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SQUARE-FULL NUMBERS MULTIPLE OF A CERTAIN SET OF PRIMES AND HYBRID NUMBERS

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Abstract. Let $h \geq 1$ be an arbitrary but fixed positive integer. Let us consider the s distinct primes $q_1, \ldots, q_s$. Let $B_{q_1\cdots q_s}(x)$ be the number of h-full numbers not exceeding x multiple of $q_1 \cdots q_s$ and let $A_{q_1\cdots q_s}(x)$ be the number of h-full numbers not exceeding x relatively prime to $q_1 \cdots q_s$. In this note we obtain asymptotic formulas for $A_{q_1\cdots q_s}(x)$ and $B_{q_1\cdots q_s}(x)$. Then we apply the results obtained in the study of hybrid h-full numbers. That is, the h-full numbers of the form $q^h Q$, where $q > 1$ is square-free, $Q > 1$ is a $(h+1)$-full number, and q, Q are relatively prime.

1. Square-full Numbers Multiple of a Certain Set of Primes

A number is square-free if either it is the product of distinct primes or 1. That is, its prime factorization is of the form $q_1 \cdots q_r$ where the q_i $(i = 1, \ldots, r)$ $(r \geq 1)$ are the distinct primes. Let $Q(x)$ be the number of square-free numbers not exceeding x. We have the following formula (see [2]),

$$Q(x) = \frac{6}{\pi^2}x + O(x^{\frac{1}{2}}). \tag{1.1}$$

Lemma 1.1. *Let $Q_{q_1\cdots q_r}(x)$ the number of square-free not exceeding x relatively prime to the square-free $q_1 \cdots q_r$. The following formula holds*

$$Q_{q_1\cdots q_r}(x) = \frac{6}{\pi^2}\frac{q_1 \cdots q_r}{(q_1+1)\cdots(q_r+1)}x + O(2^r x^{\frac{1}{2}}), \qquad (x \geq M). \tag{1.2}$$

Proof. See [5, Lemma 3]. □

Let $h \geq 1$ be an arbitrary but fixed positive integer. A number is h-full if all the distinct primes in its prime factorization have multiplicity (or exponent) greater than or equal to h. That is, the number $q_1^{s_1} \cdots q_r^{s_r}$ is h-full if $s_i \geq h$ $(i = 1, \ldots, r)$ $(r \geq 1)$. If $h = 1$we obtain all the positive integers. If $h = 2$ the numbers are called square-full.

Let $h \geq 1$ be and let $A_h(x)$ be the number of h-full numbers not exceeding x. It was proved by Ivić and Shiu (see [3, Chapter 14], or [4])

$$A_h(x) = \gamma_{0,h}x^{\frac{1}{h}} + \gamma_{1,h}x^{\frac{1}{h+1}} + \cdots + \gamma_{h-1,h}x^{\frac{1}{2h-1}} + \Delta_h(x), \tag{1.3}$$

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where $\Delta_h(x) = O\left(x^\rho\right)$ for ρ small.

We need the weaker lemma.

Lemma 1.2. *The following asymptotic formula holds*

$$A_h(x) = \gamma_{0,h} x^{\frac{1}{h}} + O\left(x^{\frac{1}{h+1}}\right), \tag{1.4}$$

where

$$\gamma_{0,h} = \frac{6}{\pi^2} C_h = \frac{6}{\pi^2} \prod_p \left(1 + \frac{1}{(p+1)(p^{\frac{1}{h}} - 1)}\right) = \prod_p \left(1 + \frac{p - p^{\frac{1}{h}}}{p^2 \left(p^{\frac{1}{h}} - 1\right)}\right). \tag{1.5}$$

Note that if $h = 1$ then we obtain the trivial formula $A_1(x) = x + o(x)$.

Proof. Equation (1.4) is a weak consequence of (1.3). For equation (1.5) see the reference [6] . The lemma is proved. □

Lemma 1.3. *Let $h \geq 1$ an arbitrary but fixed integer. The following series converges.*

$$\sum_Q \frac{1}{Q^{\frac{1}{h}}},$$

where the sum runs over all $(h+1)$-full numbers Q.

Proof. Let a_n be the n-th $(h+1)$-full number and let $A_{h+1}(x)$ be the number of $(h+1)$-full numbers not exceeding x. We have (see Lemma 1.2) $A_{h+1}(x) \sim \gamma_{0,h} \sqrt[h+1]{x}$. Therefore if $x = a_n$ we obtain $n = A_{h+1}(a_n) \sim \gamma_{0,h} \sqrt[h+1]{a_n}$, that is, $a_n \sim \frac{n^{h+1}}{\gamma_{0,h}^{h+1}}$. Now, the lemma follows by the Comparison Criterion since the series $\sum \frac{1}{n^{\frac{h+1}{h}}}$ converges. The lemma is proved. □

Theorem 1.4. *Let $h \geq 1$ an arbitrary but fixed integer. Let $r \geq 1$ an arbitrary but fixed integer. Let us consider the r distinct primes $q_1, \ldots, q_r$. Let $A_{q_1 \cdots q_r}(x)$ be the number of h-full numbers not exceeding x relatively prime to $q_1 \cdots q_r$. The following asymptotic formula holds*

$$A_{q_1 \cdots q_r}(x) = \left(\prod_{i=1}^r \frac{q_i \left(q_i^{\frac{1}{h}} - 1\right)}{q_i \left(q_i^{\frac{1}{h}} - 1\right) + q_i^{\frac{1}{h}}}\right) \gamma_{0,h} x^{\frac{1}{h}} + O\left(2^r x^{\frac{2h+1}{2h(h+1)} + \epsilon}\right)$$

$$= \left(\prod_{i=1}^r \frac{q_i - q_i^{1-\frac{1}{h}}}{q_i - q_i^{1-\frac{1}{h}} + 1}\right) \gamma_{0,h} x^{\frac{1}{h}} + O\left(2^r x^{\frac{2h+1}{2h(h+1)} + \epsilon}\right), \qquad (x \geq M), \tag{1.6}$$

where $\epsilon > 0$ can be arbitrarily small.

