

## RETRACTED ARTICLE: ON WEIGHTED GENERATOR OF TRIPLE SEQUENCES AND ITS TAUBERIAN CONDITIONS

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**ABSTRACT.** This paper presents a new perspective on the relationship between the  $(\bar{N}, p, q, r)$  method and  $P$ -convergence for triple sequences. Our primary goal is to derive Tauberian conditions that dictate the behavior of the weighted generator sequence  $(z_{lmn})$  in relation to the sequences  $(P_l)$ ,  $(Q_m)$ , and  $(R_n)$ , with the intention of establishing a fresh interpretation. These conditions control the  $O_L$ - and  $O$ -oscillatory properties and establish a connection from  $(\bar{N}, p, q, r)$  summability to  $P$ -convergence, subject to certain restrictions on the weight sequences. Furthermore, we demonstrate that specific cases, such as the  $O_L$ -condition of Landau type and the  $O$ -condition of Hardy type with respect to  $(P_l)$ ,  $(Q_m)$  and  $(R_n)$ , serve as Tauberian conditions for  $(\bar{N}, p, q, r)$  summability under additional conditions. Consequently, our findings encompass all the classical Tauberian theorems, including conditions related to slow decrease and slow oscillation in specific contexts.

### 1. INTRODUCTION

Understanding the origins and evolution of ideas that emerged in the late nineteenth century to expand summability theory from single sequences to multiple sequences is a challenging task. Prior to 1990, when Pringsheim published an article [16] titled "Zur Theorie der zweifach unendlichen Zahlenfolgen" (On the Theory of Doubly Infinite Sequences), it appeared that no researchers were actively working on the theory of multiple sequences. However, Pringsheim introduced the concept of convergence in his paper, known as  $P$ -convergence, which was further explored by Hardy [9] and Bromwich [5] in their comprehensive treatment of double sequences. This development greatly accelerated research on this new type of sequences. Regarding the application of weighted mean methods to double sequences, the work of Baron and Stadtmüller [1] is the earliest known contribution in this area. In [1], researchers scrutinized relations between  $(\bar{N}, p, q)$  method and  $P$ -convergence for double sequences and they found out that necessary conditions for (boundedly)  $P$ -convergence of a double sequence which

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is (boundedly)  $(\bar{N}, p, q, r)$  summable are  $O$ -condition of Hardy type relative to  $P = (P_l)$ ,  $Q = (Q_m)$  and  $R = (R_n)$

$$\begin{aligned} \sup_{mn \in \mathbb{N}} (\Delta_{100} z_{lmn}) &= O\left(\frac{p_l}{P_l}\right), \sup_{ln \in \mathbb{N}} (\Delta_{010} z_{lmn}) = O\left(\frac{q_m}{Q_m}\right) \text{ and } \sup_{lm \in \mathbb{N}} (\Delta_{001} z_{lmn}) \\ &= O\left(\frac{r_n}{R_n}\right) \end{aligned}$$

with regularly varying sequences  $(P_m)$ ,  $(Q_m)$  and  $(R_n)$ . Using non-factorable weights instead of factorable weights used in the preceding one as base, Stadtmüller [18] both generalized  $O_L$ -Tauberian conditions given by Móricz [14] for the  $(C, 1, 1)$  method and indicated that these conditions could be weakened. Following that, Chen and Hsu [6] constituted some Tauberian theorems for double sequences concerning with the implication from  $(\bar{N}, p, q)$  summability to  $P$ -convergence under Landau-type conditions, Schmidt-type slow decrease conditions, and more general ones including the concept of deferred means. Móricz and Stadtmüller [15] reduced assumptions claimed by Stadtmüller [18] and they investigated some conditions needed for (boundedly)  $(\bar{N}, p, q)$  summable double sequences to be (boundedly)  $P$ -convergent using the classes  $\Lambda_z$  and  $\Lambda_\ell$  constructed based on non-factorable weights. Introducing the concepts of double weighted generator sequences in certain senses, Belen [2] pointed out that certain conditions formed via these sequences such as

