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GENERALIZED FUSION FRAME IN QUATERNIONIC HILBERT SPACES

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ABSTRACT. The notion of a generalized fusion frame in quaternionic Hilbert space is introduced. A characterization of generalized fusion frame in quaternionic Hilbert space with the help of frame operator is being discussed. Finally, g-fusion frame in quaternionic Hilbert space using invertible bounded right $\mathfrak{Q}$-linear operator on quaternionic Hilbert space is constructed.

1. Introduction and preliminaries

Frames for Hilbert spaces were first introduced by Duffin and Schaeffer [4] in 1952. Thereafter, Daubechies et al. [3] popularized frame theory in Hilbert space. In recent times, several generalizations of frames such as, generalized frame [11], fusion frame [2], generalized fusion frame [10] etc. have been studied. Frame theory has so many applications in applied mathematics and physics as well as signal and image processing, filter bank theory, coding and communications, system modeling and so on.

Generalized fusion frame is a combinations of fusion frame and g-frame in Hilbert space which has been introduced and studied by Sadri et al. [10]. Let $\{W_j\}_{j \in J}$ be a family of closed subspaces of a Hilbert space H and $\{v_j\}_{j \in J}$ be a family of positive weights and for each $j \in J$, $\Lambda_j : H \to H_j$ be a bounded linear operator, where J is subset of integers $\mathbb{Z}$. Then the triplet family $\Lambda = \{(W_j, \Lambda_j, v_j)\}_{j \in J}$ is said to be a generalized fusion frame or a g-fusion frame for H with respect to $\{H_j\}_{j \in J}$ if there exist positive constants A, B such that

$$A \|f\|^2 \leq \sum_{j \in J} v_j^2 \left\| \Lambda_j P_{W_j}(f) \right\|^2 \leq B \|f\|^2, \tag{1.1}$$

for all $f \in H$, where for each $j \in J$, P_{W_j} is a orthogonal projection onto the closed subspace W_j and $\{H_j\}_{j \in J}$ is the family of Hilbert spaces. The constants A and B are called g-fusion frame bounds. If $A = B$ then Λ is called a tight g-fusion frame and if $A = B = 1$ then it is called a Parseval g-fusion frame. If Λ satisfies only the right inequality of (1.1) then it is called a g-fusion Bessel sequence in H with bound B.

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Let $\Lambda = \{ (W_j, \Lambda_j, v_j) \}_{j \in J}$ be a g-fusion Bessel sequence in H with a bound B. Then the operator T_Λ defined by

$$T_\Lambda : l^2 \left(\{ H_j \}_{j \in J} \right) \to H,$$

$$T_\Lambda \left(\{ f_j \}_{j \in J} \right) = \sum_{j \in J} v_j P_{W_j} \Lambda_j^* f_j , \text{ for all } \{ f_j \}_{j \in J} \in l^2 \left(\{ H_j \}_{j \in J} \right)$$

is called the synthesis operator and the operator given by

$$T_\Lambda^* : H \to l^2 \left(\{ H_j \}_{j \in J} \right), \; T_\Lambda^* (f) = \left\{ v_j \Lambda_j P_{W_j} (f) \right\}_{j \in J},$$

for all $f \in H$ is called the analysis operator. The operator $S_\Lambda : H \to H$ defined by

$$S_\Lambda (f) = T_\Lambda T_\Lambda^* (f) = \sum_{j \in J} v_j^2 P_{W_j} \Lambda_j^* \Lambda_j P_{W_j} (f),$$

for all $f \in H$ is called g-fusion frame operator. It is easy to verify that, for each $f \in H$

$$\langle S_\Lambda (f), f \rangle = \sum_{j \in J} v_j^2 \left\| \Lambda_j P_{W_j} (f) \right\|^2 .$$

Moreover, if Λ is a g-fusion frame with bounds A and B then from (1.1), it follows that

$$\langle A f, f \rangle \leq \langle S_\Lambda (f), f \rangle \leq \langle B f , f \rangle$$

for all $f \in H$. The operator S_Λ is bounded linear which is also self-adjoint, positive and invertible.

Khokulan et al. [6] studied frames for finite dimensional quaternionic Hilbert spaces in recent times. Sharma and Goel [8] introduced frames in a quaternionic Hilbert spaces. Various generalization of frame in quaternionic Hilbert space were introduced by S. K. Sharma et al. [9].

In this paper, the idea of a g-fusion frame in quaternionic Hilbert space is given and a characterization of generalized fusion frame in quaternionic Hilbert space using its frame operator is established. At the end, g-fusion frames in quaternionic Hilbert spaces using invertible bounded right $\mathfrak{Q}$-linear operator on quaternionic Hilbert space are being discussed.

2. Quaternionic Hilbert space

This section is started by giving some basic facts about the algebra of quaternions, right quaternionic Hilbert space and operators on right quaternionic Hilbert spaces. The non-commutative field of quaternions $\mathfrak{Q}$ is a four dimensional real algebra with unity. In $\mathfrak{Q}$, 0 denotes the null element and 1 denotes the identity with respect to multiplication. It also includes three so-called imaginary units, denoted by i, j, k. Thus,

$$\mathfrak{Q} = \{ a_0 + a_1 i + a_2 j + a_3 k : a_0, a_1, a_2, a_3 \in \mathbb{R} \}$$

where $i^2 = j^2 = k^2 = -1$; $i j = -j i = k$; $j k = -k j = i$ and $k i = -i k = j$. For each quaternion $q = a_0 + a_1 i + a_2 j + a_3 k \in \mathfrak{Q}$, the conjugate of q is denoted by $\overline{q}$ and defined by $\overline{q} = a_0 - a_1 i - a_2 j - a_3 k \in \mathfrak{Q}$. Here the real part of q is a_0 and the imaginary part of q is $a_1 i + a_2 j + a_3 k$. The modulus of quaternion q is defined as $|q| = \sqrt{a_0^2 + a_1^2 + a_2^2 + a_3^2}$. For every non-zero quaternion $q = a_0 + a_1 i + a_2 j + a_3 k \in \mathfrak{Q}$, there exists a unique inverse q^{-1} in $\mathfrak{Q}$ as

$$q^{-1} = \frac{\bar{q}}{|q|^2} = \frac{a_0 - a_1 i - a_2 j - a_3 k}{\sqrt{a_0^2 + a_1^2 + a_2^2 + a_3^2}}.$$

