SQUARE-FULL NUMBERS MULTIPLE OF A CERTAIN SET OF PRIMES AND HYBRID NUMBERS

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**Abstract.** Let $h \geq 1$ be an arbitrary but fixed positive integer. Let us consider the $s$ distinct primes $q_1, \ldots, q_s$. Let $B_{q_1 \cdots q_s}(x)$ be the number of $h$-full numbers not exceeding $x$ multiple of $q_1 \cdots q_s$ and let $A_{q_1 \cdots q_s}(x)$ be the number of $h$-full numbers not exceeding $x$ relatively prime to $q_1 \cdots q_s$. In this note we obtain asymptotic formulas for $A_{q_1 \cdots q_s}(x)$ and $B_{q_1 \cdots q_s}(x)$. Then we apply the results obtained in the study of hybrid $h$-full numbers. That is, the $h$-full numbers of the form $q^h Q$, where $q > 1$ is square-free, $Q > 1$ is a $(h+1)$-full number, and $q$, $Q$ are relatively prime.

1. Square-full Numbers Multiple of a Certain Set of Primes

A number is square-free if either it is the product of distinct primes or 1. That is, its prime factorization is of the form $q_1 \cdots q_r$ where the $q_i$ ($i = 1, \ldots, r$) ($r \geq 1$) are the distinct primes. Let $Q(x)$ be the number of square-free numbers not exceeding $x$. We have the following formula (see [2]),

$$Q(x) = \frac{6}{\pi^2} x + O(x^{\frac{1}{2}}).$$

(1.1)

**Lemma 1.1.** Let $Q_{q_1 \cdots q_s}(x)$ the number of square-free not exceeding $x$ relatively prime to the square-free $q_1 \cdots q_s$. The following formula holds

$$Q_{q_1 \cdots q_s}(x) = \frac{6}{\pi^2} q_1 \cdots q_s \frac{q_1 \cdots q_r}{(q_1 + 1) \cdots (q_r + 1)} x + O(2^r x^{\frac{1}{2}}), \quad (x \geq M).$$

(1.2)

**Proof.** See [5, Lemma 3].

Let $h \geq 1$ be an arbitrary but fixed positive integer. A number is $h$-full if all the distinct primes in its prime factorization have multiplicity (or exponent) greater than or equal to $h$. That is, the number $q_1^{s_1} \cdots q_r^{s_r}$ is $h$-full if $s_i \geq h$ ($i = 1, \ldots, r$) ($r \geq 1$). If $h = 1$ we obtain all the positive integers. If $h = 2$ the numbers are called square-full.

Let $h \geq 1$ be and let $A_h(x)$ be the number of $h$-full numbers not exceeding $x$. It was proved by Ivić and Shiue (see [3, Chapter 14], or [4])

$$A_h(x) = \gamma_{0,h} x^h + \gamma_{1,h} x^{\frac{1}{2}h} + \cdots + \gamma_{h-1,h} x^{\frac{1}{2h-1}} + \Delta_h(x),$$

(1.3)

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where $\Delta_h(x) = O(x^\rho)$ for $\rho$ small.

We need the weaker lemma.

**Lemma 1.2.** The following asymptotic formula holds

$$A_h(x) = \gamma_{0,h} x^{\frac{1}{h}} + O\left(x^{\frac{1}{h+1}}\right), \quad (1.4)$$

where

$$\gamma_{0,h} = \frac{6}{\pi^2} C_h = \frac{6}{\pi^2} \prod_p \left(1 + \frac{1}{(p+1)(p^{\frac{1}{h}} - 1)}\right) = \prod_p \left(1 + \frac{p - p^{\frac{1}{h}}}{p^2 (p^{\frac{1}{h}} - 1)}\right), \quad (1.5)$$

Note that if $h = 1$ then we obtain the trivial formula $A_1(x) = x + o(x)$.

**Proof.** Equation (1.4) is a weak consequence of (1.3). For equation (1.5) see the reference [6]. The lemma is proved. $\square$

**Lemma 1.3.** Let $h \geq 1$ an arbitrary but fixed integer. The following series converges.

$$\sum_{Q \in \mathbb{Q}} \frac{1}{Q^h},$$

where the sum runs over all $(h+1)$-full numbers $Q$.

**Proof.** Let $a_n$ be the $n$-th $(h+1)$-full number and let $A_{h+1}(x)$ be the number of $(h+1)$-full numbers not exceeding $x$. We have (see Lemma 1.2) $A_{h+1}(x) \sim \gamma_{0,h}^{h+1 \sqrt{x}}$. Therefore if $x = a_n$ we obtain $n = A_{h+1}(a_n) \sim \gamma_{0,h}^{h+1 \sqrt{a_n}}$, that is, $a_n \sim \frac{n}{\gamma_{0,h}}$. Now, the lemma follows by the Comparison Criterion since the series $\sum \frac{1}{n^{\frac{1}{h}}} \gamma_{0,h}$ converges. The lemma is proved. $\square$

**Theorem 1.4.** Let $h \geq 1$ an arbitrary but fixed integer. Let $r \geq 1$ an arbitrary but fixed integer. Let us consider the $r$ distinct primes $q_1, \ldots, q_r$. Let $A_{q_1 \cdots q_r}(x)$ be the number of $h$-full numbers not exceeding $x$ relatively prime to $q_1 \cdots q_r$. The following asymptotic formula holds

$$A_{q_1 \cdots q_r}(x) = \left(\prod_{i=1}^r q_i - q_i^{\frac{1}{h}}\right) \gamma_{0,h} x^{\frac{1}{h}} + O\left(2^{r \times 2^{h+1} M^{1+\epsilon}}\right),$$

$$\sum_{i=1}^r q_i - q_i^{\frac{1}{h}} + 1 \leq \gamma_{0,h} x^{\frac{1}{h}} + O\left(2^{r \times 2^{h+1} M^{1+\epsilon}}\right), \quad (x \geq M), \quad (1.6)$$

where $\epsilon > 0$ can be arbitrarily small.

