FOURIER TRANSFORM AND THE IDEAL OF GROUP ALGEBRA ON THE NILPOTENT ENGEL-LIE GROUP

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ABSTRACT. We study noncommutative Fourier transform on the nilpotent Engel-Lie group $G_4$ to solve some interesting problems of noncommutative analysis. In fact the finest structure of $G_4$ can be shown as a semi-direct product of two real vector groups. This helps us to form our idea by constructing a new larger group in order to define the Fourier transform and to obtain the Plancherel formula on $G_4$. Moreover we show that our methods lead us to construct several existence theorems for the invariant differential operators on $G_4$ and on $G_4 \times \mathbb{R}$. Since the heat equation is invariant on $G_4 \times \mathbb{R}$, so a fundamental solution of this equation will be obtained. Finally we establish a theorem that gives a classification of all left ideals of the noncommutative Banach algebra $L^1(G_4)$ of $G_4$.

1. Introduction and results

1.1 Let $G_4$ be the nilpotent Engel-Lie group consisting of all matrices of the form

\[
\begin{pmatrix}
1 & -x_1 & \frac{x_1^2}{2} & x_4 \\
0 & 1 & -x_1 & x_3 \\
0 & 0 & 1 & x_2 \\
0 & 0 & 0 & 1
\end{pmatrix}
\] (1.1)

where $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$. The group $G_4$ contains the 3-dimensional real vector group $H$ that consists of all matrices of the form

\[
\begin{pmatrix}
1 & 0 & 0 & x_4 \\
0 & 1 & 0 & x_3 \\
0 & 0 & 1 & x_2 \\
0 & 0 & 0 & 1
\end{pmatrix}
\] (1.2)
as normal sub-group. Let \( G \) be the group consisting of all matrices of the form

\[
\begin{pmatrix}
1 & -x_1 & \frac{x_1^2}{2} & 0 \\
0 & 1 & -x_1 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]  

(1.3)

Then \( G_4 \) can be identified with the group \( H \rtimes \gamma G \) a semi-direct product of \( G \) by \( H \) where

\[
\gamma(X_1) = X_1X_1^{-1} 
\]

(1.4)

\[
X_1 = \begin{pmatrix}
1 & -x_1 & \frac{x_1^2}{2} & 0 \\
0 & 1 & -x_1 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

(1.5)

and

\[
X = \begin{pmatrix}
1 & 0 & 0 & x_4 \\
0 & 1 & 0 & x_3 \\
0 & 0 & 1 & x_2 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

(1.6)

In view of the group isomorphism \( \Psi : \mathbb{R} \to G \) defined by

\[
\psi(x_1) = \begin{pmatrix}
1 & -x_1 & \frac{x_1^2}{2} & 0 \\
0 & 1 & -x_1 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

(1.7)

We can identify the group \( G \) with the real vector group \( \mathbb{R} \). So \( G_4 \) can be identified with the group \( K = \mathbb{R}^3 \rtimes \rho \mathbb{R}(H \simeq \mathbb{R}^3) \) a semi-direct product of \( \mathbb{R}^3 \) by \( \mathbb{R} \), via the group homomorphism \( \rho : \mathbb{R} \to Aut(\mathbb{R}^3) \), which is defined by

\[
\rho(x_1) = \begin{pmatrix}
1 & -x_1 & \frac{x_1^2}{2} \\
0 & 1 & -x_1 \\
0 & 0 & 1
\end{pmatrix}
\]

(1.8)

and

\[
\rho(x_1)(x_4, x_3, x_2) = \begin{pmatrix}
1 & -x_1 & \frac{x_1^2}{2} \\
0 & 1 & -x_1 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
x_4 \\
x_3 \\
x_2
\end{pmatrix}
\]

(1.9)

\[
= (x_4 - x_1x_3 + \frac{1}{2}x_1^2x_2, x_3 - x_1x_2, x_2)
\]

for any \((x_4, x_3, x_2, x_1) \in \mathbb{R}^4\). The multiplication of two elements \( X = (x_4, x_3, x_2, x_1) \) and \( Y = (y_4, y_3, y_2, y_1) \) in \( G_4 \) is given by

\[
X \cdot Y = (x_4, x_3, x_2, x_1)(y_4, y_3, y_2, y_1)
\]

(1.10)

\[
= (x_4 + y_4 - x_1y_3 + \frac{1}{2}x_1^2y_2, x_3 + y_3 - x_1y_2, x_2 + y_2, x_1 + y_1)
\]
The inverse $X^{-1}$ of an element $X$ in $K$ is

$$X^{-1} = (x_4, x_3, x_2, x_1)^{-1}$$

$$= (-x_4 - x_1 x_3 - \frac{1}{2} x_1^2 x_2, -x_3 - x_1 x_2, -x_2, -x_1) \quad (1.11)$$

