupoid $R$. We write $p := \theta^{(0)} : \Gamma^{(0)} \to R^{(0)}$, and say that $\theta$ is strongly
Proof. (i) $\Gamma(x) = Ker(\theta)$ the kernel $\theta^{-1}(R^{(0)})$ of $\theta$. Using the map $\theta = (r, s) : \Gamma \to R(\Gamma)$, $\Gamma$ is a extension of $R(\Gamma)$ with kernel $\Gamma(X)$ [6, page 29]. □

For an $r$-discrete groupoid $\Gamma$ with finite unit space $X$, the above proposition fits in the framework of example (ii), since in this case $\Gamma$ is decomposable in the sense of Timmerman [9, theorem 1.5] and therefore $int\Gamma(X) = \Gamma(X)$ (c.f. the proof of [9, lem 1.5]).

Definition 2.10. [16, def. 3.9] Let $\mathcal{K}$ be a topological groupoid. A $\mathcal{K}$-module is a topological groupoid $\mathcal{H}$, with source and range maps equal to a map $p : \mathcal{H} \to \mathcal{H}^{(0)}$, such that

(i) $\mathcal{H}(x)$ is an abelian group for all $x$,
(ii) As a space, $\mathcal{H}$ is endowed with a $\mathcal{K}$-action $\mathcal{K} \times_{s,p} \mathcal{H} \to \mathcal{H}$, and
(iii) For each $k \in \mathcal{K}$, the map $p_k : \mathcal{H}_{s(h)} \to \mathcal{H}_{r(h)}$, given by the action, is a group morphism.

We shall write $k(h) := ad_k(g) = kgk^{-1}$ (or $k.h$ or $kh$) similarly $h.k = k^{-1}hk$.

Lemma 2.11. $\Gamma(X)$ is an $R$-module.

Proof. (i) $\Gamma(x)$ is an abelian group for all $x$, (ii) as a space, $\Gamma(X)$ is endowed with a $R$-action $\Gamma(X) \times_{s,r} R \to \Gamma(X)$, and (iii) for each $k \in R$, the map $ad_k : \Gamma(s(k)) \to \Gamma(r(k))$, given by the action, is a group morphism. □

Definition 2.12. Let $\mathcal{H}$ and $\mathcal{K}$ be groupoids. We say that $\mathcal{K}$ acts (on the right) on the groupoid $\mathcal{H}$, or that $\mathcal{H}$ is a right $\mathcal{K}$-groupoid if

(i) $\mathcal{H}$ is a right $\mathcal{K}$-space,
(ii) The map $p : \mathcal{H} \to \mathcal{K}^{(0)}$ defining the composable pairs satisfies $p = p \circ r = p \circ s$, and
(iii) For all $(h_1, h_2, k) \in \mathcal{H} \times \mathcal{H} \times \mathcal{K}$ such that $p(h_1) = p(h_2) = r(k)$, $s(h_1) = r(h_2)$, the equality $(h_1 h_2) k = (h_1 k)(h_2 k)$ holds.

Lemma 2.13. The groupoid $\Gamma(X)$ is a right $R$-groupoid.

Proof. (i) $\Gamma(X)$ is a right $R$-space, (ii) The map $p = r : \Gamma(X) \to R^{(0)}$ defining the composable pairs satisfies $p = p \circ r = p \circ s$, and (iii) For all $(h_1, h_2, k) \in \Gamma(X) \times \Gamma(X) \times R$ such that $p(h_1) = p(h_2) = r(k)$, $s(h_1) = r(h_2)$, the equality $(h_1 h_2) k = (h_1 k)(h_2 k)$ holds. □

Definition 2.14. [1, page 80] Let $\mathcal{H}$ be a right $\mathcal{K}$-groupoid. The associated semidirect product is the space $\mathcal{H} \rtimes_{s} \mathcal{K} = \{(h, k) \in \mathcal{H} \times \mathcal{K} : p(h) = r(k)\}$ endowed with the following groupoid structure. The unit space is $\mathcal{H}^{(0)}$. The range and source maps are given by $r(h, k) = r(h)$ and $s(h, k) = s(h)k$. The multiplication is given by $(h, k)(h', k') = (hh', kk')$ (it is defined when the products $hh'$ and $kk'$ are defined) and the inverse is given by $(h, k)^{-1} = (h^{-1}k, k^{-1})$.

