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MATCHED PAIRS IN ABELIAN GROUPOIDS

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Abstract. We use a canonical matched pair in an abelian groupoid to decompose it as a semidirect product. We use this decomposition to study the corresponding full C^*-algebra.

1. Introduction

In this paper we study C^*-algebras on abelian groupoids. The structure of abelian groupoids is recently studied by the authors in [9]. Our basic reference for groupoids is [12].

A groupoid Γ is a small category in which each morphism is invertible. The unit space $X = \Gamma^{(0)}$ of Γ is the subset of elements $\gamma\gamma^{-1}$ where γ ranges over Γ. The range map $r : \Gamma \to \Gamma^{(0)}$ and source map $s : \Gamma \to \Gamma^{(0)}$ are defined by $r(\gamma) = \gamma\gamma^{-1}$, $s(\gamma) = \gamma^{-1}\gamma$, for $\gamma \in \Gamma$. For $u \in \Gamma^{(0)}$, we set $\Gamma^u = r^{-1}(u)$ and $\Gamma_u = s^{-1}(u)$. The loop space $\Gamma(u) = \Gamma^u_u = \{\gamma \in \Gamma | r(\gamma) = s(\gamma) = u\}$ is called the *isotropy group* of Γ (at u).

For subgroupoids $\mathcal{H}$ and $\mathcal{K}$ of Γ, let us denote by $\Gamma/\mathcal{K}$ (resp. $\mathcal{H}\backslash\Gamma$) the set of right (resp. left) classes in Γ modulo $\mathcal{K}$ (resp. $\mathcal{H}$), that is $\{\gamma\mathcal{K}^{s(\gamma)} | \gamma \in \Gamma\}$ (resp. $\{\mathcal{H}_{r(\gamma)}\gamma | \gamma \in \Gamma\}$). We know that $\Gamma/\mathcal{K}$ and $\mathcal{H}\backslash\Gamma$ are fibered by $\Gamma^{(0)}$: for any x in $\Gamma^{(0)}$, one can define $(\Gamma/\mathcal{K})^x = \{\gamma\mathcal{K}^{s(\gamma)} | r(\gamma) = x\}$ and $(\mathcal{H}\backslash\Gamma)^x = \{\mathcal{H}_{r(\gamma)}\gamma | s(\gamma) = x\}$.

A subgroupoid $\mathcal{N}$ of Γ is called *normal* if there exist a groupoid $\mathcal{G}$ and a surjective groupoid morphism $\pi : \Gamma \to \mathcal{G}$ such that $\pi^{-1}(\mathcal{G}^{(0)}) = \mathcal{N}$. In this case, $\Gamma/\mathcal{N}$ and $\mathcal{G}$ are isomorphic, as groupoids. The set $\Gamma(X) = \cup_{x\in X}\Gamma(x)$ is a closed normal subgroupoid of locally compact groupoid Γ. The subgroupoid $\Gamma(X)$ is closed. It is also a group bundle with locally compact topology [12, page 18]. This is an example of a continuous groupoid whose bundle map need not be open.

Remark 1.1. (i) [2, prop. 1.1.10] The quotient topology on $\Gamma/\mathcal{N}$ is induced by the continuous surjection $\pi : \Gamma \to \Gamma/\mathcal{N}$. For a topological groupoid Γ with subgroupoid $\mathcal{N}$ with induced topology, if Γ is second countable and locally compact

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and $\mathcal{N}$ is locally closed, then $\mathcal{N}$ is locally compact. If $\mathcal{N}$ is open in Γ, then its range and source maps are open.

(ii) [5, def 1.10] A normal locally compact (resp. measured) subgroupoid $\mathcal{N}$ of a locally compact (resp. measured) groupoid Γ is a continuous (resp. measurable) field $\{x \in X \to \mathcal{N}(x) \subset \Gamma(x)\}$ of subgroups such that $\gamma\mathcal{N}(x)\gamma^{-1} = \mathcal{N}(y)$, where $\gamma : x \to y$.