1.2 If $M$ is an unimodular Lie group, we denote by $L^1(M)$ the Banach algebra that consists of all complex valued functions on the group $M$, which are integrable with respect to the Haar measure of $M$ and multiplication is defined by convolution on $M$, and we denote by $L^2(M)$ its Hilbert space. Let $U$ be the complexified universal enveloping algebra of the real Lie algebra $g$ of $G_4$; which is canonically isomorphic to the algebra of all distributions on $G_4$ supported by $\{0\}$, where $0$ is the identity element of $G_4$. For any $u \in U$ one can define a differential operator $P_u$ on $G_4$ as follows:

$$P_u f(X) = u * f(X) = \int_{G_4} f(Y^{-1}X)u(Y)dY \quad (1.12)$$

for any $f \in C^\infty(G_4)$ where $dY = dy_4dy_3dy_2dy_1$ is the Haar measure on $G_4$ which is the Lebesgue measure on $\mathbb{R}^4$, $Y = (y_4, y_3, y_2, y_1)$, $X = (x_4, x_3, x_2, x_1)$ and $*$ denotes the convolution product on $G_4$. The mapping $u \mapsto P_u$ is an algebra isomorphism of $U$ onto the algebra of all invariant differential operators on $G_4$. For more details see [7] and [13].

1.3 Let $B = \mathbb{R}^3 \times \mathbb{R}$ be the commutative group of the direct product of $\mathbb{R}^3$ by $\mathbb{R}$. We denote also by $U$ the complexified enveloping algebra of the real Lie algebra $b$ of $B$. For every $u \in U$, we can associate a differential operator $Q_u$ on $B$ as follows

$$Q_u f(X) = u *_c f(X) = f *_c u(X) = \int_B f(X - Y)u(Y)dY \quad (1.13)$$

for any $f \in C^\infty(B)$, $X \in B, Y \in B$ where $*_c$ signify the convolution product on the real vector group $B$ and $dY = dy_4dy_3dy_2dy_1$ is the Lebesgue measure on $B$. The mapping $u \mapsto Q_u$ is an algebra isomorphism of $U$ onto the algebra of all invariant differential operators on $B$, which are nothing but the algebra of differential operator with constant coefficients on $B$. In this paper we will prove the following results:

I- Plancherel Formula (Theorem 2.7).

II- Existence theorems for any Invariant differential operator on $G_4$ (Theorems 3.2, 3.6 and 3.7).

III- An theorem for all left ideals of $L^1(G_4)$ (Theorem 4.2 and Corollary 4.3).
2. Invariant Functions and Fourier Transform

Let \( L = \mathbb{R}^3 \times \mathbb{R} \times [0, \infty) \) be the group with law:

\[
XY = (x_4, x_3, x_2, x_1, t)(y_4, y_3, y_2, y_1, s) = ((x_4, x_3, x_2)(\rho_1(t)(y_4, y_3, y_2)), x_1 + y_1, t + s) = (x_4 + y_4 - ty_3 + \frac{1}{2}t^2y_2, x_3 + y_3 - ty_2, x_2 + y_2, x_1 + y_1, t + s) \quad (2.1)
\]

for all \( X = (x_4, x_3, x_2, x_1, t) \in L \) and \( Y = (y_4, y_3, y_2, y_1, s) \in L \). In this case the group \( G_4 \) can be identified with the closed subgroup \( \mathbb{R}^3 \times \{0\} \times \mathbb{R} \) of \( L \) and \( B \) with the subgroup \( \mathbb{R}^3 \times \mathbb{R} \times \{0\} \) of \( L \).

**Definition 2.1.** For every \( f \in C^\infty(G_4) \), one can define a function \( \tilde{f} \in C^\infty(L) \) as follows:

\[
\tilde{f}(x_4, x_3, x_2, x_1, t) = f(\rho(x_1)(x_4, x_3, x_2), x_1 + t) \quad (2.2)
\]

for all \((x_4, x_3, x_2, x_1, t) \in L \).

So every function \( \psi(x_4, x_3, x_2, x_1) \) on \( G_4 \) extends uniquely as an invariant function \( \tilde{\psi}(x_4, x_3, x_2, x_1, t) \) on \( K \).

**Remark 2.2.** The function \( \tilde{f} \) is invariant in the following sense:

\[
\tilde{f}(\rho(s)(x_4, x_3, x_2), x_1 - s, t + s) = \tilde{f}(x_4, x_3, x_2, x_1, t) \quad (2.3)
\]

for any \((x_4, x_3, x_2, x_1, t) \in L \) and \( s \in \mathbb{R} \).

**Lemma 2.3.** For every function \( F \in C^\infty(L) \) invariant in the sense of formula (2.3) and for every \( u \in U \), we have

\[
u \ast F(x_4, x_3, x_2, x_1, t) = u \ast_c F(x_4, x_3, x_2, x_1, t) \quad (2.4)
\]

for every \((x_4, x_3, x_2, x_1, t) \in L \), where \( \ast \) signifies the convolution product on \( G_4 \) with respect to the variables \((x_4, x_3, x_2)\) and \( \ast_c \) signifies the commutative convolution product on \( B \) with respect to the variables \((x_4, x_3, x_2, x_1)\).