By the above lemma, we may define the semidirect product groupoid $\Gamma(X) \rtimes_{s} R$. The total space of $\Gamma(X) \rtimes_{s} R$ is $\Gamma(X) *_{R^{(0)}} R$. The base space $(\Gamma(X) \rtimes_{s} R)^{(0)}$ is $X$, the source and range maps are

$\tilde{s}(h, k) = s(k)$, $\tilde{r}(h, k) = r(k)$,
respectively, the inverse is \((h, k)^{-1} = (h^{-1}, k, k^{-1})\), and multiplication is given by \((h_1, k_1)(h_2, k_2) = (h_1h_2, k_1k_2)\), defined whenever the product on the right-hand side exists.

There is a canonical extension in \(\text{ext}(\mathcal{K}, \mathcal{H})\): let \(\Gamma = \mathcal{H} \times_{p, r} \mathcal{K} = \{(h, k) \in \mathcal{H} \times \mathcal{K} | p(h) = r(k)\}\). The source and range maps in \(\Gamma\) are \(s(h, k) = s(k), r(h, k) = r(k)\). The product is \((h_1, k_1)(h_2, k_2) = (h_1 + k_1h_2, k_1k_2)\) (the product in \(\mathcal{H}\) is written additively). The inclusion \(\mathcal{H} \rightarrow \Gamma\) is \(i(h) = (h, p(h))\) and the projection \(\pi : \Gamma \rightarrow \mathcal{K}\) is \(\pi(h, k) = k\). Let us call this extension the strictly trivial extension.

**Theorem 2.15.** If \(\Gamma\) is an abelian groupoid, there is a canonical extension in \(\text{ext}(R(\Gamma), \Gamma(X))\) which is denoted by \(\Gamma(X) \rtimes_s R(\Gamma)\) and is equal to \(\Gamma\). This is an strictly trivial extension when

(a) There exists a groupoid morphism \(\xi : R \rightarrow \Gamma\) which is a section of \(\theta\).

(b) There exists \(\omega : \Gamma \rightarrow \Gamma(X)\) such that \(\omega(h\gamma) = h\omega(\gamma)\) for all \((h, \gamma) \in \Gamma(X) \times_{s, r} \Gamma\) and \(\omega(\gamma_1\gamma_2) = \omega(\gamma_1) + \theta(\gamma_1)\omega(\gamma_2)\) for all composable pairs \((\gamma_1, \gamma_2) \in \Gamma^{(2)}\).

**Proof.** We know that \((R(\Gamma), \Gamma(X))\) is a matched pair of subgroupoids. When \(\Gamma(X)\) is an abelian field of abelian groups, \(ad_k(g) = kgk^{-1}\) is simply an action and \(\sigma(k_1, k_2) = r(k_1)\)
a 2-cocycle.

\[
R^{(2)} \times_X \Gamma(X) \rightarrow \Gamma(X), (k_1, k_2, h) \mapsto \sigma(k_1, k_2)h
\]
and

\[
R \times_X \Gamma(X) \rightarrow \Gamma(X), (k, h) \mapsto ad_k(h),
\]
such that \(\sigma(k_1, k_2)\) is an element of the fiber of \(\Gamma(X)\) over \(r(k_1)\), \(ad_k\) is an isomorphism from the fiber over \(s(k)\) to the fiber over \(r(k)\), and the following equations are satisfied:

\[
ad_k(k_1) \circ ad_k(k_2) = ad(\sigma(k_1, k_2)) \circ ad_k(k_1k_2)
\]
and \((ad_k(k_1) \circ \sigma(k_2, k_3))\sigma(k_1, k_2k_3) = \sigma(k_1, k_2)\sigma(k_1k_2, k_3)\) where \(ad_1(h_1)(h_2) := h_1h_2h_1^{-1}\) for elements \(h_1, h_2 \in \Gamma(X)\) that both lie in the same fiber over \(X\). Therefore we have an extension of groupoids in the sense of [3, example 4.5]. The last statement follows from [16, prop. 5.3]. \(\square\)

3. \(C^*\)-algebra of abelian groupoids

**Theorem 3.1.** If \(\Gamma\) is an \(r\)-discrete abelian groupoid with finite unit space, then \(C^*(\Gamma)\) is unital, and its unit is the characteristic function \(X\).

**Proof.** Since \(X\) is open and compact, \(\chi_X \in C_C(\Gamma)\) but for every \(f \in C_C(\Gamma)\), we have \(f \ast \chi_X = \chi_X \ast f = f\). \(\square\)

Also if \(U\) is an open and closed subset of \(X\), then there is a unique multiplier \(p_U\) of \(C^*(\Gamma, \lambda)\), such that \(p_U\) is idempotent, \(C_0(X/\Gamma)\)-linear and contractive. Moreover,

\[
p_U C^*(\Gamma, \lambda)p_U = C^*(\Gamma_U, \chi_U).
\]
Indeed, if we set in [10, def. and prop. 5.3.4] the trivial $\Gamma$-bundle, if $C^*(\Gamma, \lambda)$ is an unconditional completion of $C_c(\Gamma)$ [10, page 144], we get the above equality. By construction $p_U$ in [10, def. and prop. 5.3.3], we conclude that $p_U = \chi_U$ is the characterize function of $U$, as in [11, lemma 2.7].