Proof. Let us consider the h-full numbers of the form q^h, where q is a square-free number. The number of these h-full numbers relatively prime to $q_1 \cdots q_r$ is (see Lemma 1.1)

$$\frac{6}{\pi^2} \frac{q_1 \cdots q_r}{(q_1 + 1) \cdots (q_r + 1)} x^{\frac{1}{h}} + o\left(x^{\frac{1}{h}}\right). \tag{1.7}$$

Let us consider a $(h+1)$-full number $Q = p_1^{a_1} \cdots p_t^{a_t}$, where p_i $(i = 1, \ldots, t)$ are the distinct primes in the prime factorization of Q and $a_i \geq h+1$ $(i = 1, \ldots, t)$ are the exponents. We suppose that Q is relatively prime to $q_1 \cdots q_r$. We, for sake of simplicity, put $Q_1 = p_1 \cdots p_t$ and $Q_2 = (p_1+1) \cdots (p_t+1)$. Let us consider the h-full numbers of the form $q^h Q$, where Q relatively prime to $q_1 \cdots q_r$ is fixed and Q and q are relatively prime. The number of these h-full numbers relatively prime to $q_1 \cdots q_r$ is (see Lemma 1.1).

$$\frac{6}{\pi^2} \frac{q_1 \cdots q_r}{(q_1+1) \cdots (q_r+1)} \frac{Q_1}{Q_2} \frac{x^{\frac{1}{h}}}{Q^{\frac{1}{h}}} + o\left(x^{\frac{1}{h}}\right). \tag{1.8}$$

Let $\epsilon > 0$ be. There exist M, depending of ϵ, such that (see Lemma 1.3)

$$\sum_{Q > M} \frac{1}{Q^{\frac{1}{h}}} < \epsilon, \tag{1.9}$$

where the sum run over all $(h+1)$-full numbers $Q > M$ relatively prime to $q_1 \cdots q_r$.

Equations (1.7), (1.8) and (1.9) give

$$\begin{aligned} A_{q_1 \cdots q_r}(x) &= \frac{6}{\pi^2} \frac{q_1 \cdots q_r}{(q_1+1) \cdots (q_r+1)} \left(1 + \sum_{Q \leq M} \frac{Q_1}{Q_2} \frac{1}{Q^{\frac{1}{h}}}\right) x^{\frac{1}{h}} + o\left(x^{\frac{1}{h}}\right) \\ + \quad F(x) &= \frac{6}{\pi^2} \frac{q_1 \cdots q_r}{(q_1+1) \cdots (q_r+1)} \left(1 + \sum_{Q} \frac{Q_1}{Q_2} \frac{1}{Q^{\frac{1}{h}}}\right) x^{\frac{1}{h}} \\ - \quad & \frac{6}{\pi^2} \frac{q_1 \cdots q_r}{(q_1+1) \cdots (q_r+1)} \sum_{Q > M} \frac{Q_1}{Q_2} \frac{1}{Q^{\frac{1}{h}}} x^{\frac{1}{h}} + o\left(x^{\frac{1}{h}}\right) \\ + \quad & F(x), \end{aligned} \tag{1.10}$$

where (see (1.10) and (1.9))

$$0 \leq F(x) \leq \sum_{x \geq Q > M} \left\lfloor \frac{x^{\frac{1}{h}}}{Q^{\frac{1}{h}}} \right\rfloor \leq \sum_{Q > M} \frac{x^{\frac{1}{h}}}{Q^{\frac{1}{h}}} < \epsilon x^{\frac{1}{h}}. \tag{1.11}$$

Note that (see (1.10) and Lemma 1.2)

$$\begin{aligned}
&\frac{6}{\pi^2}\frac{q_1\cdots q_r}{(q_1+1)\cdots(q_r+1)}\left(1+\sum_{Q}\frac{Q_1}{Q_2}\frac{1}{Q^{\frac{1}{h}}}\right)\\
=\ &\frac{6}{\pi^2}\frac{q_1\cdots q_r}{(q_1+1)\cdots(q_r+1)}\prod_{\substack{p\\ p\neq q_1,\dots,q_r}}\left(1+\frac{p}{p+1}\left(\left(\frac{1}{p^{\frac{1}{h}}}\right)^{h+1}+\left(\frac{1}{p^{\frac{1}{h}}}\right)^{h+2}+\cdots\right)\right)\\
=\ &\frac{6}{\pi^2}\frac{q_1\cdots q_r}{(q_1+1)\cdots(q_r+1)}\prod_{\substack{p\\ p\neq q_1,\dots,q_r}}\left(1+\frac{1}{(p+1)\left(p^{\frac{1}{h}}-1\right)}\right)\\
=\ &\left(\prod_{i=1}^{r}\frac{q_i\left(q_i^{\frac{1}{h}}-1\right)}{q_i\left(q_i^{\frac{1}{h}}-1\right)+q_i^{\frac{1}{h}}}\right)\gamma_{0,h}. \qquad (1.12)
\end{aligned}$$

Therefore, equations (1.9), (1.10), (1.11) and (1.12) give

$$\left|\frac{A_{q_1\cdots q_r}(x)}{x^{\frac{1}{h}}}-\left(\prod_{i=1}^{r}\frac{q_i\left(q_i^{\frac{1}{h}}-1\right)}{q_i\left(q_i^{\frac{1}{h}}-1\right)+q_i^{\frac{1}{h}}}\right)\gamma_{0,h}\right|<3\epsilon, \qquad (x\geq x_\epsilon).$$

That is, since $\epsilon>0$ can be arbitrarily small, we obtain the limit

$$\lim_{x\to\infty}\frac{A_{q_1\cdots q_r}(x)}{x^{\frac{1}{h}}}=\left(\prod_{i=1}^{r}\frac{q_i\left(q_i^{\frac{1}{h}}-1\right)}{q_i\left(q_i^{\frac{1}{h}}-1\right)+q_i^{\frac{1}{h}}}\right)\gamma_{0,h}. \qquad (1.13)$$

Let G_h be the set of h-full numbers and let G'_h be the set of the h-full numbers relatively prime to $q_1\cdots q_r$.