**Proof.** In fact we have

\[
P_u F(x_4, x_3, x_2, x_1, t) = u \ast F(x_4, x_3, x_2, x_1, t) = \\
\int_{G_4} F [(y_4, y_3, y_2, s)^{-1}(x_4, x_3, x_2, x_1, t)] u(y_4, y_3, y_2, s) dy_4 dy_3 dy_2 ds
\]

\[
= \int_{G_4} F [(\rho(s^{-1})(-y_4, -y_3, y_2, y_2))(x_4, x_3, x_2), x_1, t - s] u(y_4, y_3, y_2, s) dy_4 dy_3 dy_2 ds
\]

\[
= \int_{G_4} F [(-y_4, -y_3, y_2, y_2) + (x_4, x_3, x_2), x_1 - s, t] u(y_4, y_3, y_2, s) dy_4 dy_3 dy_2 ds
\]

\[
= u \ast_c (x_4, x_3, x_2, x_1, t) = Q_u F(x_4, x_3, x_2, x_1, t) \quad (2.5)
\]

where \( P_u \) and \( Q_u \) are the invariant differential operators on \( G_4 \) and \( B \) respectively. \( \square \)
As in [5], we will define the Fourier transform on $G_4$. Therefore let $S(G_4)$ be the Schwartz space of $G_4$ which can be considered as the Schwartz space of $S(\mathbb{R}^3 \times \mathbb{R})$ and let $S'(G_4)$ be the space of all tempered distributions on $G_4$. The action $\rho$ of the group $\mathbb{R}$ on $\mathbb{R}^3$ defines a natural action $\rho$ of the dual group $(\mathbb{R})^*$ of the group $\mathbb{R}$ ($(\mathbb{R})^* \cong \mathbb{R}$) on $(\mathbb{R}^3)^*$ which is given by:

$$
\rho(x_1) = \begin{pmatrix}
1 & 0 & 0 \\
-x_1 & 1 & 0 \\
\frac{x_1^2}{2} & -x_1 & 1
\end{pmatrix}
$$

(2.6)

and

$$
\rho(x_1)(\xi_4, \xi_3, \xi_2) = \begin{pmatrix}
1 & 0 & 0 \\
-x_1 & 1 & 0 \\
\frac{x_1^2}{2} & -x_1 & 1
\end{pmatrix} \begin{pmatrix}
\xi_4 \\
\xi_3 \\
\xi_2
\end{pmatrix}

= (\xi_4, -x_1\xi_4 + \xi_3, \frac{1}{2}x_1^2\xi_4 - x_1\xi_3, \xi_2)
$$

(2.7)

for any $(\xi_4, \xi_3, \xi_2) \in \mathbb{R}^3$ and $x_1 \in \mathbb{R}$

**Definition 2.4.** If $f \in S(G_4)$, one can define its Fourier transform of $f$ by:

$$
\mathcal{F}f(\xi) = \int_{G_4} f(X) e^{-i\langle \xi, X \rangle} dX
$$

(2.8)

for any $\xi = (\xi_4, \xi_3, \xi_2, \xi_1) \in \mathbb{R}^4$ and $X = (x_4, x_3, x_2, x_1) \in \mathbb{R}^4$, where $\langle \xi, X \rangle = \xi_4x_4 + \xi_3x_3 + \xi_2x_2 + \xi_1x_1$.

It is clear that $\mathcal{F}f \in S(\mathbb{R}^4)$ and the mapping $f \rightarrow \mathcal{F}f$ is an isomorphism of the topological vector space $S(G_4)$ onto $S(\mathbb{R}^4)$.

**Definition 2.5.** If $f \in S(G_4)$, we define the Fourier transform of its invariant $\tilde{f}$ as follows

$$
\mathcal{F}\tilde{f}(\xi, 0) = \int_{L \times \mathbb{R}} \tilde{f}(X, t)e^{-i\langle \xi, X \rangle}e^{-i\langle \mu, t \rangle} dX dt d\mu
$$

(2.9)

where $(\mu, t) \in \mathbb{R}^2$ and $\langle \mu, t \rangle = \mu t$.

**Corollary 2.6.** For every $u \in S(G_4)$, and $f \in S(G_4)$, we have

$$
\mathcal{F}(\check{u} * \tilde{f})(\xi, 0) = \mathcal{F}(\check{\tilde{f}})(\xi, 0)\mathcal{F}(\check{u})(\xi)
$$

(2.10)

for any $\xi = (\xi_4, \xi_3, \xi_2, \xi_1) \in \mathbb{R}^4$ and $\mu \in \mathbb{R}$.

**Proof.** By Lemma 2.3, we have

$$
\check{u} * \tilde{f}(x_4, x_3, x_2, x_1, t) = \check{u} * \check{\tilde{f}}(x_4, x_3, x_2, x_1, t)
$$

(2.11)

So, we get

$$
\mathcal{F}(\check{u} * \tilde{f})(\xi, 0) = \mathcal{F}(\check{u} * \check{\tilde{f}})(\xi, 0) = \mathcal{F}(\tilde{f})(\xi, 0)\mathcal{F}(\check{u})(\xi)
$$

(2.12)

□
Theorem 2.7. (Plancherel’s formula)
For any $f \in L^1(G_4) \cap L^2(G_4)$, we get

$$
\int_{G_4} |f(x_4, x_3, x_2, x_1)|^2 \, dx_4 dx_3 dx_2 dx_1 = \int_{\mathbb{R}^4} |\mathcal{F}f(\xi_4, \xi_3, \xi_2, \xi_1)|^2 \, d\xi_4 d\xi_3 d\xi_2 d\xi_1 \quad (2.13)
$$