Definition 2.1. [5] A linear vector space $\mathbb{H}^R(\mathfrak{Q})$ over the filed of quaternionic $\mathfrak{Q}$ under right scalar multiplication is called a right quaternionic vector space i. e.,

$$\mathbb{H}^R(\mathfrak{Q}) \times \mathfrak{Q} \to \mathbb{H}^R(\mathfrak{Q}), \text{ this implies } (u, q) \to u q$$

and for each $u, v \in \mathbb{H}^R(\mathfrak{Q})$ and $p, q \in \mathfrak{Q}$, the right scalar multiplication satisfying the following properties:

$$(u + v)q = uq + vq,\; u(p + q) = up + uq,\; v(pq) = (vp)q.$$

Definition 2.2. [5] Let $\mathbb{H}^R(\mathfrak{Q})$ be a right quaternionic vector space. Then a binary mapping $\langle \cdot, \cdot \rangle : \mathbb{H}^R(\mathfrak{Q}) \times \mathbb{H}^R(\mathfrak{Q}) \to \mathfrak{Q}$ which satisfies the following properties:

(i) $\langle v, v \rangle > 0$ if $v \neq 0$
(ii) $\overline{\langle u, v \rangle} = \langle v, u \rangle$ for all $u, v \in \mathbb{H}^R(\mathfrak{Q})$.
(iii) $\langle u, v_1 + v_2 \rangle = \langle u, v_1 \rangle + \langle u, v_2 \rangle$ for all $u, v_1, v_2 \in \mathbb{H}^R(\mathfrak{Q})$.
(iv) $\langle u, vq \rangle = \langle u, v \rangle q$ for all $u, v \in \mathbb{H}^R(\mathfrak{Q})$ and $q \in \mathfrak{Q}$,

is called right quaternionic inner product and $\left(\mathbb{H}^R(\mathfrak{Q}), \langle \cdot, \cdot \rangle\right)$ is called a right quaternionic inner product space.

Let $\mathbb{H}^R(\mathfrak{Q})$ be a right quaternionic inner product space with respect to the right quaternionic inner product $\langle \cdot, \cdot \rangle$. Define the quaternionic norm $\|\cdot\| : \mathbb{H}^R(\mathfrak{Q}) \to \mathbb{R}^+$ on $\mathbb{H}^R(\mathfrak{Q})$ by

$$\|u\| = \sqrt{\langle u, u \rangle},\; u \in \mathbb{H}^R(\mathfrak{Q}). \tag{2.1}$$

A complete right quaternionic inner product space $\mathbb{H}^R(\mathfrak{Q})$ is called a right quaternionic Hilbert space.

Theorem 2.3. *(Cauchy-Schwarz inequality)* [5] *Let $\mathbb{H}^R(\mathfrak{Q})$ be a right quaternionic Hilbert space. Then*

$$|\langle u, v \rangle|^2 \leq \langle u, u \rangle \langle v, v \rangle,\; \textit{for all } u, v \in \mathbb{H}^R(\mathfrak{Q}).$$

The quaternionic norm defined in (2.1) satsifies the following properties:

(i) $\|u\| = 0$ for some $u \in \mathbb{H}^R(\mathfrak{Q})$, then $u = 0$.
(ii) $\|uq\| = |q|\,\|u\|$ for all $u \in \mathbb{H}^R(\mathfrak{Q})$ and $q \in \mathfrak{Q}$.
(iii) $\|u + v\| \leq \|u\| + \|v\|$ for all $u, v \in \mathbb{H}^R(\mathfrak{Q})$.

Definition 2.4. [5] Let $\mathbb{H}^R(\mathfrak{Q})$ be a right quaternionic Hilbert space and V be a subset of $\mathbb{H}^R(\mathfrak{Q})$. Define

(i) $V^{\perp} = \left\{ v \in \mathbb{H}^R(\mathfrak{Q}) : \langle v, u \rangle = 0 \;\forall\, u \in V \right\}$.

(ii) $\langle V \rangle$ be the right $\mathfrak{Q}$-linear subspace of $\mathbb{H}^R(\mathfrak{Q})$ consisting of all finite right $\mathfrak{Q}$-linear combinations of elements of V.

Definition 2.5. [5] A subset N of a quaternionic Hilbert space $\mathbb{H}^R(\mathfrak{Q})$ is called Hilbert basis or orthonormal basis of $\mathbb{H}^R(\mathfrak{Q})$, if $u, v \in N$, $\langle u, v \rangle = 0$ for $u \neq v$ and $\langle u, u \rangle = 1$.

Theorem 2.6. [5] *Let M be a Hilbert basis of a right quaternionic Hilbert space $\mathbb{H}^R(\mathfrak{Q})$. Then the following conditions are equivalent:*

(i) *The series* $\sum_{z \in N} \langle u, z \rangle \langle z, v \rangle$ *converges absolutely and* $\langle u, v \rangle = \sum_{z \in N} \langle u, z \rangle \langle z, v \rangle$, *for every* $u, v \in \mathbb{H}^R(\mathfrak{Q})$.
(ii) $\|u\|^2 = \sum_{z \in N} |\langle z, u \rangle|^2$, *for every* $u, v \in \mathbb{H}^R(\mathfrak{Q})$.
(iii) $V^{\perp} = 0$.
(iv) $\langle M \rangle$ *is dense in* $\mathbb{H}^R(\mathfrak{Q})$.

Definition 2.7. [1] Let $\mathbb{H}^R(\mathfrak{Q})$ be a right quaternionic Hilbert space and T be an operator on $\mathbb{H}^R(\mathfrak{Q})$. Then T is said to be right $\mathfrak{Q}$-linear if

$$T(u\alpha + v\beta) = \alpha T(u) + \beta T(v), \quad \text{for all } u, v \in \mathbb{H}^R(\mathfrak{Q}) \text{ and } \alpha, \beta \in \mathfrak{Q}.$$

If there exist $K > 0$ such that $\|T(v)\| \leq K\|v\|$, for all $v \in \mathbb{H}^R(\mathfrak{Q})$, then T is called bounded. The adjoint operator T^* of T is defined as $\langle v, Tu \rangle = \langle T^* v, u \rangle$, for all $u, v \in \mathbb{H}^R(\mathfrak{Q})$ and if $T = T^*$ then T is said to be a self-adjoint operator.