Let $f_h(n)$ be the characteristic function of the h-full numbers and let $f'_h(n)$ be the characteristic function of the h-full numbers relatively prime to $q_1\cdots q_r$.

Let $s=\sigma+it$ a complex number. It is well-known (see [3, Chapter 1]) that the generating Dirichlet series for h-full numbers is

$$F(s)=\sum_{n\epsilon G_h}\frac{f_h(n)}{n^s}=\prod_{p}\left(1+\frac{p^{-hs}}{1-p^{-s}}\right)=\zeta\left(hs\right)N(s),$$

where $N(s)$ is an absolutely convergent Dirichlet series for $Re(s)=\sigma>\frac{1}{h+1}$.

The generating Dirichlet series for h-full numbers relatively prime to $q_1\cdots q_r$ will be (see [3, Chapter 1])

$$\begin{aligned}
F'(s)&=\sum_{n\epsilon G'_h}\frac{f'_h(n)}{n^s}=\prod_{p\neq q_j}\left(1+\frac{p^{-hs}}{1-p^{-s}}\right)\\
&=\prod_{p\neq q_j}\frac{1}{1-\frac{1}{p^{hs}}}N'(s)=\left(\prod_{j=1}^{r}\left(1-\frac{1}{q_j^{hs}}\right)\right)\zeta(hs)N'(s),
\end{aligned}$$

where $N'(s)$ is an absolutely convergent Dirichlet series depending of $q_1, \ldots, q_r$ for $Re(s) = \sigma > \frac{1}{h+1}$. However there is a positive constant c that does not depend of $q_1, \ldots, q_r$ such that $|N'(s)| < c$. On the other hand we have

$$\left| \prod_{j=1}^{r} \left(1 - \frac{1}{q_j^{hs}} \right) \right| \leq 2^r.$$

If we put $\sigma_0 = \frac{1}{h} + \epsilon$ and $\sigma_1 = \frac{1}{h+1} + \epsilon$ and apply Perron's formula as in [5, Lemma 5] we obtain that

$$A_{q_1 \cdots q_r}(x) = c x^{\frac{1}{h}} + O\left(2^r x^{\frac{2h+1}{2h(h+1)} + \epsilon} \right). \tag{1.14}$$

Equations (1.13) and (1.14) give equation (1.6). The theorem is proved. □

Theorem 1.5. *Let $h \geq 1$ an arbitrary but fixed integer. Let $r \geq 1$ an arbitrary but fixed positive integer. Let us consider the r distinct primes $q_1, \ldots, q_r$. Let $B_{q_1 \cdots q_r}(x)$ be the number of h-full numbers not exceeding x multiple of $q_1 \cdots q_r$. The following asymptotic formula holds.*

$$B_{q_1 \cdots q_r}(x) = \left(\prod_{i=1}^{r} \frac{q_i^{\frac{1}{h}}}{q_i \left(q_i^{\frac{1}{h}} - 1 \right) + q_i^{\frac{1}{h}}} \right) \gamma_{0,h} x^{\frac{1}{h}} + O_r\left(x^{\frac{2h+1}{2h(h+1)} + \epsilon} \right), \tag{1.15}$$

where $\epsilon > 0$ can be arbitrarily small.

Proof. Suppose that $r = 1$. Theorem 1.4 gives

$$A_{q_1}(x) = \left(\frac{q_1 \left(q_1^{\frac{1}{h}} - 1 \right)}{q_1 \left(q_1^{\frac{1}{h}} - 1 \right) + q_1^{\frac{1}{h}}} \right) \gamma_{0,h} x^{\frac{1}{h}} + O\left(x^{\frac{2h+1}{2h(h+1)} + \epsilon} \right) \tag{1.16}$$

and Lemma 1.2 gives

$$A_h(x) = \gamma_{0,h} x^{\frac{1}{h}} + O\left(x^{\frac{2h+1}{2h(h+1)} + \epsilon} \right). \tag{1.17}$$

Note that we have the equation

$$A_{q_1}(x) = A_h(x) - B_{q_1}(x). \tag{1.18}$$

If we put

$$B_{q_1}(x) = \left(\frac{q_1^{\frac{1}{h}}}{q_1 \left(q_1^{\frac{1}{h}} - 1 \right) + q_1^{\frac{1}{h}}} \right) \gamma_{0,h} x^{\frac{1}{h}} + W \tag{1.19}$$

and substitute equations (1.16), (1.17) and (1.19) into (1.18) we obtain that $W = O\left(x^{\frac{2h+1}{2h(h+1)} + \epsilon} \right)$. Therefore equation (1.15) is true for $r = 1$.

Now, we proceed by mathematical induction. Suppose that equation (1.15) is true for $s = 1, 2, \ldots, r-1$ $(r \geq 2)$. We shall prove that equation (1.15) is also true for $s = r$.