Proof. For each function $f \in C_0^\infty(G_4)$, define a function $\tilde{f}$ as

$$
\tilde{f}(x_4, x_3, x_2, x_1) = f(\rho(x_1)(x_4, x_3, x_2), x_1)^{-1} \quad (2.14)
$$

Then, we have

$$
f \ast \tilde{f}(0, 0, 0, 0) = \int_{G_4} f \left[ (x_4, x_3, x_2, x_1)^{-1}(0, 0, 0, 0) \right] f(x_4, x_3, x_2, x_1) dx_4 dx_3 dx_2 dx_1 = \int_{G_4} \tilde{f} \left[ (\rho(x_1)^{-1}((x_4, -x_3, -x_2) + (0, 0, 0)), 0, -x_1) \right] f(x_4, x_3, x_2, x_1) dx_4 dx_3 dx_2 dx_1 = \int_{G_4} \tilde{f} \left[ (\rho(x_1)^{-1}((x_4, -x_3, -x_2) + (0, 0, 0)), -x_1) \right] f(x_4, x_3, x_2, x_1) dx_4 dx_3 dx_2 dx_1 \quad (2.15)
$$
We get the Plancherel’s formula on $G_4$

\[
 f \ast \tilde{f}(0,0,0,0,0) = \int_{\mathbb{R}^3} \mathcal{F}(f \ast \tilde{f})(\xi,\mu) d\xi d\mu = \int_{\mathbb{R}^5} \mathcal{F}(f \ast_c \tilde{f})(\xi,\mu) d\xi d\mu
\]

\[
= \int_{\mathbb{R}^4} \mathcal{F}(f)(\xi,0)\mathcal{F}(f)(\xi)d\xi = \int_{\mathbb{R}^4} \mathcal{F}(\tilde{f})(\xi)\mathcal{F}(f)(\xi)d\xi
\]

\[
= \int_{\mathbb{R}^4} |\mathcal{F}f(\xi_4,\xi_3,\xi_2,\xi_1)|^2 d\xi_4 d\xi_3 d\xi_2 d\xi_1
\]

\[
= \int_{G_4} |f(x_4,x_3,x_2,x_1)|^2 dx_4 dx_3 dx_2 dx_1
\]

The Fourier transform can be extended to an isometry of $L^2(G_4)$ onto $L^2(\mathbb{R}^4)$. \[\square\]

**Corollary 2.8.** In equation 2.15, replace the first $f$ by $g$ we obtain the Parseval formula on $G_4$

\[
\int_{G_4} \mathcal{F}f(\xi_4,\xi_3,\xi_2,\xi_1)\mathcal{F}g(\xi_4,\xi_3,\xi_2,\xi_1) d\xi_4 d\xi_3 d\xi_2 d\xi_1
\]

\[
\int_{\mathbb{R}^4} |\mathcal{F}f(\xi_4,\xi_3,\xi_2,\xi_1)| |\mathcal{F}g(\xi_4,\xi_3,\xi_2,\xi_1)| d\xi_4 d\xi_3 d\xi_2 d\xi_1
\]

\[
(2.16)
\]

3. **Tempered Fundamental solution.**

If we consider the group $G_4$ as a subgroup of $L$, then $\tilde{f} \in S(G_4)$ where $x_1$ is fixed, and if we consider $B$ as a subgroup of $L$, then $\tilde{f} \in S(B)$ where $t$ is fixed. Denote by $S_E(L)$ the space of all functions $\phi(x_4,x_3,x_2,x_1,t) \in C^\infty(L)$ such that $\phi(x_4,x_3,x_2,x_1,t) \in S(G_4)$ where $x_1$ is fixed, and $\phi(x_4,x_3,x_2,x_1,t) \in S(B)$ where $t$ is fixed. We equip $S_E(L)$ with the natural topology defined by the seminorms:

\[
\phi \rightarrow \sup_{(x_4,x_3,x_2,x_1) \in B} |Q(x_4,x_3,x_2,x_1,t)P(D)\phi(x_4,x_3,x_2,x_1,t)| \quad t \text{ is fixed.}
\]

\[
\phi \rightarrow \sup_{(x_4,x_3,x_2,t) \in K} |R(x_4,x_3,x_2,x_1,t)S(D)\phi(x_4,x_3,x_2,x_1,t)| \quad x_1 \text{ is fixed.}
\]

(3.1)

where $P$, $Q$, $R$ and $S$ run over the family of all complex polynomial in four variables. Let $S_E^L(L)$ be the subspace of all functions $F \in S_E(L)$, which are invariant in the sense of Remark 2.2, then we have the following result.

**Proposition 3.1.** Let $u \in \mathcal{U}$ and $Q_u$ be the invariant differential operator on the group $B$ which is associated to $u$, then we have

(i) The mapping $f \mapsto \tilde{f}$ is a topological isomorphism of $S(G_4)$ onto $S_E^L(L)$.

(ii) The mapping $F \mapsto Q_uF$ is a topological isomorphism of $S_E^L(L)$ onto its image, where $Q_u$ acts on the variables $(x_4,x_3,x_2,x_1) \in B$. 