If (continuous) Haar systems λ of Γ and β of $\Gamma(X)$ exist, then (continuous) Haar system α of $R(\Gamma) := \{(r(\gamma), s(\gamma)) : \gamma \in \Gamma\}$ exists as in Remark 6.35 in [4], for each $x \in X$, there exists a Radon measure σ^x with full support on $\Gamma_x/\Gamma(x)$ such that

$$\int_\Gamma f(\gamma)d\lambda_x(\gamma) = \int_{\Gamma_x/\Gamma(x)} \int_{\Gamma(X)} f(\gamma h)d\beta^x(h)d\sigma^x(\dot{\gamma}).$$

We know that $\Gamma_x/\Gamma(x) = R_x$. Also as in [7, lemma 4.2],

$$\int_\Gamma f(\gamma)d\lambda^x(\gamma) = \int_R \int_{\Gamma(X)} f(\gamma h)d\beta^{s(\gamma)}(h)d\alpha^x(\dot{\gamma}),$$

and the derivative of α_x with respect to σ^x at $\dot{\gamma}$ is $\omega(\gamma)^{-1}$ [7, proof of lemma 4.7], that is

$$\int_{\Gamma_x/\Gamma(x)} \omega(\gamma)^{-1}F(\dot{\gamma})d\sigma^x(\dot{\gamma}) = \int_{\Gamma_x/\Gamma(x)} F(\dot{\gamma})d\alpha_x(\dot{\gamma}),$$

where ω is a continuous $\Gamma(X)$-invariant homomorphism from Γ to $\mathbb{R}^+$ such that for all $f \in C_c(\Gamma(X))$,

$$\int_{\Gamma(X)} f(g)d\beta^{r(\gamma)}(g) = \omega(\gamma) \int_{\Gamma(X)} f(\gamma g\gamma^{-1})d\beta^{s(\gamma)}(g).$$

Conversely, if R is r-discrete (and therefore has continuous Haar system) the Haar systems on R bijectively corresponds to Haar systems on $\Gamma(X)$ [9].

The subgroupoids $\Gamma(X)$ and $R(\Gamma)$ are measured groupoids, if Γ is a measured groupoid, The continuity of their Haar systems β and α holds in some particular cases (for example see [7] and [4]), and they are locally compact groupoids in these cases.

2. Matched pairs and semidirect products in abelian groupoids

In this section we give a semidirect product decomposition for an abelian groupoid. This is done through a matched pair and leads to a C^*-diagonal (for a special case) in the next section.

Definition 2.1. An *abelian groupoid* is a groupoid whose isotropy groups are abelian.

In contrast to the group case, the C^*-algebra of an abelian groupoid is not necessarily commutative. Ramsay noted that if a groupoid is finite and has at least one element, whose source and range are different, then the corresponding C^*-algebra contains a 2×2 matrix algebra and hence is not commutative. Something of this sort is typically going to happen for étale (r-discrete) groupoids, since if one element has distinct source and range, there will be a neighborhood of that

element consisting of elements whose source and range are distinct. Then we get an algebra of functions taking 2×2 matrix values, which is not commutative. Indeed, the C^*-algebra $C^*(\Gamma)$ of a locally compact groupoid Γ is commutative if and only if Γ is a continuous field of abelian locally compact groups [2, prop. 1.5.7].

Theorem 2.2. *If Γ is a transitive abelian measured (resp. locally compact) groupoid with isotropy group G, then G constitutes a uniformity for Γ in the sense of [18], namely there is an isomorphism ι_x of G onto each isotropy $\Gamma(x)$ such that*

$$\iota_{r(\gamma)}(g)\gamma = \gamma\iota_{s(\gamma)}(g),$$

for each $\gamma \in \Gamma, g \in G$, and $(g, \gamma) \mapsto \iota_{r(\gamma)}(g)\gamma$ is a Borel (resp. continuous) map.