**Proof.** Let us consider the $h$-full numbers of the form $q^h$, where $q$ is a square-free number. The number of these $h$-full numbers relatively prime to $q_1 \cdots q_r$ is (see Lemma 1.1)

$$\frac{6}{\pi^2} \frac{q_1 \cdots q_r}{(q_1 + 1) \cdots (q_r + 1)} x^{\frac{1}{h}} + o\left(x^{\frac{1}{h}}\right). \quad (1.7)$$
Let us consider a \((h + 1)\)-full number \(Q = p_1^{a_1} \cdots p_t^{a_t}\), where \(p_i (i = 1, \ldots, t)\) are the distinct primes in the prime factorization of \(Q\) and \(a_i \geq h + 1 (i = 1, \ldots, t)\) are the exponents. We suppose that \(Q\) is relatively prime to \(q_1 \cdots q_r\). We, for sake of simplicity, put \(Q_1 = p_1 \cdots p_t\) and \(Q_2 = (p_1 + 1) \cdots (p_t + 1)\). Let us consider the \(h\)-full numbers of the form \(q^h Q\), where \(Q\) relatively prime to \(q_1 \cdots q_r\) is fixed and \(Q\) and \(q\) are relatively prime. The number of these \(h\)-full numbers relatively prime to \(q_1 \cdots q_r\) is (see Lemma 1.1).

\[
\frac{6}{\pi^2} \frac{q_1 \cdots q_r}{(q_1 + 1) \cdots (q_r + 1)} Q_1 x^\frac{1}{Q_1} + o \left( x^\frac{1}{Q_1} \right).
\]  

(1.8)

Let \(\epsilon > 0\) be. There exist \(M\), depending of \(\epsilon\), such that (see Lemma 1.3)

\[
\sum_{Q > M} \frac{1}{Q^\frac{1}{Q}} < \epsilon,
\]  

(1.9)

where the sum run over all \((h + 1)\)-full numbers \(Q > M\) relatively prime to \(q_1 \cdots q_r\).

Equations (1.7), (1.8) and (1.9) give

\[
A_{q_1 \cdots q_r}(x) = \frac{6}{\pi^2} \frac{q_1 \cdots q_r}{(q_1 + 1) \cdots (q_r + 1)} \left( 1 + \sum_{Q \leq M} \frac{Q_1}{Q_2} \frac{1}{Q^\frac{1}{Q}} \right) x^\frac{1}{Q} + o \left( x^\frac{1}{Q} \right)
+ F(x) = \frac{6}{\pi^2} \frac{q_1 \cdots q_r}{(q_1 + 1) \cdots (q_r + 1)} \left( 1 + \sum_{Q} \frac{Q_1}{Q_2} \frac{1}{Q^\frac{1}{Q}} \right) x^\frac{1}{Q}
- \frac{6}{\pi^2} \frac{q_1 \cdots q_r}{(q_1 + 1) \cdots (q_r + 1)} \sum_{Q > M} \frac{Q_1}{Q_2} \frac{1}{Q^\frac{1}{Q}} x^\frac{1}{Q} + o \left( x^\frac{1}{Q} \right)
+ F(x),
\]  

(1.10)

where (see (1.10) and (1.9))

\[
0 \leq F(x) \leq \sum_{x \geq Q > M} \left| \frac{x^\frac{1}{Q}}{Q^\frac{1}{Q}} \right| \leq \sum_{Q > M} \frac{x^\frac{1}{Q}}{Q^\frac{1}{Q}} < \epsilon x^\frac{1}{Q}.
\]  

(1.11)
Note that (see (1.10) and Lemma 1.2)

\[
\frac{6}{\pi^2} \frac{q_1 \cdots q_r}{(q_1 + 1) \cdots (q_r + 1)} \left( 1 + \sum_Q Q_1 \frac{1}{Q_2 Q^2} \right) = \frac{6}{\pi^2} \frac{q_1 \cdots q_r}{(q_1 + 1) \cdots (q_r + 1)} \prod_{p \neq q_1, \ldots, q_r} \left( 1 + \frac{p}{p+1} \left( \left( \frac{1}{p^\frac{1}{2}} \right)^{h+1} + \left( \frac{1}{p^\frac{1}{2}} \right)^{h+2} + \cdots \right) \right) = \frac{6}{\pi^2} \frac{q_1 \cdots q_r}{(q_1 + 1) \cdots (q_r + 1)} \prod_{p \neq q_1, \ldots, q_r} \left( 1 + \frac{1}{(p+1) \left( \frac{1}{p^\frac{1}{2}} - 1 \right)} \right) = \left( \prod_{i=1}^r \frac{q_i \left( \frac{1}{q\frac{1}{2}} - 1 \right)}{q_i \left( \frac{1}{q\frac{1}{2}} - 1 \right) + q_i^\frac{1}{2}} \right) \gamma_{0,h}.
\]

(1.12)

Therefore, equations (1.9), (1.10), (1.11) and (1.12) give

\[
\left| \frac{A_{q_1 \cdots q_r}(x)}{x^\frac{1}{2}} - \left( \prod_{i=1}^r \frac{q_i \left( \frac{1}{q\frac{1}{2}} - 1 \right)}{q_i \left( \frac{1}{q\frac{1}{2}} - 1 \right) + q_i^\frac{1}{2}} \right) \gamma_{0,h} \right| < 3\epsilon, \quad (x \geq x_\epsilon).
\]