Proof. (i) In fact the map is continuous and the restriction mapping \(F \mapsto RF\) on \(G_4\) is continuous from \(S'_E(L)\) into \(S(G_4)\) that satisfies \(R \circ \sim = Id_{S(G_4)}\) and \(\sim \circ R = Id_{S'_E(L)}\), where \(Id_{S(G_4)}\) (resp. \(Id_{S'_E(L)}\)) is the identity mapping of \(S(G_4)\) (resp. \(S'_E(L)\)) and \(G_4\) is considered as a subgroup of \(L\).

To prove (ii) we refer to [15] and his famous result that is: Any invariant differential operator on \(B\) is a topological isomorphism of \(S(B)\) onto its image. From this result, we obtain that

\[
Q_u : S'_E(L) \rightarrow S_E(L)
\]  

is a topological isomorphism and its restriction on \(S'_E(L)\) is a topological isomorphism of \(S'_E(L)\) onto its image. Hence the theorem is proved. 

In the following we will prove that every invariant differential operator on \(G_4 = \mathbb{R}^3 \times \{0\} \times \mathbb{R}\) has a tempered fundamental solution. As stated in the introduction, we will consider the two invariant differential operators \(P_u\) and \(Q_u\), the first on the group \(G_4 = \mathbb{R}^3 \times \{0\} \times \mathbb{R}\) and the second on the abelian group \(B = \mathbb{R}^3 \times \mathbb{R} \times \{0\}\).

Our main result is:

**Theorem 3.2.** Every nonzero invariant differential operator \(P_u\) on \(G_4\) associated to \(U\) is a topological isomorphism of \(S'_E(L)\) onto its image

Proof. By equation (2.4 we have for every \(u \in U\) and \(F \in S'_E(L)\)

\[
P_u F(x_4, x_3, x_2, x_1, t) = u \ast F(x_4, x_3, x_2, x_1, t) \\
= \int_{G_4} F[(y_4, y_3, y_2, s)^{-1}(x_4, x_3, x_2, x_1, t)] u(z', y', r', s') dy_4 dy_3 dy_2 ds \\
= \int_{G_4} F[p(s^{-1})(-y_4, -y_3, y_2, x_4, x_3, x_2, x_1, t - s)] u(y_4, y_3, y_2, s) dy_4 dy_3 dy_2 ds \\
= \int_{G_4} F[(-y_4, -y_3, y_2, x_4, x_3, x_2, x_1, s)] u(y_4, y_3, y_2, s) dy_4 dy_3 dy_2 ds \\
= u \ast_c (x_4, x_3, x_2, x_1, t) = Q_u F(x_4, x_3, x_2, x_1, t) \quad (3.3)
\]

This shows that:

\[
P_u F(x_4, x_3, x_2, x_1, t) = Q_u F(x_4, x_3, x_2, x_1, t) \quad (3.4)
\]

for all \((x_4, x_3, x_2, x_1, t) \in L\), where \(\ast\) is the convolution product on \(G_4 = \mathbb{R}^3 \times \{0\} \times \mathbb{R}\) and \(\ast_c\) is the convolution product on the group \(B = \mathbb{R}^3 \times \mathbb{R} \times \{0\}\). By Proposition 3.1 the mapping \(F \mapsto Q_u F\) is a topological isomorphism of \(S'_E(L)\) onto its image, then the mapping \(F \mapsto P_u F\) is a topological isomorphism of \(S'_E(L)\) onto its image. Since

\[
R(P_u F)(x_4, x_3, x_2, x_1, t) = P_u(RF)(x_4, x_3, x_2, x_1, t) \quad (3.5)
\]

the following diagram is commutative:
\[ S'_E(L) \xrightarrow{P_u} P_u S'_E(L) \]
\[ \sim \xrightarrow{R} \xrightarrow{R} \]
\[ S(G_4) \xrightarrow{P_u} P_u S(G_4) \]

Hence the mapping \( F \mapsto P_u F \) is a topological isomorphism of \( S(G_4) \) onto its image. \( \square \)

**Corollary 3.3.** Every nonzero invariant differential operator on \( G_4 \) has a tempered fundamental solution.

**Proof.** The transpose \( {}^tP_u \) of \( P_u \) is a continuous mapping of \( S'(G_4) \) onto \( S'(G_4) \). This means that for every tempered distribution \( T \) on \( G_4 \) there is a tempered distribution \( E \) on \( G_4 \) such that
\[ P_u E = T \] (3.6)

Indeed the Dirac measure \( \delta \) belongs to \( S'(G_4) \).

**Definition 3.4.** For every \( f \in D(L) \), one can define a function \( \hat{f} \in C^\infty(L) \) as follows:
\[ \hat{f}(x_4, x_3, x_2, x_1, t) = f(\rho(x_1)(x_4, x_3, x_2), 0, x_1 + t) = (x_4 - tx_3 + \frac{1}{2}t^2x_2, x_3 - tx_2, x_2, 0, x_1 + t) \] (3.7)

for any \((x_4, x_3, x_2, x_1, t) \in L\).

**Remark 3.5.** The function \( \hat{f} \) is invariant in the following sense:
\[ \hat{f}(\rho(s)(x_4, x_3, x_2)), x_1 - s, t + s) = \hat{f}(x_4, x_3, x_2, x_1, t) \] (3.8)

for any \((x_4, x_3, x_2, x_1, t) \in L \) and \( s \in R \).