Proof. We may define $\iota_{s(\gamma)}(g) := \gamma^{-1}\iota_{r(\gamma)}(g)\gamma$, where $g \in G = \Gamma(r(\gamma))$ and $\iota_{r(\gamma)}$ is the identity homomorphism on G. Similarly, if $G = \Gamma(s(\gamma))$ we set $\iota_{r(\gamma)}(g) := \gamma\iota_{s(\gamma)}(g)\gamma^{-1}$. This is well defined, because for each γ and γ' with $r(\gamma) = r(\gamma')$ and $s(\gamma) = s(\gamma')$, we have

$$\gamma^{-1}\iota_{r(\gamma)}(g)\gamma = {\gamma'}^{-1}\iota_{r(\gamma)}(g)\gamma',$$

since $\Gamma(r(\gamma))$ is abelian and $\gamma'\gamma^{-1}\iota_{r(\gamma)}(g) = \iota_{r(\gamma)}(g)\gamma'\gamma^{-1}$. The second assertion follows from the fact that the multiplication and inversion are Borel (resp. continuous).

In general, we can find $t_1, t_2 \in \Gamma$ with $s(t_1) = r(t_2)$, $r(t_1) = s(\gamma)$, and $s(t_2) = r(\gamma)$, such that $\iota_{s(t_1)} = \iota_{r(t_2)}$. Hence, since G is abelian, $\iota_{s(t_1)}(g) = (t_2\gamma t_1)^{-1}\iota_{r(t_2)}(g)(t_2\gamma t_1)$. Therefore if we define $\iota_{s(\gamma)}(g) = t_1\iota_{s(t_1)}(g)t_1^{-1}$ and $\iota_{r(\gamma)}(g) = t_2^{-1}\iota_{r(t_2)}(g)t_2$, then $\gamma\iota_{s(\gamma)}(g) = \iota_{r(\gamma)}(g)\gamma$, as required. □

Notation 2.3. If Γ is as the above, we put $g\gamma = \iota_{r(\gamma)}(g)\gamma$ and $\gamma g = \gamma\iota_{s(\gamma)}(g)$. Following [12, page 8], this leads to the notion of semidirect product groupoid $\Gamma \rtimes_s G$.

Definition 2.4. [17, page 18] Let Γ be any groupoid and $\mathcal{H}, \mathcal{K}$ be two subgroupoids of Γ such that $\Gamma = \mathcal{H}\mathcal{K} = \{hk : h \in \mathcal{H}, k \in \mathcal{K}^{s(h)}\}$ and $\mathcal{H} \cap \mathcal{K} \subset \Gamma^{(0)}$. Such a pair $(\mathcal{H}, \mathcal{K})$ is called a *matched pair* of (sub)groupoids.

Theorem 2.5. *The subgroupoids $R(\Gamma)$ and $\Gamma(X)$ constitute a matched pair of subgroupoids in abelian groupoid Γ.*

Proof. By Remark 6.38 of [4], since Γ_x is second countable, for each $x \in X$, we can find a Borel cross section $c : \Gamma_x/\Gamma(x) \to \Gamma_x$ for the quotient map. Furthermore we can define a Borel map $\delta : \Gamma_x \to \Gamma(x)$ such that $\gamma = c([\gamma])\delta(\gamma)$. Since $\delta(\gamma g) = c([\gamma g])^{-1}(\gamma g) = c([\gamma])^{-1}\gamma g = \delta(\gamma)g$, we have $c([\gamma g]) = c([\gamma])$, that is, $c([\gamma]) \in R$. □

Proposition 2.6. *Every subgroupoid of a abelian groupoid Γ can be constructed by a subfield of $\Gamma(X)$ and its related principal subgroupoid.*

Proof. The statement follows from the fact that for a subgroupoid the above two groupoids form a matched pair. □

Now $\Gamma = R(\Gamma)\Gamma(X)$ is a matched pair groupoid. For any $k \in R$ and $h \in \Gamma(X)^{s(k)}$, let's denote by $k \lhd h$ (resp. $k \rhd h$) the unique element in $\mathcal{H}$ (resp. $R^{s(k \rhd h)}$) such that

$$kh = (k \rhd h)(k \lhd h),$$

then $\rhd$ (resp. $\lhd$) is a left action of the groupoid R on the fibered space $\Gamma(X)$ (resp. right action of the groupoid $\Gamma(X)$ on the fibered space R).