That is, since \( \epsilon > 0 \) can be arbitrarily small, we obtain the limit

\[
\lim_{x \to \infty} \frac{A_{q_1 \cdots q_r}(x)}{x^\frac{1}{2}} = \left( \prod_{i=1}^r \frac{q_i \left( \frac{1}{q\frac{1}{2}} - 1 \right)}{q_i \left( \frac{1}{q\frac{1}{2}} - 1 \right) + q_i^\frac{1}{2}} \right) \gamma_{0,h}.
\]

(1.13)

Let \( G_h \) be the set of \( h \)-full numbers and let \( G'_h \) be the set of the \( h \)-full numbers relatively prime to \( q_1 \cdots q_r \).

Let \( f_h(n) \) be the characteristic function of the \( h \)-full numbers and let \( f'_h(n) \) be the characteristic function of the \( h \)-full numbers relatively prime to \( q_1 \cdots q_r \).

Let \( s = \sigma + it \) a complex number. It is well-known (see [3, Chapter 1]) that the generating Dirichlet series for \( h \)-full numbers is

\[
F(s) = \sum_{n \in G_h} \frac{f_h(n)}{n^s} = \prod_{p} \left( 1 + \frac{p^{-hs}}{1 - p^{-s}} \right) = \zeta(hs) N(s),
\]

where \( N(s) \) is an absolutely convergent Dirichlet series for \( Re(s) = \sigma > \frac{1}{h+1} \).

The generating Dirichlet series for \( h \)-full numbers relatively prime to \( q_1 \cdots q_r \) will be (see [3, Chapter 1])

\[
F'(s) = \sum_{n \in G'_h} \frac{f'_h(n)}{n^s} = \prod_{p \neq q_j} \left( 1 + \frac{p^{-hs}}{1 - p^{-s}} \right) = \prod_{p \neq q_j} \frac{1}{1 - \frac{1}{p^\frac{1}{2}}} N'(s) = \left( \prod_{j=1}^r \left( 1 - \frac{1}{q_j^\frac{1}{2}} \right) \right) \zeta(hs) N'(s),
\]
where $N'(s)$ is an absolutely convergent Dirichlet series depending of $q_1, \ldots, q_r$ for $\text{Re}(s) = \sigma > \frac{1}{n+1}$. However there is a positive constant $c$ that does not depend of $q_1, \ldots, q_r$ such that $|N'(s)| < c$. On the other hand we have

$$\left| \prod_{j=1}^{r} \left( 1 - \frac{1}{q_j^{\sigma}} \right) \right| \leq 2^r.$$

If we put $\sigma_0 = \frac{1}{h} + \epsilon$ and $\sigma_1 = \frac{1}{h+1} + \epsilon$ and apply Perron’s formula as in [5, Lemma 5] we obtain that

$$A_{q_1 \cdots q_r}(x) = c x^{\frac{1}{h}} + O \left( 2^r x^{\frac{2h+1}{2(h+1)} + \epsilon} \right).$$

Equations (1.13) and (1.14) give equation (1.6). The theorem is proved.

**Theorem 1.5.** Let $h \geq 1$ an arbitrary but fixed integer. Let $r \geq 1$ an arbitrary but fixed positive integer. Let us consider the $r$ distinct primes $q_1, \ldots, q_r$. Let $B_{q_1 \cdots q_r}(x)$ be the number of $h$-full numbers not exceeding $x$ multiple of $q_1 \cdots q_r$. The following asymptotic formula holds.

$$B_{q_1 \cdots q_r}(x) = \left( \prod_{i=1}^{r} \frac{q_i^{\frac{1}{h}}}{q_i \left( q_i^{\frac{1}{h}} - 1 \right) + q_i^{\frac{1}{h}}} \right) \gamma_{0,h} x^{\frac{1}{h}} + O_r \left( x^{\frac{2h+1}{2(h+1)} + \epsilon} \right),$$

where $\epsilon > 0$ can be arbitrarily small.

**Proof.** Suppose that $r = 1$. Theorem 1.4 gives

$$A_{q_1}(x) = \left( \frac{q_1 \left( q_1^{\frac{1}{h}} - 1 \right)}{q_1 \left( q_1^{\frac{1}{h}} - 1 \right) + q_1^{\frac{1}{h}}} \right) \gamma_{0,h} x^{\frac{1}{h}} + O \left( x^{\frac{2h+1}{2(h+1)} + \epsilon} \right)$$

and Lemma 1.2 gives

$$A_h(x) = \gamma_{0,h} x^{\frac{1}{h}} + O \left( x^{\frac{2h+1}{2(h+1)} + \epsilon} \right).$$

Note that we have the equation

$$A_{q_1}(x) = A_h(x) - B_{q_1}(x).$$

If we put

$$B_{q_1}(x) = \left( \frac{q_1^{\frac{1}{h}}}{q_1 \left( q_1^{\frac{1}{h}} - 1 \right) + q_1^{\frac{1}{h}}} \right) \gamma_{0,h} x^{\frac{1}{h}} + W$$

and substitute equations (1.16), (1.17) and (1.19) into (1.18) we obtain that $W = O \left( x^{\frac{2h+1}{2(h+1)} + \epsilon} \right)$. Therefore equation (1.15) is true for $r = 1$.

Now, we proceed by mathematical induction. Suppose that equation (1.15) is true for $s = 1, 2, \ldots, r - 1 \ (r \geq 2)$. We shall prove that equation (1.15) is also true for $s = r$. 