**Theorem 3.6.** For every \( u \in U \), one can find a distribution \( T \in \mathcal{D}'(G_4) \) such that
\[ u \ast T = \delta_{G_4} \] (3.9)

where \( \delta_{G_4} \) is the Dirac measure on \( G_4 \) at the identity element of \( G_4 \) and \( \ast \) denotes the convolution product on \( G_4 \).

**Proof.** Let \( P_\zeta \) be the polynomial
\[ \zeta \mapsto \mathcal{F}(\check{u})(\xi + \zeta) \]

which is obtained by translation, where \( \xi = (\xi_4, \xi_3, \xi_2, \xi_1) \in \mathbb{R}^4 \), and \( \zeta = (\zeta_4, \zeta_3, \zeta_2, \zeta_1) \in \mathbb{C}^4 \). Let \( T \) be a distribution on \( L \) defined by
\[ \langle T, f \rangle = \int_L \int_\Omega \frac{\mathcal{F}f(\xi + \zeta, \lambda)}{\mathcal{F}(\check{u})(\xi + \zeta)} \Phi(P_\zeta, \xi) d\zeta d\xi d\lambda \]

for any \( f \in \mathcal{D}(L) \), where \( \Omega \) is a ball in \( \mathbb{C}^4 \) with center 0, \( \Phi \) is the Hormander function \([12]\) and \( d\zeta = d\zeta_1 d\zeta_2 d\zeta_3 d\zeta_4 \) is the Lebesgue measure on \( \mathbb{C}^4 \). Now we can
define a distribution $\hat{T}$ on $L$, which is invariant in the sense of equation (3.8), as follows:

$$\langle \hat{T}, f \rangle = \langle T, \hat{f} \rangle = \int L \int \Omega \frac{F(\hat{f}(\xi + \zeta, \lambda))}{F(\hat{u})(\xi + \zeta)} \Phi(P_{\zeta}, \xi) d\zeta d\lambda d\mu$$

(3.10)

By Hormander construction, and Lemma 2.3, we obtain for any $u \in U$

$$\langle u \ast \hat{T}, f \rangle = \langle u \ast T, \hat{f} \rangle = \int L \int \Omega F(\hat{\nabla}^u f)(\xi + \zeta, \lambda) F(\hat{u})(\xi + \zeta) \Phi(P_{\zeta}, \xi) d\zeta d\lambda d\mu = \int G \int \Omega F(\hat{f}(\xi + \zeta, 0)) F(\hat{u})(\xi + \zeta) \Phi(P_{\zeta}, \xi) d\zeta d\lambda d\mu \Phi(P_{\zeta}, \xi) d\zeta d\lambda d\mu$$

(3.11)

where $\delta_L$ is the Dirac measure at the identity element of $L$. This gives

$$u \ast \hat{T}(x_4, x_3, x_2, x_1, t) = \delta_L(x_4, x_3, x_2, x_1, t).$$

Consequently, we have

$$u \ast T(x_4, x_3, x_2, 0, t) = \delta_{G_4}(x_4, x_3, x_2, 0, t)$$

(3.12)

Hence the theorem. □

**Theorem 3.7.** ([1]) Every invariant differential operator on $G_4$ which is not identically 0 has a tempered fundamental solution.

**Proof.** For each complex number $s$ with positive real part, we can define a distribution $T^s$ on $L$ by:

$$\langle T^s, f \rangle = \int \mathbb{R}^5 \left| \mathcal{F}(\hat{u})(\xi, \lambda) \right|^s \mathcal{F}(\hat{f})(\xi, \lambda) d\xi d\lambda d\mu$$

(3.13)

for each $f \in \mathcal{S}(L)$. By Atiyah’s theorems ([1]), the function $s \mapsto T^s$ has a meromorphic continuation in the whole complex plane, which is analytic at $s = 0$ and its value at this point is the Dirac measure on the group $L$. Now we can define another distribution, $\hat{T}^s$, as follows:

$$\langle \hat{T}^s, f \rangle = \int \mathbb{R}^6 \left| \mathcal{F}(\hat{\nabla}^u)(\xi, \lambda) \right|^s \mathcal{F}(\hat{f})(\xi, \lambda) d\xi d\lambda d\mu$$

(3.14)

for any $f \in \mathcal{S}(L)$ and $s \in \mathbb{C}$, with $\text{Re } (s) \geq 0$. 


Note that the distribution \( \hat{T}^s \) is invariant in the sense of equation (3.8), and we have

\[
\begin{align*}
\left< u * \hat{u} * c T^s, f \right> & = \left< u * \tilde{u} * c T^s, \hat{f} \right> \\
& = \left< T^s, \hat{\tilde{u}} * c \hat{u} * \hat{f} \right> \\
& = \int_{\mathbb{R}^6} \left[ \left| \mathcal{F}(\hat{u})(\xi, \lambda) \right|^2 \right]^s \mathcal{F}(\hat{\tilde{u}} * c \hat{u} * \hat{f})(\xi, \lambda) d\xi d\lambda \quad (3.15)
\end{align*}
\]

where

\[
\begin{align*}
\hat{\tilde{u}}(x_4, x_3, x_2, x_1) & = u(-x_4, -x_3, -x_2, -x_1) \\
\tilde{u}(x_4, x_3, x_2, x_1) & = \frac{1}{(x_4, x_3, x_2, x_1)^{-1}}
\end{align*}
\]