The number of h-full numbers not exceeding x multiple of some q_i $(i = 1, 2, \ldots, r)$ is (inclusion-exclusion principle)

$$\begin{aligned} C_{q_1\cdots q_r}(x) &= \sum_{1\le i\le\le r} B_{q_i}(x) - \sum_{1\le i<j\le\le r} B_{q_i,q_j}(x) + \sum_{1\le i<j<k\le\le r} B_{q_i,q_j,q_k}(x) \\ + & \cdots + (-1)^{r+1} B_{q_1\cdots q_r}(x) \\ = & \left(1 - \prod_{i=1}^{r}\left(1 - \frac{q_i^{\frac{1}{h}}}{q_i\left(q_i^{\frac{1}{h}}-1\right)+q_i^{\frac{1}{h}}}\right)\right)\gamma_{0,h}x^{\frac{1}{h}} + O\left(x^{\frac{2h+1}{2h(h+1)}+\epsilon}\right) + W \\ = & \left(1 - \left(\prod_{i=1}^{r}\frac{q_i\left(q_i^{\frac{1}{h}}-1\right)}{q_i\left(q_i^{\frac{1}{h}}-1\right)+q_i^{\frac{1}{h}}}\right)\right)\gamma_{0,h}x^{\frac{1}{h}} + O\left(x^{\frac{2h+1}{2h(h+1)}+\epsilon}\right) + W, \quad (1.20) \end{aligned}$$

where we put

$$B_{q_1\cdots q_r}(x) = \left(\prod_{i=1}^{r}\frac{q_i^{\frac{1}{h}}}{q_i\left(q_i^{\frac{1}{h}}-1\right)+q_i^{\frac{1}{h}}}\right)\gamma_{0,h}x^{\frac{1}{h}} + W. \quad (1.21)$$

Now, we have the equation

$$A_{q_1\cdots q_r}(x) = A_h(x) - C_{q_1\cdots q_r}(x), \quad (1.22)$$

where (Theorem 1.4)

$$A_{q_1\cdots q_r}(x) = \left(\prod_{i=1}^{r}\frac{q_i\left(q_i^{\frac{1}{h}}-1\right)}{q_i\left(q_i^{\frac{1}{h}}-1\right)+q_i^{\frac{1}{h}}}\right)\gamma_{0,h}x^{\frac{1}{h}} + O\left(x^{\frac{2h+1}{2h(h+1)}+\epsilon}\right). \quad (1.23)$$

Substituting equations (1.17), (1.20) and (1.23) into (1.22) we obtain that $W = O\left(x^{\frac{2h+1}{2h(h+1)}+\epsilon}\right)$. Therefore equation (1.21) becomes equation (1.15). The theorem is proved. □

2. s-full Hybrid Numbers

Let $s \ge 1$ be an arbitrary but fixed positive integer. We denote $q > 1$ a generic square-free number and $Q > 1$ a generic $(s+1)$-full number. Note that a s-full number can be of the form q^s, of the form Q or of the form q^sQ, where q and Q are relatively prime. We shall call the s-full numbers of the form q^sQ s-full hybrid numbers and, q^s will be called the s-part and Q will be called the $(s+1)$-full part. In particular, if $s = 1$ all the positive integers are of the form q, Q or qQ and consequently the 1-full hybrid numbers are of the form qQ, where q is a square-free, Q is a square-full and, q and Q are relatively prime. This case, namely $s = 1$, was studied extensively in [1].

Theorem 2.1. *Let $s \ge 1$ an arbitrary but fixed integer. The number of s-full hybrid numbers q^sQ not exceeding x will be denoted $H_s(x)$. We have*

$$H_s(x) = \sum_{q^sQ\le x} 1 = \frac{6}{\pi^2}(C_s - 1)\, x^{\frac{1}{s}} + O\left(x^{\frac{1}{s+1}}\right). \quad (2.1)$$

Proof. We have

$$A_s(x) = \sum_{q^s \leq x} 1 + H_s(x) + \sum_{Q \leq x} 1.$$

Now, $A_s(x) = \frac{6}{\pi^2} C_s x^{\frac{1}{s}} + O\left(x^{\frac{1}{s+1}}\right)$ (by Lemma 1.2), $\sum_{q^s \leq x} 1 = \frac{6}{\pi^2} x^{\frac{1}{s}} + O\left(x^{\frac{1}{2s}}\right)$ (by equation (1.1)) and $\sum_{Q \leq x} 1 = A_{s+1}(x) = \frac{6}{\pi^2} C_{s+1} x^{\frac{1}{s+1}} + O\left(x^{\frac{1}{s+2}}\right)$ (by Lemma 1.2). The theorem is proved. □

Theorem 2.2. *Let $s \geq 1$ an arbitrary but fixed integer. The sum of the s-full hybrid numbers q^sQ not exceeding x is*

$$\sum_{q^sQ \leq x} q^sQ = \frac{6}{\pi^2} \frac{1}{s+1} \left(C_s - 1\right) x^{1+\frac{1}{s}} + O\left(x^{1+\frac{1}{s+1}}\right). \tag{2.2}$$

Proof. Apply Abel's summation to equation (2.1). The theorem is proved. □

Theorem 2.3. *Let $s \geq 1$ an arbitrary but fixed integer. Let us consider the s-full hybrid numbers q^sQ. The sum of the quotients $\frac{Q}{q^s}$ of the hybrid numbers q^sQ not exceeding x is*

$$\sum_{q^sQ \leq x} \frac{Q}{q^s} = \frac{A_s}{s+2} x^{1+\frac{1}{s+1}} + O\left(x^{1+\frac{2(s+1)+1}{2(s+1)}\frac{1}{s+2}+\epsilon}\right), \tag{2.3}$$

where

$$A_s = \frac{6}{\pi^2} C_{s+1} \left(-1 + \prod_p \left(1 + \frac{p - p^{1-\frac{1}{s+1}}}{p - p^{1-\frac{1}{s+1}} + 1} \frac{1}{p^{2s+\frac{s}{s+1}}}\right)\right). \tag{2.4}$$