Theorem 2.8. [1] *Let S, T be two bounded right linear operators on a right quaternionic Hilbert space $\mathbb{H}^R(\mathfrak{Q})$. Then*

(i) *the operators $T + S$ and TS are bounded right linear on $\mathbb{H}^R(\mathfrak{Q})$. Moreover* $\|T + S\| \leq \|T\| + \|S\|$ *and* $\|TS\| \leq \|T\|\|S\|$.
(ii) $(T + S)^* = T^* + S^*$, $(TS)^* = S^* T^*$ *and* $(T^*)^* = T$.
(iii) $I_H^* = I_H$, *where I_H is an identity operator on $\mathbb{H}^R(\mathfrak{Q})$.*
(iv) *If the operator T is an invertible then* $(T^{-1})^* = (T^*)^{-1}$.

Theorem 2.9. [5] *Let T be an bounded right $\mathfrak{Q}$-linear operator on right quaternionic Hilbert space $\mathbb{H}^R(\mathfrak{Q})$. If $T \geq 0$, then exists a unique operator in $\mathcal{B}\left(\mathbb{H}^R(\mathfrak{Q})\right)$, say $\sqrt{T}$, such that $\sqrt{T} \geq 0$ and $\sqrt{T}\sqrt{T} = T$. Moreover, $\sqrt{T}$ commutes with every operator which commutes with T and if T is invertible and self-adjoint, then $\sqrt{T}$ is also invertible and self-adjoint.*

Throughout this paper, $\mathfrak{Q}$ is considered to be a non-commutative field of quaternions, J is subset of integers $\mathbb{Z}$ and $\mathbb{H}^R(\mathfrak{Q})$ is a separable right quaternionic Hilbert space. By the term "right linear operator" we mean a "right $\mathfrak{Q}$-linear operator" and $\mathcal{B}\left(\mathbb{H}^R(\mathfrak{Q})\right)$ denotes the set of all bounded (right $\mathfrak{Q}$-linear) operators on $\mathbb{H}^R(\mathfrak{Q})$.

3. Various generalizations of frame in quaternionic Hilbert space

Definition 3.1. [8] Let $\mathbb{H}^R(\mathfrak{Q})$ be a right quaternionic Hilbert space and $\{f_j\}_{j \in J}$ be a sequence in $\mathbb{H}^R(\mathfrak{Q})$. Then $\{f_j\}_{j \in J}$ is a frame for $\mathbb{H}^R(\mathfrak{Q})$ if there exist constants $A,\, B \,>\, 0$ such that

$$A \,\|\, f \,\|^{\,2} \,\leq\, \sum_{j \,\in\, J} |\, \langle\, f,\, f_{\,j} \,\rangle\, |^{\,2} \,\leq\, B \,\|\, f \,\|^{\,2},$$

for all $f \,\in\, \mathbb{H}^{\,R}\,(\,\mathfrak{Q}\,)$. The constants A and B are called frame bounds.

Example 3.2. Let N be a Hilbert basis for right separable quaternionic Hilbert space $\mathbb{H}^{\,R}\,(\,\mathfrak{Q}\,)$ such that for $z_i,\, z_k \,\in\, N,\; i,\, k \,\in\, J$, we have $\langle\, z_i,\, z_k \,\rangle \,=\, 0$ if $i \,\neq\, k$ and $\langle\, z_i,\, z_i \,\rangle \,=\, 1$. Let $\{\, f_j \,\}_{j \,\in\, J}$ be a sequence in $\mathbb{H}^{\,R}\,(\,\mathfrak{Q}\,)$ such that $u_{\,j} \,=\, u_{\,j\,+\,1} \,=\, z_{\,j}\,,\;\; j \,\in\, J$. Then $\{\, f_j \,\}_{j \,\in\, J}$ is a tight frame for $\mathbb{H}^{\,R}\,(\,\mathfrak{Q}\,)$ with bound 2.

Definition 3.3. [9] Let $\left\{\, W_j^{\,R} \,\right\}_{j \,\in\, J}$ be a family of closed subspaces of a right separable quaternionic Hilbert space $\mathbb{H}^{\,R}\,(\,\mathfrak{Q}\,)$ and $\{\, v_j \,\}_{j \,\in\, J}$ be a family of positive weights. A family of weighted closed subspaces $\left\{\, (\, W_j^{\,R},\, v_j \,) \,:\, j \,\in\, J \,\right\}$ is called a fusion frame for $\mathbb{H}^{\,R}\,(\,\mathfrak{Q}\,)$ if there exist positive constants $A,\, B$ such that

$$A \;\|\, f \,\|^{\,2} \,\leq\, \sum_{j \,\in\, J} v_j^{\,2} \,\left\|\, P_{W_j^{\,R}}\,(\, f \,) \,\right\|^{\,2} \,\leq\, B \;\|\, f \,\|^{\,2}\,,$$

for all $f \,\in\, \mathbb{H}^{\,R}\,(\,\mathfrak{Q}\,)$. The constants $A,\, B$ are called fusion frame bounds. It is called a tight fusion frame if $A \,=\, B$ and a Parseval fusion frame if $A \,=\, B \,=\, 1$.