and

\[
\tilde{u} * c f(x_4, x_3, x_2, x_1) = \int_{\mathbb{R}^4} f(x_4-a, x_3-b, x_2-c, x_1-r) \tilde{u}((a, b, c, r)) da db dc dr \quad (3.16)
\]

is the commutative convolution product on \( G_4 \). We get:

\[
\begin{align*}
\left< u * \hat{u} * c T^s, f \right> & = \int_{\mathbb{R}^6} \left[ \left| \mathcal{F}(\hat{u})(\xi, \lambda) \right|^2 \right]^s \mathcal{F}(\hat{\tilde{u}} * c \hat{u} * \hat{f})(\xi, \lambda) d\xi d\lambda \\
& \text{hence}
\end{align*}
\]

\[
u * \hat{u} * c T^s = T^{s+1} \quad (3.17)
\]

In view of the invariance in equation (3.8), the restriction of the distributions \( u * \hat{u} * c T^s = T^{s+1} \) on the sub-group \( \mathbb{R}^2 \times \{0\} \times \mathbb{R} \times \{1\} \times \mathbb{R}_+ \simeq G_4 \) are nothing but the distributions

\[
u * \hat{u} * c T^s = T^{s+1}. \quad (3.18)
\]

The distribution \( T^s \) can be expanded around \( s = -1 \) in the form

\[
T^s = \sum_{j=-5}^{\infty} \alpha_j (s+1)^j \quad (3.19)
\]

where each \( \alpha_j \) is a distribution on \( G_4 \). But \( u * \hat{u} * c T^s = T^{s+1} \) can not have a pole at \( s = -1 \) (since \( T^0 = \delta_{G_4} \)) and so we must have:

\[
\begin{align*}
u * \hat{u} * c \alpha_j & = 0 \quad \text{for} \quad j < 0 \\
u * \hat{u} * c \alpha_0 & = \delta_{G_4} \quad (3.20)
\end{align*}
\]

Hence the theorem. \( \square \)
4. IDEALS OF GROUP ALGEBRA $L^1(G_4)$

In the following we will establish a classification of all left ideals in the Banach algebra $L^1(G_5)$.

Proposition 4.1. (i) The mapping $\Gamma$ from $\widehat{L^1(G_4)}|_B$ to $\widehat{L^1(G_4)}|_{G_4}$ defined by

$$\tilde{F}|_B (x_4, x_3, x_2, x_1, 0) \to \Gamma(\tilde{F}|_B)(x_4, x_3, x_2, 0, x_1)$$

is a topological isomorphism.

(ii) For every $u \in L^1(G_4)$ and $F \in L^1(G_4)$, we obtain

$$\Gamma(u *_c \tilde{F}|_B)(x_4, x_3, x_2, 0, x_1) = u * \tilde{F}|_{G_4}(x_4, x_3, x_2, 0, x_1) \quad (4.2)$$

where

$$(u *_c \tilde{F}|_B)(x_4, x_3, x_2, x_1, 0)$$

$$= \int_B \tilde{F} [x_4 - y_4, x_3 - y_3, x_2 - y_2, x_1 - y_1, 0] u(y_5, y_4, y_3, y_2, y_1)$$

$$dy_5 dy_4 dy_3 dy_2 dy_1 \quad (4.3)$$

Proof. It is enough to see

$$\Gamma(u *_c \tilde{F}|_B)(x_4, x_3, x_2, 0, x_1)$$

$$= \int_B \tilde{F} [x_4 - y_4, x_3 - y_3, x_2 - y_2, -y_1, x_1] u(y_5, y_4, y_3, y_2, y_1)$$

$$dy_5 dy_4 dy_3 dy_2 dy_1$$

$$= \int_{G_4} F [(\rho_1(-y_1)(x_4 - y_4, x_3 - y_3, x_2 - y_2)), x_1 - y_1)](y_5, y_4, y_3, y_2, y_1)$$

$$dy_5 dy_4 dy_3 dy_2 dy_1$$

$$= u * \tilde{F}|_{G_4}(x_5, x_4, x_3, 0, x_2, 0, x_1) \quad (4.4)$$

for every $F \in L^1(G_4)$. So it is easy to show that

$$\Gamma : \widehat{L^1(G_4)}|_B \to \widehat{L^1(G_4)}|_{G_4} \quad (4.5)$$

is an topological isomorphism, and we obtain

$$\tilde{F}|_{G_4}(x_4, x_3, x_2, x_1, 0) \to \Gamma^{-1}(\tilde{F}|_{G_4})(x_4, x_3, x_2, 0, x_1)$$

$$= \tilde{F}|_B(x_4, x_3, x_2, x_1, 0) \quad (4.6)$$

$\square$

Now if $I$ is a subset of $L^1(G_4)$, we denote by $\tilde{I}$ its image by the mapping $\sim$. Let $J = \tilde{I}|_B$. Our main result is:
Theorem 4.2. Let $I$ be a subset of $L^1(G_4)$, then the following conditions are equivalent.

(i) $J = \tilde{I} |_B$ is an ideal in the Banach algebra $L^1(B)$.

(ii) $I$ is a left ideal in the Banach algebra $L^1(G_4)$.