Proof. Let us consider the s-part $q^s = (q_1 \cdots q_r)^s$, where q_i $(i = 1, \ldots, r)$ are the distinct primes in the prime factorization of q. The number of s-full hybrid numbers q^sQ not exceeding x with the fixed s-part $q^s = (q_1 \cdots q_r)^s$ will be the number of solutions Q to the inequality $q^sQ \leq x$, that is to the inequality $Q \leq \frac{x}{q^s}$. The number of solutions Q to this inequality (s-part q^s fixed) is by Theorem 1.4

$$\begin{aligned}
&\frac{6}{\pi^2} C_{s+1} \prod_{i=1}^{r} \frac{q_i - q_i^{1-\frac{1}{s+1}}}{q_i - q_i^{1-\frac{1}{s+1}} + 1} \frac{x^{\frac{1}{s+1}}}{q^{\frac{s}{s+1}}} + O\left(2^r \frac{x^{\frac{2(s+1)+1}{2(s+1)(s+2)}+\epsilon}}{q^{\frac{s(2(s+1)+1)}{2(s+1)(s+2)}+\epsilon}}\right) \\
= \ & C_q \frac{x^{\frac{1}{s+1}}}{q^{\frac{s}{s+1}}} + O\left(2^r \frac{x^{\frac{2(s+1)+1}{2(s+1)(s+2)}+\epsilon}}{q^{\frac{s(2(s+1)+1)}{2(s+1)(s+2)}+\epsilon}}\right),
\end{aligned} \tag{2.5}$$

where, by sake of simplicity, we put

$$C_q = \frac{6}{\pi^2} C_{s+1} \prod_{i=1}^{r} \frac{q_i - q_i^{1-\frac{1}{s+1}}}{q_i - q_i^{1-\frac{1}{s+1}} + 1} < C_{s+1}$$

and where $\frac{x}{q^s} \geq M$, that is $q \leq \frac{x^{\frac{1}{s}}}{M^{\frac{1}{s}}}$. Note that $2^r \leq q_1 \dots q_r = q$. Therefore equation (2.5) gives

$$\sum_{q^sQ\leq x} \frac{1}{q^{2s}} = \sum_{q\leq \frac{x^{\frac{1}{s}}}{M^{\frac{1}{s}}}} C_q \frac{x^{\frac{1}{s+1}}}{q^{\frac{s}{s+1}+2s}} + \sum_{q\leq \frac{x^{\frac{1}{s}}}{M^{\frac{1}{s}}}} O\left(\frac{x^{\frac{2(s+1)+1}{2(s+1)(s+2)}+\epsilon}}{q^{\frac{s(2(s+1)+1)}{2(s+1)(s+2)}+2s-1+\epsilon}}\right) + F(x)$$

$$= \left(\sum_q \frac{C_q}{q^{2s+\frac{s}{s+1}}}\right) x^{\frac{1}{s+1}} - \left(\sum_{q>\frac{x^{\frac{1}{s}}}{M^{\frac{1}{s}}}} \frac{C_q}{q^{2s+\frac{s}{s+1}}}\right) x^{\frac{1}{s+1}} + O\left(x^{\frac{2(s+1)+1}{2(s+1)(s+2)}+\epsilon}\right)$$

$$+ \quad F(x). \tag{2.6}$$

Let $A_{s+1}(x)$ be the number of $(s+1)$-full numbers not exceeding x (see Lemma 1.2). We have $A_{s+1}(x) \leq cx^{\frac{1}{s+1}}$, where c is a positive constant and $x \geq 1$. Therefore (see (2.6))

$$0 \leq F(x) \leq \sum_{x^{\frac{1}{s}}\geq q>\frac{x^{\frac{1}{s}}}{M^{\frac{1}{s}}}} \frac{1}{q^{2s}} A_{s+1}\left(\frac{x}{q^s}\right) \leq \sum_{q>\frac{x^{\frac{1}{s}}}{M^{\frac{1}{s}}}} \frac{1}{q^{2s}} c \left(\frac{x}{q^s}\right)^{\frac{1}{s+1}}$$

$$\leq \quad cx^{\frac{1}{s+1}} \sum_{q>\frac{x^{\frac{1}{s}}}{M^{\frac{1}{s}}}} \frac{1}{q^{2s+\frac{s}{s+1}}}. \tag{2.7}$$

Abel's summation gives

$$x^{\frac{1}{s+1}} \sum_{q>\frac{x^{\frac{1}{s}}}{M^{\frac{1}{s}}}} \frac{1}{q^{2s+\frac{s}{s+1}}} = o(1). \tag{2.8}$$

Therefore, equations (2.6), (2.7) and (2.8) give

$$\sum_{q^sQ\leq x} \frac{1}{q^{2s}} = \left(\sum_q \frac{C_q}{q^{2s+\frac{s}{s+1}}}\right) x^{\frac{1}{s+1}} + O\left(x^{\frac{2(s+1)+1}{2(s+1)(s+2)}+\epsilon}\right)$$

$$= \quad A_s x^{\frac{1}{s+1}} + O\left(x^{\frac{2(s+1)+1}{2(s+1)(s+2)}+\epsilon}\right), \tag{2.9}$$

where $A_s = \sum_q \frac{C_q}{q^{2s+\frac{s}{s+1}}}$ and the sum runs on all square-free number $q > 1$. Therefore A_s can be written as an infinite product on the primes in the form (2.4).