Definition 3.4. [9] Let $\mathbb{H}^{\,R}\,(\,\mathfrak{Q}\,)$ be a right quaternionic Hilbert space and $\left\{\, \mathbb{H}_j^{\,R}\,(\,\mathfrak{Q}\,) \,\right\}_{j \,\in\, J}$ be a collection of right quaternionic Hilbert spaces. Then the sequence of bounded right linear operator $\left\{\, \Lambda_j \,\in\, \mathcal{B}\,(\,\mathbb{H}^{\,R}\,(\,\mathfrak{Q}\,),\, \mathbb{H}_j^{\,R}\,(\,\mathfrak{Q}\,)\,) \,:\, j \,\in\, J \,\right\}$ is called frame of operator for $\mathbb{H}^{\,R}\,(\,\mathfrak{Q}\,)$ with respect to $\left\{\, \mathbb{H}_j^{\,R}\,(\,\mathfrak{Q}\,) \,\right\}_{j \,\in\, J}$ if there are two positive constants A and B such that

$$A \;\|\, f \,\|^{\,2} \,\leq\, \sum_{j \,\in\, J} \|\, \Lambda_j\, f \,\|^{\,2} \,\leq\, B \;\|\, f \,\|^{\,2}\,,$$

for all $f \,\in\, \mathbb{H}^{\,R}\,(\,\mathfrak{Q}\,)$. The constants A and B are called frame bounds.

Example 3.5. Let $N \,=\, \{\, u_{\,j} \,\}_{j \in \mathbb{N}}$ be an orthonormal basis for a right quaternionic Hilbert space $\mathbb{H}^{\,R}\,(\,\mathfrak{Q}\,)$. Let us define $\Lambda_j \,:\, \mathbb{H}^{\,R}\,(\,\mathfrak{Q}\,) \,\to\, \mathfrak{Q}$ by $\Lambda_j\,(\, f \,) \,=\, \langle\, z_j,\, f \,\rangle$, for all $f \,\in\, \mathbb{H}^{\,R}\,(\,\mathfrak{Q}\,),\; j \,\in\, \mathbb{N}$. Then $\left\{\, \Lambda_j \,:\, \mathbb{H}^{\,R}\,(\,\mathfrak{Q}\,) \,\to\, \mathfrak{Q} \,\right\}_{j \in \mathbb{N}}$ is frame of operator for $\mathbb{H}^{\,R}\,(\,\mathfrak{Q}\,)$ with respect to $\mathfrak{Q}$.

4. g-fusion frame in Quaternionic Hilbert space

In this section, the concept of generalized fusion frame or g-fusion frame in a right quaternionic Hilbert space is presented and some properties are discussed.

Definition 4.1. Let $W \,=\, \left\{\, W_j^{\,R} \,\right\}_{j \,\in\, J}$ be a family of closed subspaces of right quaternionic Hilbert space $\mathbb{H}^{\,R}\,(\,\mathfrak{Q}\,)$ and $\{\, v_j \,\}_{j \,\in\, J}$ be a family of positive weights and $\left\{\, \Lambda_j \,:\, \mathbb{H}^{\,R}\,(\,\mathfrak{Q}\,) \,\to\, \mathbb{H}_j^{\,R}\,(\,\mathfrak{Q}\,) \,\right\}$ be a family of bounded right linear operators. Then the family $\Lambda \,=\, \left\{\, \left(\, W_j^{\,R},\, \Lambda_j,\, v_j \,\right) \,\right\}_{j \,\in\, J}$ is called a generalized fusion frame or a g-fusion frame for $\mathbb{H}^R(\mathfrak{Q})$ with respect to $\left\{\mathbb{H}_j^R(\mathfrak{Q})\right\}_{j \in J}$ if there exist positive constants A and B such that

$$A \|f\|^2 \leq \sum_{j \in J} v_j^2 \left\| \Lambda_j P_{W_j^R}(f) \right\|^2 \leq B \|f\|^2, \tag{4.1}$$

for all $f \in \mathbb{H}^R(\mathfrak{Q})$, where each $P_{W_j^R}$ is an orthogonal projection onto the closed subspace W_j^R and $\left\{\mathbb{H}_j^R(\mathfrak{Q})\right\}_{j \in J}$ is the family of right quaternionic Hilbert spaces. The constants A and B are called the g-fusion frame bounds. Λ is a called tight g-fusion frame for $\mathbb{H}^R(\mathfrak{Q})$ if $A = B$ and a Parseval g-fusion frame for $\mathbb{H}^R(\mathfrak{Q})$ if $A = B = 1$. If Λ satisfies only the right inequality of (4.1) then it is called a g-fusion Bessel sequence in $\mathbb{H}^R(\mathfrak{Q})$ with bound B.

Example 4.2. Let $\{z_j\}_{j \in \mathbb{N}}$ be an orthonormal basis for a right separable quaternionic Hilbert space $\mathbb{H}^R(\mathfrak{Q})$. Let us define

$$W_1^R = \mathbb{H}_1^R(\mathfrak{Q}) = \operatorname{span}\{z_1\}, \; W_j^R = \mathbb{H}_j^R(\mathfrak{Q}) = \operatorname{span}\{z_{j-1}\}, \; j \geq 2$$

and $v_j = 1$, $j \in \mathbb{N}$. Now, for each $j \in \mathbb{N}$, define $\Lambda_j : \mathbb{H}^R(\mathfrak{Q}) \to \mathbb{H}_j^R(\mathfrak{Q})$ by

$$\Lambda_j f = \langle z_j, f \rangle z_j, \text{ for all } f \in \mathbb{H}^R(\mathfrak{Q}).$$

Then it is easy to verify that

$$\|f\|^2 \leq \sum_{j \in \mathbb{N}} \left\| \Lambda_j P_{W_j^R}(f) \right\|^2 \leq 2 \|f\|^2,$$

for all $f \in \mathbb{H}^R(\mathfrak{Q})$. Thus, $\Lambda = \left\{\left(W_j^R, \Lambda_j, 1\right)\right\}_{j \in \mathbb{N}}$ is a g-fusion frame for $\mathbb{H}^R(\mathfrak{Q})$ with bounds 1 and 2.