$$\Delta_{10} V_{mn}^{11(0)} (\Delta_{11}(u)) = O_L\left(\frac{p_m}{P_{m-1}}\right) \text{ and } \Delta_{01} V_{mn}^{11(0)} (\Delta_{11}(u)) = O_L\left(\frac{q_n}{Q_{n-1}}\right)$$

are Tauberian conditions for  $(\bar{N}, p, q)$  method with some additional conditions imposed on the sequences of weights  $(p_m)$  and  $(q_n)$ .

This paper focuses on investigating the relationship between the  $(\bar{N}, p, q, r)$  method and  $P$ -convergence for triple sequences. Our objective is to establish Tauberian conditions that govern the behavior of the weighted generator sequence  $(z_{lmn})$  with respect to  $(P_l)$ ,  $(Q_m)$ , and  $(R_n)$ , in terms of  $O_L$ - and  $O$ -oscillation. We achieve this by exploring the transition from  $(\bar{N}, p, q, r)$  summability to  $P$ -convergence, while imposing certain restrictions on the weight sequences. Within this framework, we demonstrate that specific cases, such as the  $O_L$ -condition of Landau type relative to  $(P_l)$ ,  $(Q_m)$ , and  $(R_n)$ , as well as the  $O$ -condition of Hardy type relative to  $(P_m)$ ,  $(Q_m)$ , and  $(R_n)$ , can be regarded as Tauberian conditions for  $(\bar{N}, p, q, r)$  summability under additional conditions. As a result, these findings encompass the classical Tauberian theorems, including conditions related to slow decrease and slow oscillation in certain contexts.

## 2. PRELIMINARIES

In this section, we will start by providing basic definitions and notations related to double sequences and their weighted means. We will then introduce the weighted generator sequences and the weighted Kronecker identities, which are based on the sequence  $(z_{lmn})$ . We will discuss their weighted means and generator ones in certain senses, as well as the weighted de la Vallée Poussin means for triple sequences.

Additionally, we will introduce the concepts of slow decrease relative to  $(P_l)$ ,  $(Q_m)$ , and  $(R_n)$ , as well as slow oscillation relative to  $(P_m)$ ,  $(Q_m)$ , and  $(R_n)$  for triple sequences. We will demonstrate how a relationship exists between these newly described concepts.

Finally, we will conclude this section by identifying the class  $SV A$ , providing its characterization, and discussing two of its subclasses.

Let  $\mathbb{K} := \mathbb{R}$  or  $\mathbb{K} := \mathbb{C}$  be the field of all real or complex numbers, respectively. Further, let  $\mathbb{N}$  be the set of all nonnegative integers.

The function  $X : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{R}(C)$  allows for the creation of a sequence of triples consisting of real or complex numbers. Initially, Sahiner et al. [17] introduced and examined various concepts related to triple sequences and their statistical convergence. For the further information on triple sequence, refer ([8, 10, 11, 12]).

If the set  $w^3(\mathbb{K})$  denotes the set of all triple sequences, then  $w^3(\mathbb{K})$  together with coordinate-wise addition and scalar multiplication defined by  $((z_{lmn}), (\omega_{lmn})) \rightarrow (z_{lmn} + \omega_{lmn})$  and  $(\lambda, (z_{lmn})) \rightarrow (\lambda z_{lmn})$ ,

$$w^3(\mathbb{K}) = \mathbb{K}^{\mathbb{N} \times \mathbb{N} \times \mathbb{N}} := \{z = (z_{lmn}) \mid z : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{K}, (l, m, n) \rightarrow z(l, m, n) := z_{lmn}\}$$

is a linear space over  $\mathbb{K}$ . Each linear subspace of  $w^3(\mathbb{K})$  is called a triple sequence space. Besides, the following subsets of  $w^3(\mathbb{K})$  are obviously triple sequences spaces:

$$c^3(\mathbb{K}) := \left\{ z = (z_{lmn}) \mid (z_{lmn}) \text{ is convergent in Pringsheim's sense, that is, } \right. \\ \left. P - \lim_{l, m, n \rightarrow \infty} z_{lmn} \text{ has a finite value} \right\},$$

or equivalently,

$$c^3(\mathbb{K}) = \{z = (z_{lmn}) \mid \forall \epsilon > 0, \exists n_0 = n_0(\epsilon) \in \mathbb{N} \text{ such that } |z_{lmn} - \ell| < \epsilon \\ \text{holds for all } l, m, n \geq n_0\},$$

$$\ell_\infty^3(\mathbb{K}) := \{z = (z_{lmn}) \mid \|z_{lmn}\|_\infty = \sup |z_{lmn}| < \infty \text{ for all } l, m, n \in \mathbb{N}\}.$$

These spaces represent the set of all  $P$ -convergent triple sequences and the set of all bounded triple sequences, respectively.

Note that  $(z_{lmn})$  may converge without  $(z_{lmn})$  being a bounded function of  $l, m$  and  $n$ . To put it more explicitly,  $P$ -convergence of  $(z_{lmn})$  may not imply boundedness of its term in contrast to the case in single sequences. For instance, the sequence  $(z_{lmn})$  defined by

$$z_{lmn} = \begin{cases} 5^n & \text{if } l = 1, m, n \in \mathbb{N} \\ 5^{l+m+3} & \text{if } n = 3, l, n \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

is  $P$ -convergent, but it is unbounded.

Some notations that will be used in places throughout this paper are given below.

**Notation 2.1:** Let  $(z_{lmn})$  be a triple sequence.

- The symbol  $z_{lmn} = O(1)$  means that  $|z_{lmn}| \leq H$  for some constant  $H > 0$  and each  $l, m, n \geq n_0$ .
- The symbol  $z_{lmn} = O_L(1)$  means that  $z_{lmn} \geq M$  for some constant  $M > 0$  and each  $l, m, n \geq n_0$ .
- The symbol  $z_{lmn} = o(1)$  means that  $z_{lmn} \rightarrow 0$  as  $l, m, n \rightarrow \infty$ .

Let  $z = (z_{lmn}) \in w^3(\mathbb{K})$  and let  $(p_l), (q_m), (r_n) \in w(\mathbb{R}^{>0})$  such that

$$P_l := \sum_{i=0}^l p_i \rightarrow \infty, Q_m := \sum_{j=0}^m q_j \rightarrow \infty \quad \text{and} \quad R_n := \sum_{k=0}^n r_k \rightarrow \infty \quad \text{as } l, m, n \rightarrow \infty \tag{2.1}$$

where  $w(\mathbb{R}^{>0})$  represents the set of all single sequences of positive real numbers.

The weighted means of  $(z_{lmn})$  determined by the sequences of weights  $(p_l), (q_m)$  and  $(r_n)$  are defined by

$$\begin{aligned} \sigma_{lmn}^{111} &:= \frac{1}{P_l Q_m R_n} \sum_{i=0}^l \sum_{j=0}^m \sum_{r=0}^n p_i q_j r_k z_{ijk}, \\ \sigma_{lmn}^{100} &:= \frac{1}{P_l} \sum_{i=0}^l p_i z_{imn}, \quad \sigma_{lmn}^{010} := \frac{1}{Q_m} \sum_{j=0}^m q_j z_{ljn}, \\ \sigma_{lmn}^{001} &:= \frac{1}{R_n} \sum_{k=0}^n r_k z_{lmk} \end{aligned}$$

for all  $(l, m, n) \in \mathbb{N} \times \mathbb{N}$  and  $P_l Q_m R_n > 0$ .