The map $h \to hR^{s(h)}$ (resp. $k \to \Gamma(X)_{r(k)}k$) is a natural bijection between $\Gamma(X)$ and Γ/R (resp. R and $\Gamma(X)\backslash\Gamma$). Also R (resp. $\Gamma(X)$) has a left action on Γ/R (resp. $\Gamma(X)\backslash\Gamma$) by multiplication: for any h in $\Gamma(X)$, k in R, γ in $\Gamma_{r(h)}$ and γ' in $\Gamma^{s(k)}$, one can define actions $\delta_h(\Gamma(X)_{r(\gamma)}\gamma) = \Gamma(X)_{r(\gamma)}\gamma h$ and $\Delta_k(\gamma' R^{s(\gamma')}) = k\gamma' R^{s(\gamma')}$. By a slight abuse of notation, one can see that the right action of $\Gamma(X)$ on R (resp. left action of R on $\Gamma(X)$) is exactly the one coming from the above paragraph.

Next we consider the extension problem for (abelian) groupoids. Some of the upcoming results are motivated by [14].

Definition 2.7. [1, def. 5.3.7] Let $c : \Gamma \to \mathcal{K}$ be a homomorphism from a groupoid Γ to a groupoid $\mathcal{K}$. We write $p := c^{(0)} : \Gamma^{(0)} \to \mathcal{K}^{(0)}$, and say that c is *strongly surjective* if p is surjective and for every $x \in \Gamma^{(0)}$, $c(\Gamma^x) = \mathcal{K}^{p(x)}$. We denote by $\mathcal{H} = ker(c)$ the kernel $c^{-1}(\mathcal{K}^{(0)})$ of c. In this situation, we say that Γ is an *extension* of $\mathcal{K}$ by $\mathcal{H}$.

Let us first give some examples to motivate the construction.

Example 2.8. (i) [15] (central extension of a groupoid R by an abelian group G) Let $p : R^{(0)} \times G \to R^{(0)}$ denote the first factor projection. By taking $s = r = p$ and identifying $R^{(0)}$ with $R^{(0)} \times \{e\}$ for the unit element $e \in G$, $R^{(0)} \times G$ could be regarded as a groupoid on $R^{(0)}$. A central extension Γ of the groupoid R by the abelian group G is a sequence

$$R^{(0)} \times G \to^i \Gamma \to^\pi R,$$

where i and π are injective and surjective groupoid morphisms over the identifying map $R^{(0)} \to \cong\Gamma^{(0)}$ and its inverse respectively, satisfying the conditions $im(i) = ker(\pi)$ and $(i(r \circ \pi(\gamma), g))\gamma = \gamma(i(s \circ \pi(\gamma), g))$ for any $\gamma \in \Gamma$ and $g \in G$. The above equation is abbreviated as $g\gamma = \gamma g$. Note that $\pi|_{\Gamma^{(0)}}$ is injective. We choose a section ξ of π such that $\xi|_{R^{(0)}}$ coincides with the identifying map $R^{(0)} \to \cong\Gamma^{(0)}$.

(ii) [13, prop. 3.2] Let Γ be an r-discrete groupoid over X and let α be the canonical action of the inverse semigroup of its open bisections $\mathcal{S}$ on X. Let $\mathcal{K}$ be the groupoid of germs of the pseudogroup $\alpha(\mathcal{S})$. Then we have a short exact sequence of r-discrete groupoids

$$int(\Gamma(X)) \to \Gamma \to \mathcal{K},$$

where $int(\Gamma(X))$ is the interior of the isotropy bundle.

Proposition 2.9. *Γ is an extension of R by $\Gamma(X)$.*

Proof. The map $\theta : \Gamma \to R$ is a continuous homomorphism from a groupoid Γ to the groupoid R. We write $p := \theta^{(0)} : \Gamma^{(0)} \to R^{(0)}$, and say that θ is *strongly*

surjective if p is surjective and for every $x \in \Gamma^{(0)}$, $\theta(\Gamma^x) = R^{p(x)}$. We know $\Gamma(X) = Ker(\theta)$ the kernel $\theta^{-1}(R^{(0)})$ of θ. Using the map $\theta = (r, s) : \Gamma \to R(\Gamma)$, Γ is a extension of $R(\Gamma)$ with kernel $\Gamma(X)$ [6, page 29]. □

For an r-discrete groupoid Γ with finite unit space X, the above proposition fits in the framework of example (ii), since in this case Γ is decomposable in the sense of Timmerman [9, theorem 1.5] and therefore $int\Gamma(X) = \Gamma(X)$ (c.f. the proof of [9, lem 1.5]).