Define the space

$$\mathcal{H}_2 = \bigoplus_{j \in J} \mathbb{H}_j^R(\mathfrak{Q}) = \left\{ \{f_j\}_{j \in J} : f_j \in \mathbb{H}_j^R(\mathfrak{Q}), \sum_{j \in J} \|f_j\|^2_{\mathbb{H}_j^R(\mathfrak{Q})} < \infty \right\}$$

under right multiplications by quaternionic scalars together with the quaternionic inner product is given by

$$\langle \{f_j\}_{j \in J}, \{g_j\}_{j \in J} \rangle = \sum_{j \in J} \langle f_j, g_j \rangle_{\mathbb{H}_j^R(\mathfrak{Q})},$$

and the norm is defined as $\|\{f_j\}_{j \in J}\|_{\mathcal{H}_2} = \sum_{j \in J} \|f_j\|_{\mathbb{H}_j^R(\mathfrak{Q})}$, for all $\{f_j\}_{j \in J} \in \mathcal{H}_2$. It is easy to very that $\mathcal{H}_2$ is a right quaternionic Hilbert space with respect to the quaternionic inner product given by above.

Remark 4.3. Let Λ be g-fusion Bessel sequence for $\mathbb{H}^R(\mathfrak{Q})$ with bound B. Then for every sequence $\{f_j\}_{j \in J} \in \mathcal{H}_2$, the series $\sum_{j \in J} v_j P_{W_j^R} \Lambda_j^* f_j$ converges unconditionally.

Theorem 4.4. *The family Λ is a g-fusion Bessel sequence for $\mathbb{H}^{R}(\mathfrak{Q})$ with bound B if and only if the right linear operator $T_{\mathfrak{Q}} : \mathcal{H}_2 \to \mathbb{H}^{R}(\mathfrak{Q})$ defined by*

$$T_{\mathfrak{Q}}\left(\{f_j\}_{j \in J}\right) = \sum_{j \in J} v_j P_{W_j^R} \Lambda_j^* f_j,$$

for all $\{f_j\}_{j \in J} \in \mathcal{H}_2$, is a well-defined and bounded linear operator with $\|T_{\mathfrak{Q}}\| \leq \sqrt{B}$.

Proof. Suppose Λ is a g-fusion Bessel sequence for $\mathbb{H}^{R}(\mathfrak{Q})$ with bound B. Let I be a finite subset of J. Then

$$\begin{aligned}
\left\|\sum_{j \in I} v_j P_{W_j^R} \Lambda_j^* f_j\right\|^2 &= \sup_{\|g\|=1}\left|\left\langle \sum_{j \in I} v_j P_{W_j^R} \Lambda_j^* f_j, g\right\rangle\right|^2 \\
&= \sup_{\|g\|=1} \sum_{j \in I}\left|\left\langle f_j, v_j \Lambda_j P_{W_j^R}(g)\right\rangle\right|^2 \\
&\leq \sum_{j \in I}\|f_j\|^2_{\mathbb{H}_j^R(\mathfrak{Q})} \sup_{\|g\|=1} \sum_{j \in I} v_j^2\left\|\Lambda_j P_{W_j^R}(g)\right\|^2 \\
&\leq B \sum_{j \in I}\|f_j\|^2_{\mathbb{H}_j^R(\mathfrak{Q})} < \infty.
\end{aligned}$$

Thus, the series $\sum\limits_{j \in J} v_j P_{W_j^R} \Lambda_j^* f_j$ converges unconditionally. Hence, the right linear operator $T_{\mathfrak{Q}}$ is well-defined. By the above similar calculation it is easy to verify that $T_{\mathfrak{Q}}$ is bounded and $\|T_{\mathfrak{Q}}\| \leq \sqrt{B}$.

Conversely, suppose that $T_{\mathfrak{Q}}$ is well-defined and bounded right linear operator with $\|T_{\mathfrak{Q}}\| \leq \sqrt{B}$. Then the adjoint $T_{\mathfrak{Q}}^*$ of a bounded right linear operator $T_{\mathfrak{Q}}$ is itself bounded and $\|T_{\mathfrak{Q}}\| = \|T_{\mathfrak{Q}}^*\|$. Now, for $f \in \mathbb{H}^{R}(\mathfrak{Q})$, we have

$$\sum_{j \in J} v_j^2\left\|\Lambda_j P_{W_j^R}(f)\right\|^2 = \|T_{\mathfrak{Q}}^* f\|^2 \leq \|T_{\mathfrak{Q}}\|^2\|f\|^2 \leq B\|f\|^2.$$

Thus, Λ is a g-fusion Bessel sequence for the right quaternionic Hilbert space $\mathbb{H}^{R}(\mathfrak{Q})$ with bound B. □

Let Λ be a g-fusion Bessel sequence for $\mathbb{H}^{R}(\mathfrak{Q})$. Then the right linear operator $T_{\mathfrak{Q}} : \mathcal{H}_2 \to \mathbb{H}^{R}(\mathfrak{Q})$ given by

$$T_{\mathfrak{Q}}\left(\{f_j\}_{j \in J}\right) = \sum_{j \in J} v_j P_{W_j^R} \Lambda_j^* f_j,$$

for all $\{f_j\}_{j \in J} \in \mathcal{H}_2$, is called the (right) synthesis operator and the adjoint of $T_{\mathfrak{Q}}$ given by

$$T_{\mathfrak{Q}}^* : \mathbb{H}^{R}(\mathfrak{Q}) \to \mathcal{H}_2, \; T_{\mathfrak{Q}}^*(f) = \left\{v_j \Lambda_j P_{W_j^R}(f)\right\}_{j \in J},$$

for all $f \in H$, is called the (right) analysis operator.

Definition 4.5. Let Λ be a g-fusion frame for the right quaternionic Hilbert space $\mathbb{H}^R(\mathfrak{Q})$. The right linear operator $S_{\mathfrak{Q}} : \mathbb{H}^R(\mathfrak{Q}) \to \mathbb{H}^R(\mathfrak{Q})$ defined by

$$S_{\mathfrak{Q}} f = T_{\mathfrak{Q}} T_{\mathfrak{Q}}^* f = \sum_{j \in J} v_j^2 P_{W_j^R} \Lambda_j^* \Lambda_j P_{W_j^R} f, \quad f \in \mathbb{H}^R(\mathfrak{Q}),$$

is called the (right) g-fusion frame operator for Λ.

In the next Theorem, a few properties of the frame operator for the g-fusion frame in right quaternionic Hilbert space is discussed.