A triple sequence  $(z_{lmn})$  is called  $(\bar{N}, p, q, r)$  summable to  $\ell$  if  $P\text{-}\lim \sigma_{lmn}^{111} = \ell$ . Similarly,  $(\bar{N}, p, *, r), (\bar{N}, *, q, r)$  and  $(\bar{N}, p, q, *)$  summable sequences are defined via triple sequences  $(\sigma_{lmn}^{10}), (\sigma_{lmn}^{010})$  and  $(\sigma_{lmn}^{001})$ , respectively. It can be easily seen that necessary and sufficient condition for regularity of the  $(\bar{N}, p, q, r)$  method is condition (2.1). To put it another way,  $(z_{lmn}) \in c^3(\mathbb{K}) \cap \ell_\infty^3(\mathbb{K})$  is also  $(\bar{N}, p, q, r)$

summable to same number under condition (2.1). Nevertheless, the opposite of this proposition is not true in general. The question of whether some conditions on the terms  $z_{lmn}$  under which its  $(\bar{N}, p, q, r)$  summability implies its  $P$ -convergence exist comes to mind at this point. The condition  $T\{z_{lmn}\}$  making such a situation possible is called a Tauberian condition. The resulting theorem stating that  $P$ -convergence follows from its  $(\bar{N}, p, q, r)$  summability and  $T\{z_{lmn}\}$  is called a Tauberian Theorem.

In conjunction with the weighted means, there are many special means occurring depends on choosing of the sequences of weights  $(p_l)$ ,  $(q_m)$  and  $(r_n)$ . Included by the weighted means and also commonly used by researchers in literature, some means are listed as follows.

- In case  $p_l = q_m = r_n = 1$ , it leads to the arithmetic means (or called Cesàro means of order  $(1, 1, 1)$ ) of a triple sequence where  $P_l = l + 1$ ,  $Q_m = m + 1$  and  $R_n = n + 1$  for all  $l, m, n \in \mathbb{N}$ .
- In case  $p_l = 1/(l + 1)$ ,  $q_m = 1/(m + 1)$  and  $r_n = 1/(n + 1)$ , it leads to the harmonic means (or called the logarithmic means) of a triple sequence where  $P_l \sim \log l$ ,  $Q_m \sim \log m$  and  $R_n \sim \log n$  for all  $l, m, n \in \mathbb{N}$ .
- In case  $p_l = 1/((l + 1) \log(l + 1))$ ,  $q_m = 1/((m + 1) \log(m + 1))$  and  $r_n = 1/((n + 1) \log(n + 1))$ , it leads to the harmonic means of second order (or called the iterated logarithmic means) of a double sequence where  $P_l \sim \log l$ ,  $Q_m \sim \log m$  and  $R_n \sim \log n$  for all  $l, m, n \in \mathbb{N}$ .
- In case  $p_l = (l + 1)^\alpha$ ,  $q_m = (m + 1)^\beta$  and  $r_n = (n + 1)^\gamma$  with  $\alpha, \beta, \gamma > -1$ , it leads to untitle means of a double sequence where  $P_l \sim (l + 1)^{\alpha+1}/(\alpha + 1)$ ,  $Q_m \sim (m + 1)^{\beta+1}/(\beta + 1)$  and  $r_n \sim (n + 1)^{\gamma+1}/(\gamma + 1)$  for all  $l, m, n \in \mathbb{N}$ .

For  $(z_{lmn}) \in w^3(\mathbb{K})$ , we define

$$\begin{aligned} \Delta_{111}z_{lmn} &:= \Delta_{100}\Delta_{010}\Delta_{001}z_{lmn} = \Delta_{100}(\Delta_{010}\Delta_{001}z_{lmn}) = \Delta_{010}(\Delta_{100}\Delta_{001}z_{lmn}) \\ &= z_{l,mn} - z_{l,m,n-1} - z_{l-1,m,n} - z_{l,m-1,n} + z_{l-1,m-1,n-1}, \end{aligned}$$

$$\Delta_{100}z_{lmn} := z_{lmn} - z_{l,m,n-1},$$

$$\Delta_{010}z_{lmn} := z_{lmn} - z_{l,m-1,n}$$

$$\Delta_{001}z_{lmn} := z_{lmn} - z_{l-1,m,n}$$

for all  $l, m, n \in \mathbb{N}$ .