Definition 2.10. [16, def. 3.9] Let $\mathcal{K}$ be a topological groupoid. A $\mathcal{K}$-module is a topological groupoid $\mathcal{H}$, with source and range maps equal to a map $p : \mathcal{H} \to \mathcal{H}^{(0)}$, such that
(i) $\mathcal{H}(x)$ is an abelian group for all x,
(ii) As a space, $\mathcal{H}$ is endowed with a $\mathcal{K}$-action $\mathcal{K} \times_{s,p} \mathcal{H} \to \mathcal{H}$, and
(iii) For each $k \in \mathcal{K}$, the map $\rho_k : \mathcal{H}_{s(h)} \to \mathcal{H}_{r(h)}$, given by the action, is a group morphism.

We shall write $k(h) := ad_k(g) = kgk^{-1}$ (or $k.h$ or kh) similarly $h.k = k^{-1}hk$.

Lemma 2.11. $\Gamma(X)$ *is an* R*-module.*

Proof. (i) $\Gamma(x)$ is an abelian group for all x, (ii) as a space, $\Gamma(X)$ is endowed with a R-action $\Gamma(X) \times_{s,r} R \to \Gamma(X)$, and (iii) for each $k \in R$, the map $ad_k : \Gamma(s(k)) \to \Gamma(r(k))$, given by the action, is a group morphism. □

Definition 2.12. Let $\mathcal{H}$ and $\mathcal{K}$ be groupoids. We say that $\mathcal{K}$ acts (on the right) on the groupoid $\mathcal{H}$, or that $\mathcal{H}$ is a right $\mathcal{K}$-groupoid if
(i) $\mathcal{H}$ is a right $\mathcal{K}$-space,
(ii) The map $p : \mathcal{H} \to \mathcal{K}^{(0)}$ defining the composable pairs satisfies $p = p \circ r = p \circ s$, and
(iii) For all $(h_1, h_2, k) \in \mathcal{H} \times \mathcal{H} \times \mathcal{K}$ such that $p(h_1) = p(h_2) = r(k)$, $s(h_1) = r(h_2)$, the equality $(h_1h_2)k = (h_1k)(h_2k)$ holds.

Lemma 2.13. *The groupoid* $\Gamma(X)$ *is a right* R*-groupoid.*

Proof. (i) $\Gamma(X)$ is a right R-space, (ii) The map $p = r : \Gamma(X) \to R^{(0)}$ defining the composable pairs satisfies $p = p \circ r = p \circ s$, and (iii) For all $(h_1, h_2, k) \in \Gamma(X) \times \Gamma(X) \times R$ such that $p(h_1) = p(h_2) = r(k)$, $s(h_1) = r(h_2)$, the equality $(h_1h_2)k = (h_1k)(h_2k)$ holds.

□

Definition 2.14. [1, page 80] Let $\mathcal{H}$ be a right $\mathcal{K}$-groupoid. The associated *semidirect product* is the space $\mathcal{H} \rtimes_s \mathcal{K} = \{(h, k) \in \mathcal{H} \times \mathcal{K} : p(h) = r(k)\}$ endowed with the following groupoid structure. The unit space is $\mathcal{H}^{(0)}$. The range and source maps are given by $r(h, k) = r(h)$ and $s(h, k) = s(h)k$. The multiplication is given by $(h, k)(h'k, k') = (hh', kk')$ (it is defined when the products hh' and kk' are defined) and the inverse is given by $(h, k)^{-1} = (h^{-1}k, k^{-1})$.

By the above lemma, we may define the semidirect product groupoid $\Gamma(X) \rtimes_s R$. The total space of $\Gamma(X) \rtimes_s R$ is $\Gamma(X) *_{R^{(0)}} R$. The base space $(\Gamma(X) \rtimes_s R)^{(0)}$ is X, the source and range maps are

$$\tilde{s}(h, k) = s(k), \ \tilde{r}(h, k) = r(k),$$

respectively, the inverse is $(h,k)^{-1} = (h^{-1}.k, k^{-1})$, and multiplication is given by $(h_1,k_1)(h_2.k_1,k_2) = (h_1h_2, k_1k_2)$, defined whenever the product on the right-hand side exists.