The number of $h$-full numbers not exceeding $x$ multiple of some $q_i$ ($i = 1, 2, \ldots, r$) is (inclusion-exclusion principle)

\[
C_{q_1 \ldots q_r}(x) = \sum_{1 \leq i \leq r} B_{q_i}(x) - \sum_{1 \leq i < j \leq r} B_{q_i q_j}(x) + \sum_{1 \leq i < j < k \leq r} B_{q_i q_j q_k}(x) + \cdots + (-1)^{r+1} B_{q_1 \ldots q_r}(x)
\]

where we put

\[
B_{q_1 \ldots q_r}(x) = \left( \prod_{i=1}^r \frac{q_i^{\frac{1}{h}}}{q_i \left( q_i^{\frac{1}{h}} - 1 \right) + q_i^{\frac{1}{h}}} \right) \gamma_{0,h} x^{\frac{1}{h}} + O \left( x^{\frac{2h+1}{2(h+1)} + \epsilon} \right) + W.
\]

Now, we have the equation

\[
A_{q_1 \ldots q_r}(x) = A_h(x) - C_{q_1 \ldots q_r}(x),
\]

where (Theorem 1.4)

\[
A_{q_1 \ldots q_r}(x) = \left( \prod_{i=1}^r \frac{q_i \left( q_i^{\frac{1}{h}} - 1 \right)}{q_i \left( q_i^{\frac{1}{h}} - 1 \right) + q_i^{\frac{1}{h}}} \right) \gamma_{0,h} x^{\frac{1}{h}} + O \left( x^{\frac{2h+1}{2(h+1)} + \epsilon} \right).\]

Substituting equations (1.17), (1.20) and (1.23) into (1.22) we obtain that $W = O \left( x^{\frac{2h+1}{2(h+1)} + \epsilon} \right)$. Therefore equation (1.21) becomes equation (1.15). The theorem is proved.

2. $s$-full Hybrid Numbers

Let $s \geq 1$ be an arbitrary but fixed positive integer. We denote $q > 1$ a generic square-free number and $Q > 1$ a generic $(s+1)$-full number. Note that a $s$-full number can be of the form $q^a$, of the form $Q$ or of the form $q^a Q$, where $q$ and $Q$ are relatively prime. We shall call the $s$-full numbers of the form $q^a Q$ $s$-full hybrid numbers and, $q^a$ will be called the $s$-part and $Q$ will be called the $(s+1)$-full part. In particular, if $s = 1$ all the positive integers are of the form $q$, $Q$ or $qQ$ and consequently the 1-full hybrid numbers are of the form $qQ$, where $q$ is a square-free, $Q$ is a square-full and, $q$ and $Q$ are relatively prime. This case, namely $s = 1$, was studied extensively in [1].

**Theorem 2.1.** Let $s \geq 1$ an arbitrary but fixed integer. The number of $s$-full hybrid numbers $q^a Q$ not exceeding $x$ will be denoted $H_s(x)$. We have

\[
H_s(x) = \sum_{q^a Q \leq x} 1 = \frac{6}{\pi^2} (C_s - 1) x^{\frac{1}{s}} + O \left( x^{\frac{1}{s+1}} \right).
\]
Proof. We have

\[ A_s(x) = \sum_{q^s \leq x} 1 + H_s(x) + \sum_{Q \leq x}. \]

Now, \( A_s(x) = \frac{6}{\pi^2} C_s x^\frac{1}{s} + O \left( x^{\frac{1}{s+1}} \right) \) (by Lemma 1.2), \( \sum_{q^s \leq x} 1 = \frac{6}{\pi^2} x^\frac{1}{s} + O \left( x^{\frac{1}{s+2}} \right) \) (by equation (1.1)) and \( \sum_{Q \leq x} 1 = A_{s+1}(x) = \frac{6}{\pi^2} C_{s+1} x^\frac{1}{s+1} + O \left( x^{\frac{1}{s+2}} \right) \) (by Lemma 1.2). The theorem is proved.

Theorem 2.2. Let \( s \geq 1 \) an arbitrary but fixed integer. The sum of the \( s \)-full hybrid numbers \( q^s Q \) not exceeding \( x \) is

\[ \sum_{q^s Q \leq x} q^s Q = \frac{6}{\pi^2} \frac{1}{s+1} \left( C_s - 1 \right) x^{1+\frac{1}{s}} + O \left( x^{1+\frac{1}{s+1}} \right). \]  
\[ (2.2) \]

Proof. Apply Abel’s summation to equation (2.1). The theorem is proved. \( \square \)

Theorem 2.3. Let \( s \geq 1 \) an arbitrary but fixed integer. Let us consider the \( s \)-full hybrid numbers \( q^s Q \). The sum of the quotients \( \frac{Q}{q^s} \) of the hybrid numbers \( q^s Q \) not exceeding \( x \) is

\[ \sum_{q^s Q \leq x} \frac{Q}{q^s} = \frac{A_s}{s+2} x^{1+\frac{1}{s+1}} + O \left( x^{1+\frac{1}{s+2}} \right), \]  
\[ (2.3) \]

where

\[ A_s = \frac{6}{\pi^2} C_{s+1} \left( -1 + \prod_p \left( 1 + \frac{p-p^{1-\frac{1}{s+1}}}{p-p^{1-\frac{1}{s+1}} + 1} \right) \right). \]  
\[ (2.4) \]

Proof. Let us consider the \( s \)-part \( q^s = (q_1 \cdots q_r)^s \), where \( q_i \) \((i = 1, \ldots, r)\) are the distinct primes in the prime factorization of \( q \). The number of \( s \)-full hybrid numbers \( q^s Q \) not exceeding \( x \) with the fixed \( s \)-part \( q^s = (q_1 \cdots q_r)^s \) will be the number of solutions \( Q \) to the inequality \( q^s Q \leq x \), that is to the inequality \( Q \leq \frac{x}{q^s} \).