The weighted Kronecker identities for a sequence  $(z_{lmn})$  are defined by:

$$z_{lmn} - \sigma_{lmn}^{100}(z) = \frac{1}{P_l} \sum_{i=1}^l P_{i-1} \Delta_{100} z_{imn} =: V_{lmn}^{10\sigma(0)}(\Delta_{100}z),$$

$$z_{lmn} - \sigma_{lmn}^{010}(z) = \frac{1}{Q_m} \sum_{j=1}^m Q_{j-1} \Delta_{010} u_{ljn} =: V_{lmn}^{010^{(0)}}(\Delta_{010}z)$$

and

$$z_{lmn} - \sigma_{lmn}^{001}(z) = \frac{1}{R_n} \sum_{j=1}^m R_{j-1} \Delta_{001} z_{lmk} =: V_{lmn}^{010^{(0)}}(\Delta_{001}z)$$

for all  $l, m, n \in \mathbb{N}$ .

The sequence  $(V_{lmn}^{100^{(0)}}(\Delta_{100}z))$  is the  $(\bar{N}, p, *, *)$  mean of  $(P_{l-1} \Delta_{100} z_{lmn})$  and called the weighted generator sequence of  $(z_{lmn})$  in the sense  $(1, 0, 0)$ . Concor-  
dantly, the sequence  $(V_{lmn}^{010^{(0)}}(\Delta_{010}z))$  is the  $(\bar{N}, *, q, *)$  mean of  $(Q_{m-1} \Delta_{010} z_{lmn})$   
and  $(V_{lmn}^{001^{(0)}}(\Delta_{001}z))$  is the  $(\bar{N}, *, *, r)$  mean of  $(R_{n-1} \Delta_{001} z_{lmn})$  and called the  
weighted generator sequence of  $(z_{lmn})$  in the sense  $(0, 1, 0)$  and  $(0, 0, 1)$  respec-  
tively. More generally, the triple weighted Kronecker identity for a sequence  
 $(z_{lmn})$  are defined via  $(V_{lmn}^{100^{(0)}}(\Delta_{100}z))$ ,  $(V_{lmn}^{010^{(0)}}(\Delta_{010}z))$  and  $(V_{lmn}^{001^{(0)}}(\Delta_{001}z))$   
as follows:

$$z_{lmn} - \sigma_{lmn}^{111}(z) = V_{lmn}^{111^{(0)}}(\Delta_{111}z)$$

where

$$\begin{aligned} V_{lmn}^{111^{(0)}}(\Delta_{111}z) &:= V_{lmn}^{100^{(0)}}(\Delta_{100}z) + V_{lmn}^{010^{(0)}}(\Delta_{010}z) + V_{lmn}^{001^{(0)}}(\Delta_{001}z) \\ &\quad - \frac{1}{P_l Q_m R_n} \sum_{i=1}^l \sum_{j=1}^m \sum_{k=1}^n P_{i-1} Q_{j-1} R_{k-1} \Delta_{111} z_{ijk} \end{aligned}$$

for all  $l, m, n \in \mathbb{N}$ .

The sequence  $(V_{lmn}^{111^{(0)}}(\Delta_{111}z))$  is called the weighted generator sequence of  
 $(z_{lmn})$  in the sense  $(1, 1, 1)$ .