There is a canonical extension in $ext(\mathcal{K},\mathcal{H})$: let $\Gamma = \mathcal{H} \times_{p,r} \mathcal{K} = \{(h,k) \in \mathcal{H}\times\mathcal{K} | p(h) = r(k)\}$. The source and range maps in Γ are $s(h,k) = s(k)$, $r(h,k) = r(k)$. The product is $(h_1,k_1)(h_2,k_2) = (h_1 + k_1.h_2, k_1k_2)$ (the product in $\mathcal{H}$ is written additively). The inclusion $\mathcal{H} \to \Gamma$ is $i(h) = (h, p(h))$ and the projection $\pi : \Gamma \to \mathcal{K}$ is $\pi(h,k) = k$. Let us call this extension the *strictly trivial extension.*

Theorem 2.15. *If Γ is an abelian groupoid, there is a canonical extension in $ext(R(\Gamma),\Gamma(X))$ which is denoted by $\Gamma(X) \rtimes_s R(\Gamma)$ and is equal to Γ. This is an strictly trivial extension when*

(a) There exists a groupoid morphism $\xi : R \to \Gamma$ which is a section of θ.

(b) There exists $\omega : \Gamma \to \Gamma(X)$ such that $\omega(h\gamma) = h\omega(\gamma)$ for all $(h,\gamma) \in \Gamma(X) \times_{s,r} \Gamma$ and $\omega(\gamma_1\gamma_2) = \omega(\gamma_1) + \theta(\gamma_1).\omega(\gamma_2)$ for all composable pairs $(\gamma_1,\gamma_2) \in \Gamma^{(2)}$.

Proof. We know that $(R(\Gamma),\Gamma(X))$ is a matched pair of subgroupoids. When $\Gamma(X)$ is an abelian field of abelian groups, $ad_k(g) = kgk^{-1}$ is simply an action and

$$\sigma(k_1,k_2) = r(k_1)$$

a 2-cocycle.

$$R^{(2)} \times_X \Gamma(X) \to \Gamma(X), (k_1,k_2,h) \to \sigma(k_1,k_2)h$$

and

$$R \times_X \Gamma(X) \to \Gamma(X), (k,h) \to ad_k(h),$$

such that $\sigma(k_1,k_2)$ is an element of the fiber of $\Gamma(X)$ over $r(k_1)$, ad_k is an isomorphism from the fiber over $s(k)$ to the fiber over $r(k)$, and the following equations are satisfied:

$$ad(k_1) \circ ad(k_2) = ad(\sigma(k_1,k_2)) \circ ad(k_1k_2)$$

and $(ad(k_1) \circ \sigma(k_2,k_3))\sigma(k_1,k_2k_3) = \sigma(k_1,k_2)\sigma(k_1k_2,k_3)$ where $ad(h_1)(h_2) := h_1h_2h_1^{-1}$ for elements $h_1,h_2 \in \Gamma(X)$ that both lie in the same fiber over X. Therefore we have an extension of groupoids in the sense of [3, example 4.5]. The last statement follows from [16, prop. 5.3]. □

3. C^*-algebra of abelian groupoids

Theorem 3.1. *If Γ is an r-discrete abelian groupoid with finite unit space, then $C^*(\Gamma)$ is unital, and its unit is the characteristic function X.*

Proof. Since X is open and compact, $\chi_X \in C_C(\Gamma)$ but for every $f \in C_C(\Gamma)$, we have $f * \chi_X = \chi_X * f = f$. □

Also if U is an open and closed subset of X, then there is a unique multiplier p_U of $C^*(\Gamma,\lambda)$, such that p_U is idempotent, $C_0(X/\Gamma)$-linear and contractive. Moreover,

$$p_U C^*(\Gamma,\lambda)p_U = C^*(\Gamma_U^U, \lambda_U^U).$$

Indeed, if we set in [10, def. and prop. 5.3.4] the trivial Γ-bundle, if $C^*(\Gamma,\lambda)$ is an unconditional completion of $C_c(\Gamma)$ [10, page 144], we get the above equality. By construction p_U in [10, def. and prop. 5.3.3], we conclude that $p_U = \chi_U$ is the characterize function of U, as in [11, lemma 2.7].