Finally, equation (2.9) and Abel's summation give

$$\sum_{q^sQ\leq x} \frac{Q}{q^s} = \sum_{q^sQ\leq x} \frac{q^sQ}{q^{2s}} = \frac{A_s}{s+2} x^{1+\frac{1}{s+1}} + O\left(x^{1+\frac{2(s+1)+1}{2(s+1)}\frac{1}{s+2}+\epsilon}\right),$$

that is, equation (2.3). The theorem is proved. □

Theorem 2.4. *Let $s \geq 1$ an arbitrary but fixed integer. Let us consider the s-full hybrid numbers $q^s Q$. The sum of the $(s+1)$-full parts Q of the hybrid numbers $q^s Q$ not exceeding x is*

$$\sum_{q^s Q \leq x} Q = \frac{B_s}{s+2} x^{1+\frac{1}{s+1}} + O\left(x^{1+\frac{2(s+1)+1}{2(s+1)}\frac{1}{s+2}+\epsilon}\right) \qquad (s \geq 2), \tag{2.10}$$

$$\sum_{qQ \leq x} Q = \frac{B_1}{3} x^{1+\frac{1}{2}} + O\left(x^{1+\frac{6}{13}+\epsilon}\right) \qquad (s = 1), \tag{2.11}$$

where

$$B_s = \frac{6}{\pi^2} C_{s+1} \left(-1 + \prod_p \left(1 + \frac{p - p^{1-\frac{1}{s+1}}}{p - p^{1-\frac{1}{s+1}} + 1} \frac{1}{p^{s+\frac{s}{s+1}}}\right)\right), \qquad (s \geq 1). \tag{2.12}$$

Proof. In this case we study the sum

$$\sum_{q^s Q \leq x} \frac{1}{q^s}.$$

If $s \geq 2$ then the proof is the same as the proof of Theorem 2.3.

Suppose that $s = 1$. In this case equation (2.5) becomes

$$C_q \frac{x^{\frac{1}{2}}}{q^{\frac{1}{2}}} + O\left(2^r \frac{x^{\frac{5}{12}+\epsilon}}{q^{\frac{5}{12}+\epsilon}}\right)$$

and, since $2^r \leq q_1 \dots q_r = q$, equation (2.6) becomes

$$\sum_{qQ \leq x} \frac{1}{q} = \sum_{q \leq x^{\frac{1}{13}}} C_q \frac{x^{\frac{1}{2}}}{q^{\frac{1}{2}+1}} + \sum_{q \leq x^{\frac{1}{13}}} O\left(\frac{x^{\frac{5}{12}+\epsilon}}{q^{\frac{5}{12}+\epsilon}}\right) + F(x)$$

$$= \left(\sum_q \frac{C_q}{q^{1+\frac{1}{2}}}\right) x^{\frac{1}{2}} - \left(\sum_{q > x^{\frac{1}{13}}} \frac{C_q}{q^{1+\frac{1}{2}}}\right) x^{\frac{1}{2}} + O\left(x^{\frac{6}{13}+\epsilon}\right)$$

$$+ \quad F(x),$$

where we have applied Abel's summation to the divergent sum $\sum_{q \leq x^{\frac{1}{13}}} \frac{1}{q^{\frac{5}{12}+\epsilon}}$.

Equation (2.7) becomes

$$0 \leq F(x) \leq c x^{\frac{1}{2}} \sum_{q > x^{\frac{1}{13}}} \frac{1}{q^{1+\frac{1}{2}}}.$$

Abel's summation gives

$$x^{\frac{1}{2}} \sum_{q > x^{\frac{1}{13}}} \frac{1}{q^{1+\frac{1}{2}}} = O\left(x^{\frac{6}{13}}\right).$$

The theorem is proved. ∎

Theorem 2.5. *Let $s \geq 1$ an arbitrary but fixed integer. Let us consider the s-full hybrid numbers q^sQ. The sum of the s-parts q^s of the hybrid numbers q^sQ not exceeding x is*

$$\sum_{q^sQ\leq x} q^s = \frac{D_s}{s+1} x^{1+\frac{1}{s}} + O\left(x^{1+\frac{1}{2s}}\right), \tag{2.13}$$

where

$$D_s = \frac{6}{\pi^2}\left(-1+\prod_p\left(1+\frac{1}{(p+1)p^s\left(p^{1+\frac{1}{s}}-1\right)}\right)\right).$$

Proof. Let us consider the $(s+1)$-full part $Q = q_1^{a_1}\cdots q_r^{a_r}$, where q_i $(i=1,\ldots,r)$ are the distinct primes in the prime factorization of Q. The number of s-full hybrid numbers q^sQ not exceeding x with the fixed $(s+1)$-full part $Q = q_1^{a_1}\cdots q_r^{a_r}$ will be the number of solutions q to the inequality $q^sQ \leq x$, that is to the inequality $q \leq \frac{x^{\frac{1}{s}}}{Q^{\frac{1}{s}}}$. The number of solutions q to this inequality ($(s+1)$-full part Q fixed) is by Lemma 1.1

$$\frac{6}{\pi^2}\prod_{i=1}^r \frac{q_i}{q_i+1}\frac{x^{\frac{1}{s}}}{Q^{\frac{1}{s}}} + O\left(2^r\frac{x^{\frac{1}{2s}}}{Q^{\frac{1}{2s}}}\right), \tag{2.14}$$

where $\frac{x^{\frac{1}{s}}}{Q^{\frac{1}{s}}} \geq M$, that is $Q \leq \frac{x}{M^s}$. Now, we have $2^r \leq q_1\cdots q_r \leq Q^{\frac{1}{s+1}}$. Therefore equation (2.14) gives

$$\begin{aligned}\sum_{q^sQ\leq x}\frac{1}{Q} &= \frac{6}{\pi^2}\sum_{Q\leq\frac{x}{M^s}}\prod_{i=1}^r\frac{q_i}{q_i+1}\frac{x^{\frac{1}{s}}}{Q^{\frac{1}{s}}}\frac{1}{Q} + \sum_{Q\leq\frac{x}{M^s}} O\left(\frac{Q^{\frac{1}{s+1}}}{Q^{\frac{1}{2s}}Q}x^{\frac{1}{2s}}\right) + F(x)\\ &= \left(\sum_Q\frac{6}{\pi^2}\prod_{i=1}^r\frac{q_i}{q_i+1}\frac{1}{Q^{1+\frac{1}{s}}}\right)x^{\frac{1}{s}} + O\left(x^{\frac{1}{2s}}\right) - \left(\sum_{Q>\frac{x}{M^s}}\frac{6}{\pi^2}\prod_{i=1}^r\frac{q_i}{q_i+1}\frac{1}{Q^{1+\frac{1}{s}}}\right)x^{\frac{1}{s}}\\ &+ F(x).\end{aligned} \tag{2.15}$$