Theorem 4.6. *Let Λ be a generalized fusion frame for $\mathbb{H}^R(\mathfrak{Q})$ with bounds A, B and $S_{\mathfrak{Q}}$ be the corresponding right g-fusion frame operator. Then $S_{\mathfrak{Q}}$ is bounded, positive, invertible and self-adjoint right linear operator on $\mathbb{H}^R(\mathfrak{Q})$.*

Proof. For each $f \in \mathbb{H}^R(\mathfrak{Q})$, we have

$$\langle S_{\mathfrak{Q}} f, f \rangle = \sum_{j \in J} v_j^2 \left\| \Lambda_j P_{W_j^R}(f) \right\|^2$$

and from (4.1), we get

$$A \|f\|^2 \leq \langle S_{\mathfrak{Q}} f, f \rangle \leq B \|f\|^2.$$

This implies that $A I_H \leq S_{\mathfrak{Q}} \leq B I_H$. Hence, $S_{\mathfrak{Q}}$ is bounded and positive right linear operator on $\mathbb{H}^R(\mathfrak{Q})$ and consequently it is a invertible.

Furthermore, for any $f, g \in \mathbb{H}^R(\mathfrak{Q})$, we have

$$\begin{aligned}\langle S_{\mathfrak{Q}} f, g \rangle &= \left\langle \sum_{j \in J} v_j^2 P_{W_j^R} \Lambda_j^* \Lambda_j P_{W_j^R} f, g \right\rangle \\ &= \sum_{j \in J} \left\langle f, v_j^2 P_{W_j^R} \Lambda_j^* \Lambda_j P_{W_j^R} g \right\rangle \\ &= \left\langle f, \sum_{j \in J} v_j^2 P_{W_j^R} \Lambda_j^* \Lambda_j P_{W_j^R}(g) \right\rangle = \langle f, S_{\mathfrak{Q}} g \rangle.\end{aligned}$$

Thus, $S_{\mathfrak{Q}}$ is also self-adjoint right linear operator on $\mathbb{H}^R(\mathfrak{Q})$. □

Corollary 4.7. *For every $f \in \mathbb{H}^R(\mathfrak{Q})$, we get the reconstruction formula as:*

$$f = \sum_{j \in J} v_j^2 S_{\mathfrak{Q}}^{-1} P_{W_j^R} \Lambda_j^* \Lambda_j P_{W_j^R} f = \sum_{j \in J} v_j^2 P_{W_j^R} \Lambda_j^* \Lambda_j P_{W_j^R} S_{\mathfrak{Q}}^{-1} f.$$

In the following Theorem, a characterization of a Parseval g-fusion frame for the right quaternionic Hilbert space $\mathbb{H}^R(\mathfrak{Q})$ is established.

Theorem 4.8. *Let Λ be a g-fusion frame for $\mathbb{H}^R(\mathfrak{Q})$ with the corresponding right g-fusion frame operator $S_{\mathfrak{Q}}$. Then Λ is a Parseval g-fusion frame for $\mathbb{H}^R(\mathfrak{Q})$ if and only if $S_{\mathfrak{Q}}$ is an identity operator on $\mathbb{H}^R(\mathfrak{Q})$.*

Proof. Let Λ be a Parseval g-fusion frame for $\mathbb{H}^R(\mathfrak{Q})$. Then, for each $f \in \mathbb{H}^R(\mathfrak{Q})$, we get

$$\sum_{j \in J} v_j^2 \left\| \Lambda_j P_{W_j^R}(f) \right\|^2 = \| f \|^2 \Rightarrow \langle S_{\mathfrak{Q}} f, f \rangle = \langle f, f \rangle .$$

This is shows that $S_{\mathfrak{Q}}$ is an identity operator on $\mathbb{H}^R(\mathfrak{Q})$.

Conversely, suppose that $S_{\mathfrak{Q}}$ is an identity operator on $\mathbb{H}^R(\mathfrak{Q})$. Then, for $f \in \mathbb{H}^R(\mathfrak{Q})$, we get

$$f = S_{\mathfrak{Q}} f = \sum_{j \in J} v_j^2 P_{W_j^R} \Lambda_j^* \Lambda_j P_{W_j^R} f.$$

Therefore, for $f \in \mathbb{H}^R(\mathfrak{Q})$, we have

$$\begin{aligned} \| f \|^2 = \langle f, f \rangle &= \left\langle \sum_{j \in J} v_j^2 P_{W_j^R} \Lambda_j^* \Lambda_j P_{W_j^R} f, f \right\rangle \\ &= \sum_{j \in J} v_j^2 \left\langle \Lambda_j P_{W_j^R} f, \Lambda_j P_{W_j^R} f \right\rangle = \sum_{j \in J} v_j^2 \left\| \Lambda_j P_{W_j^R}(f) \right\|^2 . \end{aligned}$$

Thus, Λ is a Parseval g-fusion frame for the right quaternionic Hilbert space $\mathbb{H}^R(\mathfrak{Q})$. □

Next, a characterization of a g-fusion frame for the right quaternionic Hilbert space $\mathbb{H}^R(\mathfrak{Q})$ with the help of its right synthesis operator is given.

Theorem 4.9. *The family Λ is a g-fusion frame for $\mathbb{H}^R(\mathfrak{Q})$ if and only if the right synthesis operator $T_{\mathfrak{Q}}$ is well-defined and bounded mapping from $\mathcal{H}_2$ onto $\mathbb{H}^R(\mathfrak{Q})$.*

Proof. Let Λ is a g-fusion frame for $\mathbb{H}^R(\mathfrak{Q})$. Then it is easy to verify that $T_{\mathfrak{Q}}$ is well-defined and bounded mapping from $\mathcal{H}_2$ onto $\mathbb{H}^R(\mathfrak{Q})$.