The number of solutions \( Q \) to this inequality \((s\text{-part } q^s \text{ fixed})\) is by Theorem 1.4

\[ \frac{6}{\pi^2} C_{s+1} \prod_{i=1}^{r} \frac{q_i - q_i^{1-\frac{1}{s+1}}}{q_i^{1-\frac{1}{s+1}} + 1} x^{\frac{1}{s+1}} + O \left( 2^r x^{\frac{2(s+1)+1}{s(2s+1)(s+2)^2}} + \epsilon \right) \]
\[ = C_q x^{\frac{1}{s+1}} + O \left( 2^r x^{\frac{2(s+1)+1}{s(2s+1)(s+2)^2}} + \epsilon \right), \]  
\[ (2.5) \]

where, by sake of simplicity, we put

\[ C_q = \frac{6}{\pi^2} C_{s+1} \prod_{i=1}^{r} \frac{q_i - q_i^{1-\frac{1}{s+1}}}{q_i^{1-\frac{1}{s+1}} + 1} < C_{s+1}. \]
and where $\frac{x}{q} \geq M$, that is $q \leq \frac{1}{M^2}$. Note that $2^r \leq q_1 \ldots q_r = q$. Therefore equation (2.5) gives

$$
\sum_{q^s Q \leq x} \frac{1}{q^{2s}} = \sum_{q \leq \frac{1}{M^2}} \frac{1}{q^{2s}} C_q \frac{x^{1+\frac{1}{s}+2+\epsilon}}{x^{1+\frac{1}{s}+2+\epsilon} + 2} + \sum_{q \leq \frac{1}{M^2}} O \left( \frac{2^{(s+1)+1} x^{2(s+1)(s+2)+\epsilon}}{x^{2(s+1)(s+2)+\epsilon} + 2} \right) + F(x)
$$

$$
= \left( \sum_{q} \frac{C_q}{q^{2s+\frac{1}{s+1}}} \right) x^{1+\frac{1}{s}} - \left( \sum_{q > \frac{1}{M^2}} \frac{C_q}{q^{2s+\frac{1}{s+1}}} \right) x^{1+\frac{1}{s}} + O \left( x^{2(s+1)(s+2)+\epsilon} \right)
$$

$$
+ F(x).
$$

(2.6)

Let $A_{s+1}(x)$ be the number of $(s + 1)$-full numbers not exceeding $x$ (see Lemma 1.2). We have $A_{s+1}(x) \leq c x^{\frac{1}{s+1}}$, where $c$ is a positive constant and $x \geq 1$. Therefore (see (2.6))

$$
0 \leq F(x) \leq \sum_{q \leq \frac{1}{M^2}} \frac{1}{q^{2s}} A_{s+1} \left( \frac{x}{q^s} \right) \leq \sum_{q > \frac{1}{M^2}} \frac{1}{q^{2s}} C_q \left( \frac{x}{q^s} \right)^{\frac{1}{s+1}}
$$

$$
\leq c x^{\frac{1}{s+1}} \sum_{q > \frac{1}{M^2}} \frac{1}{q^{2s+\frac{1}{s+1}}}.
$$

(2.7)

Abel’s summation gives

$$
x^{1+\frac{1}{s}} \sum_{q > \frac{1}{M^2}} \frac{1}{q^{2s+\frac{1}{s+1}}} = o(1).
$$

(2.8)

Therefore, equations (2.6), (2.7) and (2.8) give

$$
\sum_{q^s Q \leq x} \frac{1}{q^{2s}} = \left( \sum_{q} \frac{C_q}{q^{2s+\frac{1}{s+1}}} \right) x^{1+\frac{1}{s}} + O \left( x^{2(s+1)(s+2)+\epsilon} \right)
$$

$$
= A_s x^{1+\frac{1}{s}} + O \left( x^{2(s+1)(s+2)+\epsilon} \right),
$$

(2.9)

where $A_s = \sum_q \frac{C_q}{q^{s+\frac{1}{s+1}}}$ and the sum runs on all square-free number $q > 1$. Therefore $A_s$ can be written as an infinite product on the primes in the form (2.4).

Finally, equation (2.9) and Abel’s summation give

$$
\sum_{q^s Q \leq x} \frac{Q}{q^s} = \sum_{q^s Q \leq x} \frac{Q}{q^{2s}} = \frac{A_s}{s+2} x^{1+\frac{1}{s+1}} + O \left( x^{1+\frac{2(s+1)+1}{2(s+1)+1} \frac{1}{s+2} + \epsilon} \right),
$$

that is, equation (2.3). The theorem is proved. \hfill \Box
Theorem 2.4. Let $s \geq 1$ an arbitrary but fixed integer. Let us consider the $s$-full hybrid numbers $q^s Q$. The sum of the $(s + 1)$-full parts $Q$ of the hybrid numbers $q^s Q$ not exceeding $x$ is
\[
\sum_{q^s Q \leq x} Q = \frac{B_s}{s + 2} x^{1 + \frac{1}{s + 1}} + O \left( x^{1 + \frac{2(s + 1) + 1}{2(s + 1)} + \epsilon} \right) \quad (s \geq 2),
\]
(2.10)

\[
\sum_{qQ \leq x} Q = \frac{B_1}{3} x^{1 + \frac{3}{2}} + O \left( x^{1 + \frac{6}{13} + \epsilon} \right) \quad (s = 1),
\]
(2.11)

where
\[
B_s = \frac{6}{\pi^2} C_{s + 1} \left( -1 + \prod_p \left( 1 + \frac{p - p^{-1} - \frac{1}{s + 1}}{p - p^{-1} + \frac{1}{s + 1}} \right) \right), \quad (s \geq 1). \quad (2.12)
\]

Proof. In this case we study the sum
\[
\sum_{q^s Q \leq x} \frac{1}{q^s}.
\]
If $s \geq 2$ then the proof is the same as the proof of Theorem 2.3.