Proof. (i) implies (ii) Let $I$ be a subset of the algebra $L^1(B)$ such that $J = \tilde{I}|_B$ is an ideal in $L^1(B)$, then we have:

$$u *_c \tilde{I} |_B(x_4, x_3, x_2, x_1, 0) \subseteq \tilde{I} |_B(x_4, x_3, x_2, x_1, 0)$$  \hspace{1cm} (4.7)

for any $u \in L^1(B)$ and $(x_4, x_3, x_2, x_1) \in B$, where

$$u *_c \tilde{I} |_B(x_4, x_3, x_2, 0, x_1)$$

$$= \left\{ \int_B \tilde{f} |_B [x_4 - y_4, x_3 - y_3, x_2 - y_2, 0, x_1 - y_1]u(y_5, y_4, y_3, y_2, y_1) \right\} dy_5dy_4dy_3dy_2dy_1, \ f \in I$$  \hspace{1cm} (4.8)

It shows that

$$u *_c \tilde{f} |_B(x_4, x_3, x_2, 0, x_1) \in \tilde{I} |_B(x_4, x_3, x_2, 0, x_1)$$  \hspace{1cm} (4.9)

for any $\tilde{f} \in \tilde{I}$. Then we get

$$\Gamma(u *_c \tilde{f}|_B)(x_5, x_4, x_3, x_2, 0, x_1)$$

$$= u * \tilde{f}|_{G_4}(x_4, x_3, x_2, 0, x_1) \in \Gamma(\tilde{I} |_B)(x_4, x_3, x_2, 0, x_1)$$

$$= \tilde{I} |_{G_4}(x_4, x_3, x_2, 0, x_1) = I(x_4, x_3, x_2, x_1)$$  \hspace{1cm} (4.10)

(ii) implies (i) If $I$ is an ideal in $L^1(G_4)$, then we get

$$u * \tilde{I} |_{G_4}(x_4, x_3, x_2, 0, x_1)$$

$$= u * I (x_4, x_3, x_2, x_1) \subseteq \tilde{I} |_{G_4}(x_4, x_3, x_2, 0, x_1)$$

$$= I(x_4, x_3, x_2, x_1)$$  \hspace{1cm} (4.11)

where

$$u * \tilde{I} |_{G_4}(x_4, x_3, x_2, 0, x_1)$$

$$= \left\{ \int_{G_4} F([\rho_1(-y_1)(x_4 - y_4, x_3 - y_3, x_2 - y_2)], x_1 - y_1) \right\}$$

$$= u * \tilde{F}|_{G_5}(x_4, x_3, x_2, 0, x_1)$$

$$= u * F(x_4, x_3, x_2, x_1)$$  \hspace{1cm} (4.12)

This leads us

$$\Gamma^{-1}(u * \tilde{F} |_{G_4})(x_4, x_3, x_2, x_1, 0)$$

$$= u *_c \tilde{F}|_B(x_4, x_3, x_2, x_1, 0) \in \Gamma^{-1}(u * \tilde{I} |_{G_4})(x_4, x_3, x_2, 0, x_1)$$

$$= u * \tilde{I} |_B(x_4, x_3, x_2, x_1, 0)$$  \hspace{1cm} (4.13)

$\square$
Corollary 4.3. Let $I$ be a subset of the Banach algebra $L^1(G_4)$ and $\tilde{I}$ its image by the mapping $\sim$ such that $J = \tilde{I}|_B$ is an ideal in $L^1(B)$, then the following conditions are verified:

(i) $J$ is a maximal ideal in the algebra $L^1(B)$ if and only if $I$ is a left ideal closed in the algebra $L^1(G_4)$.

(ii) $J$ is a closed ideal in the algebra $L^1(B)$ if and only if $I$ is a left prime ideal in the algebra $L^1(G_4)$.

(iii) $J$ is a maximal ideal in the algebra $L^1(B)$ if and only if $I$ is a left maximal ideal in the algebra $L^1(G_4)$.

(iv) $J$ is a dense ideal in the algebra $L^1(B)$ if and only if $I$ is a left dense ideal in the algebra $L^1(G_4)$.

5. Conclusion

The noncommutative Fourier transform does exist, and is being used, in the representation theory of non-abelian compact groups. But still, from the Fourier analysis point of view, it is not really satisfactory. In this paper we show how the classical Fourier transform on $\mathbb{R}^n$ can be defined on the nilpotent Engel-Lie group $G_4$, which is noncommutative and non-compact, to obtain the Plancherel formula and two interesting results.

The first one is the solvability of the Lewy and Mizohata operators, see [3] and [10].

The second is the solvability of the following heat equation on the Lie group $G_4 \times \mathbb{R}$ direct product of $G_4$ and the real vector group $\mathbb{R}$

$$
\left( \frac{\partial}{\partial x_1} \right)^2 + \left( \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3} + \frac{x_1^2}{2\partial x_4} \right)^2 \frac{\partial}{\partial t} = 0
$$

(5.1)

U. Boscain, in [2], wrote that there is no general solution. Since equation (5.1) is among the elements of the enveloping algebra $\mathcal{U}$ of the group $G_4 \times \mathbb{R}$, so we obtain the existence of the fundamental solutions of this equation.

References


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