Proposition 3.2. *An approximate unit of $C^*(\Gamma)$ is contained in $C^*(\Gamma(X))$, for decomposable abelian groupoid Γ.*

Proof. When $\Gamma(X)$ is open, $C_c(\Gamma(X))$ is an abelian subalgebra of $C_c(\Gamma)$, and the restriction of the continuous Haar system of Γ gives a continuous Haar system of $\Gamma(X)$. We know that $C^*(\Gamma,\lambda)$ has an approximate identity (e_i) in $C_c(\Gamma)$ [4, lemma 5.10]. If moreover $e_i^* = e_i$ and $e_i * e_i = e_i$, the subalgebra $e_iC^*(\Gamma,\lambda)e_i$, which is isomorphic to $C^*(\Gamma,\lambda|_{suppe_i})$, has a compact unit, that is the characteristic function of the support of e_i [10, def. and prop. 5.3.3]. In particular, this is the case for an abelian r-discrete groupoids whose algebraic components have compact unit space.

□

The following result also appeared in [8].

Proposition 3.3. *The center of C^*-algebra $C^*(\Gamma)$ (resp. $C^*_\mu(\Gamma)$) is $C^*(\Gamma(X))$ if Γ is an r-discrete abelian groupoid with finite unit space, namely $Z(C^*(\Gamma)) = C^*(\Gamma(X))$.*

Proof. If Γ is an r-discrete abelian groupoid with finite unit space (which is decomposable abelian groupoid when has finite unit space [9]), the lemma follows from [12, prop. II.4.7(i)], because $C^*(X) \subset C^*(\Gamma(X))$ and $C^*(\Gamma(X))$ is abelian. An element $f \in C^*(\Gamma)$ is central if it commutes with every element of $C^*(\Gamma)$. In this case, f commutes with each element of $C^*(X)$, therefore f vanishes off $\Gamma(X)$. □

Theorem 3.4. *If Γ is an decomposable abelian groupoid, $C^*(\Gamma)$ is $C^*(\Gamma(X))$-bimodule.*

Proof. The maximal tensor product of $C^*(\Gamma(X))$ and $C_0(X)$ is equal to $C^*(\Gamma(X))$ [12, page 81]. By [12, prop. II.2.4], $C^*(\Gamma)$ can be considered as a $C^*(\Gamma(X))$-bimodule. Also there is a $*$-homomorphism of $C^*(\Gamma(X))$ into the multiplier algebra of $C^*(\Gamma)$, because $\Gamma(X)$ is a closed subgroupoid of Γ containing X, admitting a continuous Haar system. □

A C^*-algebra A is decomposable if the inclusion of the center $Z(A)$ in A is non-degenerate, that is $Z(A)A = A$. This definition is declared by Timmerman.

Theorem 3.5. *If Γ is an r-discrete abelian groupoid with finite unit space, then $C^*(\Gamma)$ is decomposable C^*-algebra.*

Proof. Each element $f \in C^*(\Gamma)$ can be written as $g * \chi_X$, where $\chi_X \in C^*(\Gamma(X))$ and $g \in C^*(\Gamma)$. On the other hand, obviously $C^*(\Gamma(X))C^*(\Gamma) \subset C^*(\Gamma)$. □

We know that abelian C^*-algebras are semisimple but $C^*(\Gamma)$ for abelian groupoid Γ is not necessary abelian. But we now show that this is correct for $C^*(\Gamma)$.

Proposition 3.6. *For a r-discrete abelian groupoid with finite unit space* Γ*, the* C^**-algebra* $C^*(\Gamma)$ *is semisimple.*

Proof. In [4, page 237] it is asserted that every representation of Γ induced from an irreducible representation of a stability group is irreducible. This implies that $radC^*(\Gamma) \subset radC^*(\Gamma(X)) = \{0\}$. □

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