In addition, the  $(\bar{N}, p, q, r)$  means of order  $v \in \mathbb{N}$  of sequences  $(z_{lmn})$  and  
 $(V_{lmn}^{111^{(0)}}(\Delta_{111}z))$  are defined by

$$\sigma_{lmn}^{111^{(v)}}(z) := \begin{cases} \frac{1}{P_l Q_m R_n} \sum_{i=0}^l \sum_{j=1}^m \sum_{k=1}^n P_i Q_j R_k \sigma_{ijk}^{111^{(v-1)}}(z) & \text{if } v \geq 1 \\ z_{lmn} & \text{if } v = 0 \end{cases}$$

and

$$V_{lmn}^{111^{(v)}}(\Delta_{111}z) := \begin{cases} \frac{1}{P_l Q_m R_n} \sum_{i=0}^l \sum_{j=0}^m \sum_{k=0}^n p_i q_j r_k V_{ijk}^{111^{(v-1)}}(\Delta_{111}z) & \text{if } v \geq 1 \\ V_{lmn}^{111^{(0)}}(\Delta_{111}z) & \text{if } v = 0 \end{cases}$$

respectively.

Throughout this paper,  $\sigma_{lmn}^{111}$  and  $V_{lmn}^{111(1)}$  will be used instead of  $\sigma_{lmn}^{111(1)}(z)$  and  $V_{lmn}^{111(1)}(\Delta_{111}z)$  for the sake of convenience.

The weighted de la Vallée Poussin means of  $(z_{lmn})$  are defined by:

$$\tau_{lmn}^{\mu\eta\chi}(z) := \frac{1}{(P_\mu - P_l)(Q_\eta - Q_m)(R_\chi - R_n)} \sum_{i=l+1}^{\mu} \sum_{j=m+1}^{\eta} \sum_{k=n+1}^{\chi} p_i q_j r_k z_{ijk},$$

$$\mu > l, \eta > m, \chi > n$$

and

$$\tau_{\mu\eta\chi}^{lmn}(z) := \frac{1}{(P_l - P_\mu)(Q_m - Q_\eta)(R_m - R_\chi)} \sum_{i=\mu+1}^l \sum_{j=\eta+1}^m \sum_{k=\chi+1}^n p_i q_j r_k z_{ijk},$$

$$\mu < l, \eta < m, \chi < n$$

for all  $l, m, n \in \mathbb{N}$ .

At present, we define concepts of slow decrease relative to  $(P_l)$ ,  $(Q_m)$  and  $(R_n)$  and slow oscillation relative to  $(P_l)$ ,  $(Q_m)$  and  $(R_n)$  for triple sequences. In the wake of defining of that, we mention a relation between them.

A sequence  $(z_{lmn}) \in w^3(\mathbb{R})$  is said to be slowly decreasing relative to both  $(P_l)$ ,  $(Q_m)$  and  $(R_n)$  provided that

$$\lim_{\lambda \rightarrow 1^+} \liminf_{l, m, n \rightarrow \infty} \min_{\substack{P_l \leq P_i \leq \lambda P_l \\ Q_m \leq Q_j \leq \kappa \\ R_n \leq R_k \leq \delta}} (z_i - z_{lmn}) \geq 0; \quad (2.2)$$

that is, for each  $\epsilon > 0$  there exist  $n_0 = n_0(\epsilon) \in \mathbb{N}$ ,  $\lambda = \lambda(\epsilon) > 1$ ,  $\kappa = \kappa(\epsilon)$  and  $\delta = \delta(\epsilon) > 1$  such that

$$z_{ijk} - z_{lmn} \geq -\epsilon \text{ whenever } n_0 \leq l \leq i, n_0 \leq m \leq j, n_0 \leq n \leq k$$

$$\text{and } 1 \leq \frac{P_i}{P_l} \leq \lambda, 1 \leq \frac{Q_j}{Q_m} \leq \kappa, 1 \leq \frac{R_k}{R_n} \leq \delta.$$

Condition (2.2) is equivalent to

$$\lim_{\lambda \rightarrow 1^+} \liminf_{l, m, n \rightarrow \infty} \min_{\substack{P_l \leq P_i \leq \lambda P_l \\ Q_m \leq Q_j \leq \kappa \\ R_n \leq R_k \leq \delta}} (u_{mn} - u_{ij}) \geq 0.$$

The set of all slowly decreasing sequences relative to both  $(P_l)$ ,  $(Q_m)$  and  $(R_n)$  is denoted by  $\mathfrak{SD}_{(P, Q, R)}$ .