**Proposition 3.2.** An approximate unit of $C^*(\Gamma)$ is contained in $C^*(\Gamma(X))$, for decomposable abelian groupoid $\Gamma$.

**Proof.** When $\Gamma(X)$ is open, $C_c(\Gamma(X))$ is an abelian subalgebra of $C_c(\Gamma)$, and the restriction of the continuous Haar system of $\Gamma$ gives a continuous Haar system of $\Gamma(X)$. We know that $C^*(\Gamma, \lambda)$ has an approximate identity $(e_i)$ in $C_c(\Gamma)$ [4, lemma 5.10]. If moreover $e_i^* = e_i$ and $e_i * e_i = e_i$, the subalgebra $e_i C^*(\Gamma, \lambda)e_i$, which is isomorphic to $C^*(\Gamma, \lambda|_{\text{supp}(e_i)})$, has a compact unit, that is the characteristic function of the support of $e_i$ [10, def. and prop. 5.3.3]. In particular, this is the case for an abelian r-discrete groupoids whose algebraic components have compact unit space. □

The following result also appeared in [8].

**Proposition 3.3.** The center of $C^*$-algebra $C^*(\Gamma)$ (resp. $C^*_\mu(\Gamma)$) is $C^*(\Gamma(X))$, if $\Gamma$ is an r-discrete abelian groupoid with finite unit space, namely $Z(C^*(\Gamma)) = C^*(\Gamma(X))$.

**Proof.** If $\Gamma$ is an r-discrete abelian groupoid with finite unit space (which is decomposable abelian groupoid when has finite unit space [9]), the lemma follows from [12, prop. II.4.7(i)], because $C^*(X) \subset C^*(\Gamma(X))$ and $C^*(\Gamma(X))$ is abelian. An element $f \in C^*(\Gamma)$ is central if it commutes with every element of $C^*(\Gamma)$. In this case, $f$ commutes with each element of $C^*(X)$, therefore $f$ vanishes off $\Gamma(X)$. □

**Theorem 3.4.** If $\Gamma$ is a decomposable abelian groupoid, $C^*(\Gamma)$ is $C^*(\Gamma(X))$-bimodule.

**Proof.** The maximal tensor product of $C^*(\Gamma(X))$ and $C_0(X)$ is equal to $C^*(\Gamma(X))$ [12, page 81]. By [12, prop. II.2.4], $C^*(\Gamma)$ can be considered as a $C^*(\Gamma(X))$-bimodule. Also there is a $*$-homomorphism of $C^*(\Gamma(X))$ into the multiplier algebra of $C^*(\Gamma)$, because $\Gamma(X)$ is a closed subgroupoid of $\Gamma$ containing $X$, admitting a continuous Haar system. □

A $C^*$-algebra $A$ is decomposable if the inclusion of the center $Z(A)$ in $A$ is non-degenerate, that is $Z(A)A = A$. This definition is declared by Timmerman.

**Theorem 3.5.** If $\Gamma$ is an r-discrete abelian groupoid with finite unit space, then $C^*(\Gamma)$ is decomposable $C^*$-algebra.

**Proof.** Each element $f \in C^*(\Gamma)$ can be written as $g * \chi_X$, where $\chi_X \in C^*(\Gamma(X))$ and $g \in C^*(\Gamma)$. On the other hand, obviously $C^*(\Gamma(X))C^*(\Gamma) \subset C^*(\Gamma)$. □

We know that abelian $C^*$-algebras are semisimple but $C^*(\Gamma)$ for abelian groupoid $\Gamma$ is not necessary abelian. But we now show that this is correct for $C^*(\Gamma)$.
Proposition 3.6. For a r-discrete abelian groupoid with finite unit space $\Gamma$, the $C^*$-algebra $C^*(\Gamma)$ is semisimple.

Proof. In [4, page 237] it is asserted that every representation of $\Gamma$ induced from an irreducible representation of a stability group is irreducible. This implies that $\text{rad} C^*(\Gamma) \subset \text{rad} C^*(\Gamma(X)) = \{0\}$. \hfill $\square$

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