Suppose that $s = 1$. In this case equation (2.5) becomes
\[
C_q x^{\frac{3}{2}} q^{-\frac{1}{2}} + O \left( \frac{2^r x^{\frac{7}{12} + \epsilon}}{q^{\frac{5}{12} + \epsilon}} \right)
\]
and, since $2^r \leq q_1 \ldots q_r = q$, equation (2.6) becomes
\[
\sum_{qQ \leq x} \frac{1}{q} = \sum_{q \leq x \frac{1}{13}} C_q \frac{x^{\frac{7}{2}}}{q^{\frac{1}{2} + \frac{1}{13}}} + \sum_{q \leq x \frac{1}{13}} O \left( \frac{x^{\frac{5}{12} + \epsilon}}{q^{\frac{5}{12} + \epsilon}} \right) + F(x)
\]
\[
= \left( \sum_q \frac{C_q}{q^{1 + \frac{1}{2}}} \right) x^{\frac{1}{2}} - \left( \sum_{q > x \frac{1}{13}} \frac{C_q}{q^{1 + \frac{1}{2}}} \right) x^{\frac{1}{2}} + O \left( x^{\frac{6}{13} + \epsilon} \right)
\]
\[
+ F(x),
\]
where we have applied Abel’s summation to the divergent sum $\sum_{q \leq x \frac{1}{13}} \frac{1}{q^{\frac{1}{2} + \frac{1}{13}}}$.

Equation (2.7) becomes
\[
0 \leq F(x) \leq c x^{\frac{1}{2}} \sum_{q > x \frac{1}{13}} \frac{1}{q^{1 + \frac{1}{2}}},
\]

Abel’s summation gives
\[
x^{\frac{1}{2}} \sum_{q > x \frac{1}{13}} \frac{1}{q^{1 + \frac{1}{2}}} = O \left( x^{\frac{6}{13}} \right).
\]

The theorem is proved. 
\[\square\]
Theorem 2.5. Let \( s \geq 1 \) an arbitrary but fixed integer. Let us consider the \( s \)-full hybrid numbers \( q^s Q \). The sum of the \( s \)-parts \( q^s \) of the hybrid numbers \( q^s Q \) not exceeding \( x \) is

\[
\sum_{q^s Q \leq x} q^s = \frac{D_s}{s+1} x^{1+\frac{1}{s}} + O \left( x^{1+\frac{1}{2s}} \right),
\]

where

\[
D_s = \frac{6}{\pi^2} \left( -1 + \prod_p \left( 1 + \frac{1}{(p+1)p^s (p^{1+\frac{1}{s}} - 1)} \right) \right).
\]

Proof. Let us consider the \((s+1)\)-full part \( Q = q_1^{a_1} \cdots q_r^{a_r} \), where \( q_i \ (i = 1, \ldots, r) \) are the distinct primes in the prime factorization of \( Q \). The number of \( s \)-full hybrid numbers \( q^s Q \) not exceeding \( x \) with the fixed \((s+1)\)-full part \( Q = q_1^{a_1} \cdots q_r^{a_r} \) will be the number of solutions \( q \) to the inequality \( q^s Q \leq x \), that is to the inequality \( q \leq x^{1/s} Q^{1/2} \). The number of solutions \( q \) to this inequality \((s+1)\)-full part \( Q \) fixed) is by Lemma 1.1

\[
\frac{6}{\pi^2} \prod_{i=1}^r \frac{q_i}{q_i + 1} x^{1/2} + O \left( 2^r \frac{x^{1/2}}{Q^{1/2}} \right),
\]

where \( x^{1/2} \geq M \), that is \( Q \leq x^{M^{-1}} \). Now, we have \( 2^r \leq q_1 \cdots q_r \leq Q^{1+1/s} \). Therefore equation (2.14) gives

\[
\sum_{q^s Q \leq x} \frac{1}{Q} = \frac{6}{\pi^2} \sum_{q \leq M^{1/2}} \prod_{i=1}^r \frac{q_i}{q_i + 1} x^{1/2} + \sum_{Q > M^{1/2}} O \left( \frac{Q^{1+1/2}}{Q^{1/2}} \cdot x^{1/2} \right) + F(x)
\]

\[
= \left( \sum_{Q > M^{1/2}} \frac{6}{\pi^2} \prod_{i=1}^r \frac{q_i}{q_i + 1} \right) x^{1/2} + O \left( x^{1/2} \right) - \left( \sum_{Q > M^{1/2}} \frac{6}{\pi^2} \prod_{i=1}^r \frac{q_i}{q_i + 1} \right) x^{1/2}
\]

\[
+ F(x). \tag{2.15}
\]

Now, we have (see (2.15))

\[
0 \leq F(x) \leq \sum_{q \geq Q > M^{1/2}} \frac{1}{Q} \left( x^{1/2} \frac{1}{Q^{1/2}} \right) \leq \sum_{Q > M^{1/2}} \frac{1}{Q} \frac{x^{1/2}}{Q^{1/2}}. \tag{2.16}
\]

Abel’s summation gives

\[
x^{1/2} \sum_{Q > M^{1/2}} \frac{1}{Q^{1+1/2}} = o(1). \tag{2.17}
\]

Therefore, equations (2.15), (2.16) and (2.17) give

\[
\sum_{q^s Q \leq x} \frac{1}{Q} = D_s x^{1/2} + O \left( x^{1/2} \right) \tag{2.18}
\]
where

$$D_s = \sum_{Q} \frac{6}{\pi^2} \prod_{i=1}^{r} \frac{q_i}{q_i + 1} \frac{1}{Q^{1+\frac{1}{s}}}$$  \hspace{1cm} (2.19)$$

and the sum runs on all $(s+1)$-full number $Q > 1$.