A sequence  $(z_{lmn}) \in w^3(\mathbb{C})$  is said to be slowly oscillating relative to both  $(P_l)$ ,  $(Q_m)$  and  $(R_n)$  provided that

$$\lim_{\lambda \rightarrow 1^+} \lim_{l, m, n \rightarrow \infty} \sup_{\substack{P_l \leq P_i \leq \lambda P_l \\ Q_m \leq Q_j \leq \kappa Q_m \\ R_n \leq R_k \leq \delta R_n}} |z_{ijk} - z_{lmn}| = 0 \quad (2.3)$$

that is, for each  $\epsilon > 0$  there exist  $n_0 = n_0(\epsilon) \in \mathbb{N}$ ,  $\lambda = \lambda(\epsilon) > 1$ ,  $\kappa = \kappa(\epsilon) > 1$  and  $\delta = \delta(\epsilon) > 1$  such that

$$|z_{ijk} - z_{lmn}| \leq \epsilon \text{ whenever } n_0 \leq l \leq i, n_0 \leq m \leq j, n_0 \leq n \leq k, \\ \text{and } 1 \leq \frac{P_i}{P_l} \leq \lambda, 1 \leq \frac{Q_j}{Q_m} \leq \kappa, 1 \leq \frac{R_k}{R_n} \leq \delta.$$

Condition (2.3) is equivalent to

The set of all slowly oscillating sequences relative to both  $(P_l)$ ,  $(Q_m)$  and  $(R_n)$  is denoted by  $\mathfrak{SO}_{(P, Q, R)}$ .

A sequence  $(z_{lmn}) \in w^3(\mathbb{R})$  is said to be slowly decreasing relative to  $(P_l)$  provided that

$$\lim_{\lambda \rightarrow 1^+} \liminf_{l, m, n \rightarrow \infty} \min_{P_l \leq P_i \leq \lambda P_l} (z_{imn} - z_{lmn}) \geq 0, \quad (2.4)$$

or equivalently,

$$\lim_{\lambda \rightarrow 1^-} \liminf_{l, m, n \rightarrow \infty} \min_{\lambda P_l < P_i \leq P_l} (z_{lmn} - z_{imn}) \geq 0.$$

The set of all slowly decreasing sequences relative to  $(P_l)$  is denoted by  $\mathfrak{SD}_{(P)}$ . Besides, a sequence  $(z_{lmn})$  is said to be slowly decreasing relative to  $(P_l)$  in the strong sense if (2.4) is satisfied with

$$\min_{\substack{P_l \leq P_i \leq \lambda P_l \\ Q_m \leq Q_j \leq \kappa Q_m \\ R_n \leq R_k \leq \delta R_n}} (z_{ijk} - z_{ljk}) \quad \text{instead of} \quad \min_{P_l \leq P_i \leq \lambda P_l} (z_{imn} - z_{lmn}).$$

The set of all slowly decreasing sequences relative to  $(P_l)$  in the strong sense is denoted by  $\mathfrak{SDSI}(P)$ .