Conversely, suppose that the right synthesis operator $T_{\mathfrak{Q}}$ is well-defined and bounded mapping from $\mathcal{H}_2$ onto $\mathbb{H}^R(\mathfrak{Q})$. Then by Theorem 4.4, Λ is a g-fusion Bessel sequence for $\mathbb{H}^R(\mathfrak{Q})$. Since $T_{\mathfrak{Q}}$ is onto, there exists a right linear bounded operator $T_{\mathfrak{Q}}^{\dagger} : \mathbb{H}^R(\mathfrak{Q}) \to \mathcal{H}_2$ such that

$$f = T_{\mathfrak{Q}} T_{\mathfrak{Q}}^{\dagger} f = \sum_{j \in J} v_j P_{W_j^R} \Lambda_j^* \left(T_{\mathfrak{Q}}^{\dagger} f \right)_j , \quad f \in \mathbb{H}^R(\mathfrak{Q}),$$

where $\left(T_{\mathfrak{Q}}^{\dagger} f\right)_j$ denotes the j-th coordinate of $T_{\mathfrak{Q}}^{\dagger} f$. Now, for each $f \in \mathbb{H}^R(\mathfrak{Q})$, we have

$$\begin{aligned}\|f\|^4 = |\langle f, f\rangle|^2 &= \left|\left\langle \sum_{j\in J} v_j P_{W_j^R} \Lambda_j^* \left(T_{\mathfrak{Q}}^{\dagger} f\right)_j, f\right\rangle\right|^2 \\ &\leq \sum_{j\in J}\left|\left(T_{\mathfrak{Q}}^{\dagger} f\right)_j\right|^2 \sum_{j\in J} v_j^2 \left\|\Lambda_j P_{W_j^R}(f)\right\|^2 \\ &\leq \left\|T_{\mathfrak{Q}}^{\dagger}\right\|^2 \|f\|^2 \sum_{j\in J} v_j^2 \left\|\Lambda_j P_{W_j^R}(f)\right\|^2.\end{aligned}$$

This implies that

$$\frac{1}{\left\|T_{\mathfrak{Q}}^{\dagger}\right\|^2}\|f\|^2 \leq \sum_{j\in J} v_j^2 \left\|\Lambda_j P_{W_j^R}(f)\right\|^2.$$

Thus, Λ is a g-fusion frame for the right quaternionic Hilbert space $\mathbb{H}^R(\mathfrak{Q})$. □

Remark 4.10. Let $V \subset \mathbb{H}^R(\mathfrak{Q})$ be a closed subspace and $T \in \mathcal{B}\left(\mathbb{H}^R(\mathfrak{Q})\right)$. Then $P_V T^* = P_V T^* P_{\overline{TV}}$.

In the next Theorem, a g-fusion frame with the help of a given g-fusion frame in a right quaternionic Hilbert space is constructed.

Theorem 4.11. *Let $\Lambda = \{\left(W_j^R, \Lambda_j, v_j\right)\}_{j\in J}$ be g-fusion frame for $\mathbb{H}^R(\mathfrak{Q})$ with bounds A, B and $S_{\mathfrak{Q}}$ be the corresponding right g-fusion frame operator. Then $\Gamma = \left\{\left(S_{\mathfrak{Q}}^{-1} W_j^R, \Lambda_j P_{W_j^R} S_{\mathfrak{Q}}^{-1}, v_j\right)\right\}_{j\in J}$ is a g-fusion frame for $\mathbb{H}^R(\mathfrak{Q})$ with bounds $1/B$ and $1/A$.*

Proof. For each $f \in \mathbb{H}^R(\mathfrak{Q})$, we have

$$\begin{aligned}\sum_{j\in J} v_j^2 \left\|\Lambda_j P_{W_j^R} S_{\mathfrak{Q}}^{-1} P_{S_{\mathfrak{Q}}^{-1} W_j^R}(f)\right\|^2 &= \sum_{j\in J} v_j^2 \left\|\Lambda_j P_{W_j^R} S_{\mathfrak{Q}}^{-1}(f)\right\|^2 \\ &\leq B\left\|S_{\mathfrak{Q}}^{-1}\right\|^2 \|f\|^2.\end{aligned}$$

Thus, Γ is a g-fusion Bessel sequence for $\mathbb{H}^R(\mathfrak{Q})$. So, the right g-fusion frame operator for Γ is well-defined. Now, it is easy to verify that the right g-fusion frame operator for Γ is $S_{\mathfrak{Q}}^{-1}$. The operator $S_{\mathfrak{Q}}^{-1}$ commutes with both $S_{\mathfrak{Q}}$ and I_H. Thus, multiplying the inequality $A I_H \leq S_{\mathfrak{Q}} \leq B I_H$ with $S_{\mathfrak{Q}}^{-1}$, we get

$$B^{-1} I_H \leq S_{\mathfrak{Q}}^{-1} \leq A^{-1} I_H.$$

This implies that

$$B^{-1}\|f\|^2 \leq \left\langle S_{\mathfrak{Q}}^{-1} f, f\right\rangle \leq A^{-1}\|f\|^2, \quad f \in \mathbb{H}^R(\mathfrak{Q}). \tag{4.2}$$

Now, $f \in \mathbb{H}^R(\mathfrak{Q})$, we have

$$\begin{aligned}
S_{\mathfrak{Q}}^{-1} f &= S_{\mathfrak{Q}}^{-1}\left(S_{\mathfrak{Q}}\left(S_{\mathfrak{Q}}^{-1} f\right)\right) \\
&= S_{\mathfrak{Q}}^{-1}\left(\sum_{j \in J} v_j^2 P_{W_j^R} \Lambda_j^* \Lambda_j P_{W_j^R} S_{\mathfrak{Q}}^{-1} f\right) \\
&= \sum_{j \in J} v_j^2 S_{\mathfrak{Q}}^{-1} P_{W_j^R} \Lambda_j^* \Lambda_j P_{W_j^R} S_{\mathfrak{Q}}^{-1} f \\
&= \sum_{j \in J} v_j^2 \left(\Lambda_j P_{W_j^R} S_{\mathfrak{Q}}^{-1}\right)^* \Lambda_j P_{W_j^R} S_{\mathfrak{Q}}^{-1} f \\
&= \sum_{j \in J} v_j^2 P_{S_{\mathfrak{Q}}^{-1} W_j^R} S_{\mathfrak{Q}}^{-1} P_{W_j^R} \Lambda_j^* \Lambda_j P_{W_j^R} S_{\mathfrak{Q}}^{-1} P_{S_{\mathfrak{Q}}^{-1} W_j^R} f.
\end{aligned}$$

Therefore, from (4.2), for $f \in \mathbb{H}^R(\mathfrak{Q})$, we get

$$B^{-1}\|f\|^2 \leq \sum_{j \in J} v_j^2 \left\|\Lambda_j P_{W_j^R} S_{\mathfrak{Q}}^{-1} P_{S_{\mathfrak{Q}}^{-1} W_j^R}(f)\right\|^2 \leq A^{-1}\|f\|^2.$$

This shows that Γ is a g-fusion frame for $\mathbb{H}^R(\mathfrak{Q})$ with bounds $1/B$ and $1/A$. □

In Theorem 4.11, the g-fusion frame Γ for $\mathbb{H}^R(\mathfrak{Q})$ is called the canonical dual of the g-fusion frame of Λ.