Equation (2.18) and Abel’s summation give

$$\sum_{q^s Q \leq x} q^s = \sum_{q^s Q \leq x} \frac{q^s Q}{Q} = \frac{D_s}{s+1} x^{1+\frac{1}{s}} + O \left( x^{1+\frac{1}{2s}} \right),$$

that is, equation (2.13).

Finally, equation (2.19) gives

$$D_s = \sum_{Q} \frac{6}{\pi^2} \prod_{i=1}^{r} \frac{q_i}{q_i + 1} \frac{1}{Q^{1+\frac{1}{s}}}$$

$$= \frac{6}{\pi^2} \left( -1 + \prod_{p} \left( 1 + \frac{p}{p+1} \left( \sum_{i=s+1}^{\infty} \frac{1}{(p^i)^{1+\frac{1}{s}}} \right) \right) \right)$$

$$= \frac{6}{\pi^2} \left( -1 + \prod_{p} \left( 1 + \frac{1}{(p+1)p^s(p^{1+\frac{1}{s}} - 1)} \right) \right).$$

The theorem is proved. \hfill \Box

We recall that (Theorem 2.1)

$$H_s(x) = \frac{6}{\pi^2} (C_s - 1) x^{\frac{1}{s}} + o \left( x^{\frac{1}{s}} \right)$$  \hspace{1cm} (2.20)$$

Let $D_{s,h}(x)$ be the number of $s$-full hybrid numbers $q^s Q$ not exceeding $x$ such that $\frac{Q}{q^s} < h$, where $0 < h < 1$ is an arbitrary but fixed real number. We have the following theorem

**Theorem 2.6.** Let $s \geq 1$ an arbitrary but fixed integer. The following asymptotic formulas hold

$$D_{s,h}(x) = \frac{6}{\pi^2} (C_s - 1) x^{\frac{1}{s}} + o \left( x^{\frac{1}{s}} \right)$$  \hspace{1cm} (2.21)$$

$$D_{s,h}(x) \sim H_s(x)$$  \hspace{1cm} (2.22)$$

$$H_s(x) - D_{s,h}(x) = o \left( x^{\frac{1}{s}} \right) = o (H_s(x))$$  \hspace{1cm} (2.23)$$

Therefore, almost all $s$-full hybrid number $q^s Q$ has its $(s+1)$-full part $Q$ very small compared with its $s$-part $q^s$, since $h$ can be arbitrarily small.

**Proof.** Equations (2.22) and (2.23) are an immediate consequence of equations (2.20) and (2.21). Therefore we shall prove equation (2.21). Let us consider the convergent series (see Lemma 1.3)