Now, we have (see (2.15))

$$0 \leq F(x) \leq \sum_{x\geq Q>\frac{x}{M^s}}\frac{1}{Q}\left\lfloor\frac{x^{\frac{1}{s}}}{Q^{\frac{1}{s}}}\right\rfloor \leq \sum_{Q>\frac{x}{M^s}}\frac{1}{Q}\frac{x^{\frac{1}{s}}}{Q^{\frac{1}{s}}}. \tag{2.16}$$

Abel's summation gives

$$x^{\frac{1}{s}}\sum_{Q>\frac{x}{M^s}}\frac{1}{Q^{1+\frac{1}{s}}} = o(1). \tag{2.17}$$

Therefore, equations (2.15), (2.16) and (2.17) give

$$\sum_{q^sQ\leq x}\frac{1}{Q} = D_s x^{\frac{1}{s}} + O\left(x^{\frac{1}{2s}}\right) \tag{2.18}$$

where

$$D_s = \sum_Q \frac{6}{\pi^2} \prod_{i=1}^{r} \frac{q_i}{q_i+1} \frac{1}{Q^{1+\frac{1}{s}}} \tag{2.19}$$

and the sum runs on all $(s+1)$-full number $Q > 1$.

Equation (2.18) and Abel's summation give

$$\sum_{q^sQ \leq x} q^s = \sum_{q^sQ \leq x} \frac{q^sQ}{Q} = \frac{D_s}{s+1} x^{1+\frac{1}{s}} + O\left(x^{1+\frac{1}{2s}}\right),$$

that is, equation (2.13).

Finally, equation (2.19) gives

$$\begin{aligned} D_s &= \sum_Q \frac{6}{\pi^2} \prod_{i=1}^{r} \frac{q_i}{q_i+1} \frac{1}{Q^{1+\frac{1}{s}}} \\ &= \frac{6}{\pi^2}\left(-1 + \prod_p \left(1 + \frac{p}{p+1}\left(\sum_{i=s+1}^{\infty} \frac{1}{(p^i)^{1+\frac{1}{s}}}\right)\right)\right) \\ &= \frac{6}{\pi^2}\left(-1 + \prod_p \left(1 + \frac{1}{(p+1)p^s\left(p^{1+\frac{1}{s}}-1\right)}\right)\right). \end{aligned}$$

The theorem is proved. □

We recall that (Theorem 2.1)

$$H_s(x) = \frac{6}{\pi^2}\left(C_s - 1\right) x^{\frac{1}{s}} + o\left(x^{\frac{1}{s}}\right) \tag{2.20}$$

Let $D_{s,h}(x)$ be the number of s-full hybrid numbers q^sQ not exceeding x such that $\frac{Q}{q^s} < h$, where $0 < h < 1$ is an arbitrary but fixed real number. We have the following theorem

Theorem 2.6. *Let $s \geq 1$ an arbitrary but fixed integer. The following asymptotic formulas hold*

$$D_{s,h}(x) = \frac{6}{\pi^2}\left(C_s - 1\right) x^{\frac{1}{s}} + o\left(x^{\frac{1}{s}}\right) \tag{2.21}$$

$$D_{s,h}(x) \sim H_s(x) \tag{2.22}$$

$$H_s(x) - D_{s,h}(x) = o\left(x^{\frac{1}{s}}\right) = o\left(H_s(x)\right) \tag{2.23}$$

Therefore, almost all s-full hybrid number q^sQ has its $(s+1)$-full part Q very small compared with its s-part q^s, since h can be arbitrarily small.

Proof. Equations (2.22) and (2.23) are an immediate consequence of equations (2.20) and (2.21). Therefore we shall prove equation (2.21). Let us consider the convergent series (see Lemma 1.3)

$$1 + \sum_Q \frac{q_1 \cdots q_r}{(q_1+1)\cdots(q_r+1)} \frac{1}{Q^{\frac{1}{s}}},$$

where the sum runs on all $(s+1)$-full number $Q = q_1^{a_1} \cdots q_r^{a_r}$ and where q_i $(i = 1, \ldots, r)$ are the distinct primes in the prime factorization of Q. The sum of this series is C_s (see Lemma 1.2), since

$$1 + \sum_{Q} \frac{q_1 \cdots q_r}{(q_1+1)\cdots(q_r+1)} \frac{1}{Q^{\frac{1}{s}}} = \prod_p \left(1 + \frac{p}{p+1}\left(\sum_{i=s+1}^{\infty} \frac{1}{(p^i)^{\frac{1}{s}}}\right)\right)$$

$$= \prod_p \left(1 + \frac{1}{(p+1)\left(p^{\frac{1}{s}} - 1\right)}\right) = C_s. \tag{2.24}$$

Let $\epsilon > 0$. By definition of $D_{s,h}(x)$ and $H_s(x)$ we have

$$D_{s,h}(x) \leq H_s(x) = \frac{6}{\pi^2}(C_s - 1)\, x^{\frac{1}{s}} + o\left(x^{\frac{1}{s}}\right) \leq \frac{6}{\pi^2}(C_s - 1)\, x^{\frac{1}{s}} + 2\epsilon x^{\frac{1}{s}}. \tag{2.25}$$