Remark 4.12. Let Λ be a g-fusion frame for $\mathbb{H}^R(\mathfrak{Q})$ with the corresponding right g-fusion frame operator $S_{\mathfrak{Q}}$. Then

$$\left\{\left(S_{\mathfrak{Q}}^{-1/2} W_j^R,\, \Lambda_j P_{W_j^R} S_{\mathfrak{Q}}^{-1/2},\, v_j\right)\right\}_{j \in J}$$

is a Parseval g-fusion frame for $\mathbb{H}^R(\mathfrak{Q})$.

Theorem 4.13. *Let $U \in \mathcal{B}\left(\mathbb{H}^R(\mathfrak{Q})\right)$ be an invertible bounded right $\mathfrak{Q}$-linear operator on $\mathbb{H}^R(\mathfrak{Q})$ and $\Lambda = \{(W_j, \Lambda_j, v_j)\}_{j \in J}$ be a g-fusion frame for $\mathbb{H}^R(\mathfrak{Q})$ with bounds A and B. Then the family $\Gamma = \left\{\left(U W_j, \Lambda_j P_{W_j} U^*, v_j\right)\right\}_{j \in J}$ is a $U U^*$-g-fusion frame for $\mathbb{H}^R(\mathfrak{Q})$.*

Proof. Since U is an bounded right $\mathfrak{Q}$-linear operator on $\mathbb{H}^R(\mathfrak{Q})$ for any $j \in J$, $U W_j$ is closed in $\mathbb{H}^R(\mathfrak{Q})$. Now, for each $f \in \mathbb{H}^R(\mathfrak{Q})$, using Note 4.10, we obtain

$$\begin{aligned}
\sum_{j \in J} v_j^2 \left\|\Lambda_j P_{W_j^R} U^* P_{U W_j^R}(f)\right\|^2 &= \sum_{j \in J} v_j^2 \left\|\Lambda_j P_{W_j^R} U^*(f)\right\|^2 \\
&\leq B\,\|U^* f\|^2 \leq B\,\|U\|^2\,\|f\|^2.
\end{aligned}$$

On the other hand, for each $f \in \mathbb{H}^R(\mathfrak{Q})$, we get

$$\begin{aligned}\frac{A}{\|U\|^2}\,\|(UU^*)^* f\|^2 &= \frac{A}{\|U\|^2}\,\|UU^* f\|^2 \leq A\,\|U^* f\|^2 \\ &\leq \sum_{j \in J} v_j^2 \left\|\Lambda_j P_{W_j^R}(U^* f)\right\|^2 \\ &= \sum_{j \in J} v_j^2 \left\|\Lambda_j P_{W_j^R} U^* P_{UW_j^R}(f)\right\|^2.\end{aligned}$$

Therefore, Γ is a UU^*-g-fusion frame for $\mathbb{H}^R(\mathfrak{Q})$. □

Theorem 4.14. *Let U be an bounded right $\mathfrak{Q}$-linear operator on $\mathbb{H}^R(\mathfrak{Q})$ and $\Gamma = \left\{\left(UW_j, \Lambda_j P_{W_j} U^*, v_j\right)\right\}_{j \in J}$ be a g-fusion frame for $\mathbb{H}^R(\mathfrak{Q})$. Then $\Lambda = \{(W_j, \Lambda_j, v_j)\}_{j \in J}$ is a g-fusion frame for $\mathbb{H}^R(\mathfrak{Q})$.*

Proof. For each $f \in \mathbb{H}^R(\mathfrak{Q})$, we have

$$\begin{aligned}\frac{A}{\|U\|^2}\,\|f\|^2 &= \frac{A}{\|U\|^2}\left\|U^*(U^{-1})^* f\right\|^2 \leq A\left\|\left(U^{-1}\right)^* f\right\|^2 \\ &\leq \sum_{j \in J} v_j^2 \left\|\Lambda_j P_{W_j^R} U^* P_{UW_j^R}\left(\left(U^{-1}\right)^* f\right)\right\|^2 \\ &= \sum_{j \in J} v_j^2 \left\|\Lambda_j P_{W_j^R}\left(U^*\left(U^{-1}\right)^* f\right)\right\|^2 \\ &= \sum_{j \in J} v_j^2 \left\|\Lambda_j P_{W_j^R}(f)\right\|^2.\end{aligned}$$

Also, for each $f \in \mathbb{H}^R(\mathfrak{Q})$, we have

$$\begin{aligned}\sum_{j \in J} v_j^2 \left\|\Lambda_j P_{W_j^R}(f)\right\|^2 &= \sum_{j \in J} v_j^2 \left\|\Lambda_j P_{W_j^R}\left(U^*\left(U^{-1}\right)^* f\right)\right\|^2 \\ &= \sum_{j \in J} v_j^2 \left\|\Lambda_j P_{W_j^R} U^* P_{UW_j^R}\left(\left(U^{-1}\right)^* f\right)\right\|^2 \\ &\leq B\left\|\left(U^{-1}\right)^* f\right\|^2 \leq B\left\|U^{-1}\right\|^2 \|f\|^2.\end{aligned}$$

Thus, Λ is a g-fusion frame for $\mathbb{H}^R(\mathfrak{Q})$ with bounds $\dfrac{A}{\|U\|^2}$ and $B\,\|U^{-1}\|^2$.
□

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