$$1 + \sum_{Q} \frac{q_1 \cdots q_r}{(q_1 + 1) \cdots (q_r + 1)} \frac{1}{Q^{\frac{1}{s}}}.$$
where the sum runs on all \((s + 1)\)-full number \(Q = q_1^{a_1} \cdots q_r^{a_r}\) and where \(q_i\) 
\((i = 1, \ldots, r)\) are the distinct primes in the prime factorization of \(Q\). The sum of
this series is \(C_s\) (see Lemma 1.2), since
\[
1 + \sum_Q \frac{q_1 \cdots q_r}{(q_1 + 1) \cdots (q_r + 1)} \frac{1}{Q^{1/2}} = \prod_p \left(1 + \frac{p}{p + 1} \left(\sum_{i=s+1}^{\infty} \frac{1}{(p)^{1/2}}\right)\right)
\]
\[
= \prod_p \left(1 + \frac{1}{(p + 1) \left(\frac{1}{p} - 1\right)}\right) = C_s. \tag{2.24}
\]
Let \(\epsilon > 0\). By definition of \(D_{s,h}(x)\) and \(H_s(x)\) we have
\[
D_{s,h}(x) \leq H_s(x) = \frac{6}{\pi^2} (C_s - 1) x^{\frac{1}{2}} + o \left(\frac{1}{x^{\frac{1}{2}}}\right) \leq \frac{6}{\pi^2} (C_s - 1) x^{\frac{1}{2}} + 2\epsilon x^{\frac{1}{2}}. \tag{2.25}
\]
Let us consider a fixed \((s + 1)\)-full part \(Q = q_1^{a_1} \cdots q_r^{a_r}\), where \(q_i\) 
\((i = 1, \ldots, r)\) are the distinct primes in its prime factorization. Let us consider the
\(s\)-full hybrid numbers \(q^sQ\) not exceeding \(x\), that is \(q^sQ \leq x\), that is
\[
q \leq \frac{x^{\frac{1}{2}}}{Q^{\frac{1}{2}}}. \tag{2.26}
\]
where \(Q = q_1^{a_1} \cdots q_r^{a_r}\) is fixed. Inequality (2.26) implies that inequality \(\frac{Q}{q^s} < h\) holds for all hybrid \(q^sQ\) except by a finite number of cases. Since \(q \to \infty\) if \(x \to \infty\)
and \(Q\) is fixed. Therefore the number of hybrid numbers \(q^sQ\) not exceeding \(x\) such that inequality \(\frac{Q}{q^s} < h\) holds will be (see Lemma 1.1)
\[
\frac{6}{\pi^2} \frac{q_1 \cdots q_r}{(q_1 + 1) \cdots (q_r + 1)} \frac{1}{Q^{1/2}} x^{\frac{1}{2}} + o \left(\frac{1}{x^{\frac{1}{2}}}\right). \tag{2.27}
\]
We choose the number \(B\) such that
\[
\frac{6}{\pi^2} \sum_{Q > B} \frac{q_1 \cdots q_r}{(q_1 + 1) \cdots (q_r + 1)} \frac{1}{Q^{1/2}} < \epsilon. \tag{2.28}
\]
Now, equations (2.27), (2.28) and (2.24) give
\[
D_{s,h}(x) \geq \left(\frac{6}{\pi^2} \sum_{Q \leq B} \frac{q_1 \cdots q_r}{(q_1 + 1) \cdots (q_r + 1)} \frac{1}{Q^{1/2}}\right) x^{\frac{1}{2}} + o \left(\frac{1}{x^{\frac{1}{2}}}\right)
\]
\[
= \left(\frac{6}{\pi^2} \sum_Q \frac{q_1 \cdots q_r}{(q_1 + 1) \cdots (q_r + 1)} \frac{1}{Q^{1/2}}\right) x^{\frac{1}{2}}
\]
\[
- \left(\frac{6}{\pi^2} \sum_{Q > B} \frac{q_1 \cdots q_r}{(q_1 + 1) \cdots (q_r + 1)} \frac{1}{Q^{1/2}}\right) x^{\frac{1}{2}}
\]
\[
+ o \left(\frac{1}{x^{\frac{1}{2}}}\right) \geq \frac{6}{\pi^2} (C_s - 1) x^{\frac{1}{2}} - \epsilon x^{\frac{1}{2}} - \epsilon x^{\frac{1}{2}}
\]
\[
= \frac{6}{\pi^2} (C_s - 1)) x^{\frac{1}{2}} - 2\epsilon x^{\frac{1}{2}}. \tag{2.29}
\]
Equations (2.25) and (2.29) give
\[
\left| D_s,h(x) - \frac{6}{\pi^2} (C_s - 1) x^{s+\frac{1}{2}} \right| = \left| D_s,h(x) \frac{x^{\frac{1}{2}}}{x^{s+\frac{1}{2}}} - \frac{6}{\pi^2} (C_s - 1) \right| \leq 2\epsilon,
\]
that is, equation (2.21), since \( \epsilon \) can be arbitrarily small. The theorem is proved. \( \square \)

**Theorem 2.7.** Let \( s \geq 1 \) an arbitrary but fixed integer. Let us consider the \( s \)-full hybrid numbers \( q^*Q \). The sum of the quotients \( \frac{q^s}{Q} \) of the hybrid numbers \( q^*Q \) not exceeding \( x \) is
\[
\sum_{q^*Q \leq x} \frac{q^s}{Q} = \frac{E_s}{s+1} x^{1+\frac{1}{s}} + O \left( x^{1+\frac{1}{2s}} \right)
\]
(2.30)

where
\[
E_s = \frac{6}{\pi^2} \left( -1 + \prod_p \left( 1 + \frac{1}{(p+1)p^{2s} \left( p^{2\frac{s}{s}+s} - 1 \right)} \right) \right)
\]

**Proof.** The proof is the same as the proof of Theorem 2.5. In this case we study the sum
\[
\sum_{q^*Q \leq x} \frac{1}{Q^2}
\]
The theorem is proved. \( \square \)

**Remark 2.8.** Note that the sums \( \sum_{q^*Q \leq x} q^s Q \), \( \sum_{q^*Q \leq x} q^s \) and \( \sum_{q^*Q \leq x} \frac{q^s}{Q} \) have the same magnitude order \( x^{1+\frac{1}{s}} \) (see Theorem 2.2, Theorem 2.5 and Theorem 2.7). Note that the sums \( \sum_{q^sQ \leq x} \frac{Q}{q^s} \) and \( \sum_{q^sQ \leq x} Q \) have the same smaller magnitude order \( x^{1+\frac{1}{s+1}} \) (see Theorem 2.3 and Theorem 2.4). Consequently, for example, the sum of \( (s+1) \)-full parts \( Q \) of the \( s \)-full hybrid numbers not exceeding \( x \) \( (\sum_{q^sQ \leq x} Q) \) is negligible compared with the sum of \( s \)-parts \( q^s \) of the \( s \)-full hybrid numbers not exceeding \( x \) \( (\sum_{q^sQ \leq x} q^s) \). However, there is a surprising fact, if we compare Theorem 2.3 with Theorem 2.6. The contribution to the sum \( \sum_{q^sQ \leq x} \frac{Q}{q^s} \) of almost all \( s \)-full hybrid numbers is smaller than \( hD_{s,h(x)} < cx^{\frac{1}{2}} \), where \( c \) can be arbitrarily small (Theorem 2.6). However the order of the sum \( \sum_{q^sQ \leq x} \frac{Q}{q^s} \) is greater, namely \( x^{1+\frac{1}{s+1}} \) (Theorem 2.3). Therefore there is a negligible number \( o \left( x^{\frac{1}{2}} \right) \) of \( s \)-full hybrid numbers \( q^sQ \) responsible of this greater magnitude order and consequently where the quotient \( \frac{Q}{q^s} \) must be very big.

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References

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