Let us consider a fixed $(s+1)$-full part $Q = q_1^{a_1} \cdots q_r^{a_r}$, where q_i $(i = 1, \ldots, r)$ are the distinct primes in its prime factorization. Let us consider the s-full hybrid numbers $q^s Q$ not exceeding x, that is $q^s Q \leq x$, that is

$$q \leq \frac{x^{\frac{1}{s}}}{Q^{\frac{1}{s}}}, \tag{2.26}$$

where $Q = q_1^{a_1} \cdots q_r^{a_r}$ is fixed. Inequality (2.26) implies that inequality $\frac{Q}{q^s} < h$ holds for all hybrid $q^s Q$ except by a finite number of cases. Since $q \to \infty$ if $x \to \infty$ and Q is fixed. Therefore the number of hybrid numbers $q^s Q$ not exceeding x such that inequality $\frac{Q}{q^s} < h$ holds will be (see Lemma 1.1)

$$\frac{6}{\pi^2} \frac{q_1 \cdots q_r}{(q_1+1)\cdots(q_r+1)} \frac{x^{\frac{1}{s}}}{Q^{\frac{1}{s}}} + o\left(x^{\frac{1}{s}}\right). \tag{2.27}$$

We choose the number B such that

$$\frac{6}{\pi^2} \sum_{Q > B} \frac{q_1 \cdots q_r}{(q_1+1)\cdots(q_r+1)} \frac{1}{Q^{\frac{1}{s}}} < \epsilon. \tag{2.28}$$

Now, equations (2.27), (2.28) and (2.24) give

$$D_{s,h}(x) \geq \left(\frac{6}{\pi^2} \sum_{Q \leq B} \frac{q_1 \cdots q_r}{(q_1+1)\cdots(q_r+1)} \frac{1}{Q^{\frac{1}{s}}}\right) x^{\frac{1}{s}} + o\left(x^{\frac{1}{s}}\right)$$

$$= \left(\frac{6}{\pi^2} \sum_{Q} \frac{q_1 \cdots q_r}{(q_1+1)\cdots(q_r+1)} \frac{1}{Q^{\frac{1}{s}}}\right) x^{\frac{1}{s}}$$

$$- \left(\frac{6}{\pi^2} \sum_{Q > B} \frac{q_1 \cdots q_r}{(q_1+1)\cdots(q_r+1)} \frac{1}{Q^{\frac{1}{s}}}\right) x^{\frac{1}{s}}$$

$$+ o\left(x^{\frac{1}{s}}\right) \geq \frac{6}{\pi^2}(C_s - 1)\, x^{\frac{1}{s}} - \epsilon x^{\frac{1}{s}} - \epsilon x^{\frac{1}{s}}$$

$$= \frac{6}{\pi^2}(C_s - 1))\, x^{\frac{1}{s}} - 2\epsilon x^{\frac{1}{s}}. \tag{2.29}$$

Equations (2.25) and (2.29) give

$$\left| \frac{D_{s,h}(x) - \frac{6}{\pi^2}(C_s - 1)x^{\frac{1}{s}}}{x^{\frac{1}{s}}} \right| = \left| \frac{D_{s,h}(x)}{x^{\frac{1}{s}}} - \frac{6}{\pi^2}(C_s - 1) \right| \leq 2\epsilon,$$

that is, equation (2.21), since ϵ can be arbitrarily small. The theorem is proved. □

Theorem 2.7. *Let $s \geq 1$ an arbitrary but fixed integer. Let us consider the s-full hybrid numbers q^sQ. The sum of the quotients $\frac{q^s}{Q}$ of the hybrid numbers q^sQ not exceeding x is*

$$\sum_{q^sQ \leq x} \frac{q^s}{Q} = \frac{E_s}{s+1} x^{1+\frac{1}{s}} + O\left(x^{1+\frac{1}{2s}}\right) \tag{2.30}$$

where

$$E_s = \frac{6}{\pi^2}\left(-1 + \prod_p \left(1 + \frac{1}{(p+1)p^{2s}\left(p^{2+\frac{1}{s}} - 1\right)}\right)\right)$$

Proof. The proof is the same as the proof of Theorem 2.5. In this case we study the sum

$$\sum_{q^sQ \leq x} \frac{1}{Q^2}.$$

The theorem is proved. □

Remark 2.8. Note that the sums $\sum_{q^sQ \leq x} q^sQ$, $\sum_{q^sQ \leq x} q^s$ and $\sum_{q^sQ \leq x} \frac{q^s}{Q}$ have the same magnitude order $x^{1+\frac{1}{s}}$ (see Theorem 2.2, Theorem 2.5 and Theorem 2.7). Note that the sums $\sum_{q^sQ \leq x} \frac{Q}{q^s}$ and $\sum_{q^sQ \leq x} Q$ have the same smaller magnitude order $x^{1+\frac{1}{s+1}}$ (see Theorem 2.3 and Theorem 2.4). Consequently, for example, the sum of $(s+1)$-full parts Q of the s-full hybrid numbers not exceeding x ($\sum_{q^sQ \leq x} Q$) is negligible compared with the sum of s-parts q^s of the s-full hybrid numbers not exceeding x ($\sum_{q^sQ \leq x} q^s$). However, there is a surprising fact, if we compare Theorem 2.3 with Theorem 2.6. The contribution to the sum $\sum_{q^sQ \leq x} \frac{Q}{q^s}$ of almost all s-full hybrid numbers is smaller than $hD_{s,h(x)} < cx^{\frac{1}{s}}$, where c can be arbitrarily small (Theorem 2.6). However the order of the sum $\sum_{q^sQ \leq x} \frac{Q}{q^s}$ is greater, namely $x^{1+\frac{1}{s+1}}$ (Theorem 2.3). Therefore there is a negligible number $o\left(x^{\frac{1}{s}}\right)$ of s-full hybrid numbers q^sQ responsible of this greater magnitude order and consequently where the quotient $\frac{Q}{q^s}$ must be very big.

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