A sequence  $(z_{lmn}) \in w^3(\mathbb{C})$  is said to be slowly oscillating relative to  $(P_l)$  provided that

$$\lim_{\lambda \rightarrow 1^+} \limsup_{l, m, n \rightarrow \infty} \max_{P_l \leq P_i \leq \lambda P_l} |z_{imn} - z_{lmn}| = 0 \quad (2.5)$$

or equivalently,



$$\lim_{\lambda \rightarrow 1^-} \limsup_{l, m, n \rightarrow \infty} \max_{\lambda P_l < P_i \leq P_l} |z_{lmn} - z_{imn}| = 0$$

Besides, a sequence  $(z_{lmn})$  is said to be slowly oscillating relative to  $(P_l)$  in the strong sense if (2.5) is satisfied with

$$\begin{aligned} & \max_{\substack{P_l \leq P_i \leq \lambda P_l \\ Q_m \leq Q_j \leq \kappa Q_m \\ R_n \leq R_k \leq \delta R_n}} |z_{ijk} - z_{ljk}| \quad \text{instead of} \quad \max_{P_l \leq P_i \leq \lambda P_l} |z_{imn} - z_{lmn}|. \end{aligned}$$

The set of all slowly oscillating sequences relative to  $(P_l)$  in the strong sense is denoted by  $\mathfrak{SO}\mathfrak{ST}(P)$ .

Similarly, the sets  $\mathfrak{SO}_{(Q)}$ ,  $\mathfrak{SO}\mathfrak{ST}(Q)$ ,  $\mathfrak{SO}_{(Q)}$ , and  $\mathfrak{SO}\mathfrak{ST}(Q)$  can be analogously defined.

Indeed, for all large enough  $l, m$  and  $n$ , that is,  $l, m, n \geq n_0, \lambda > 1, \kappa > 1$  and  $\delta > 1$ , we find

$$\begin{aligned} & \min_{\substack{P_l \leq P_i \leq \lambda P_l \\ Q_m \leq Q_j \leq \kappa Q_m \\ R_n \leq R_k \leq \delta R_n}} (z_{ijk} - z_{lmn}) = \min_{\substack{P_l \leq P_i \leq \lambda P_l \\ Q_m \leq Q_j \leq \kappa Q_m \\ R_n \leq R_k \leq \delta R_n}} (z_{ijk} - u_{ljk} + u_{ljk} - u_{lmn}) \\ & \geq \min_{\substack{P_l \leq P_i \leq \lambda P_l \\ Q_m \leq Q_j \leq \kappa Q_m \\ R_m \leq R_j \leq \kappa R_n}} (z_{ijk} - u_{ljk}) + \min_{Q_m \leq j \leq \kappa Q_m} (u_{ljk} - z_{lmn}). \end{aligned} \quad (2.6)$$

Taking  $\liminf$  and limit of both sides of (2.6) as  $l, m, n \rightarrow \infty$  and  $\lambda, \kappa, \delta \rightarrow 1^+$  respectively, we get that the terms on right-hand side of (2.6) are greater than 0. Therefore, we reach  $(z_{lmn}) \in \mathfrak{SO}_{(P, Q)}$ .

It can be also said that if  $(z_{lmn}) \in \mathfrak{SO}\mathfrak{ST}(Q) \cap \mathfrak{SO}_{(P)} \cap \mathfrak{SO}_{(R)}$ , then  $(z_{lmn}) \in \mathfrak{SO}_{(P, Q, R)}$ . Similarly, if  $(z_{lmn}) \in \mathfrak{SO}\mathfrak{ST}(P) \cap \mathfrak{SO}_{(Q)}$  or  $(z_{lmn}) \in \mathfrak{SO}\mathfrak{ST}(Q) \cap \mathfrak{SO}_{(P)} \cap \mathfrak{SO}_{(R)}$ , then  $(z_{lmn}) \in \mathfrak{SO}_{(P, Q, R)}$ .

In the remainder of this section, we mention the classes including all positive sequences  $(p_l)$  whose sequence of partial sum  $(P_l)$  is

- a regularly varying sequence of positive index,
- a rapidly varying sequence of index  $\infty$  (see [3] for more details).

Let  $p = (p_l)$  be a sequence that satisfies  $(p_l) = (P_l - P_{l-1})$ , where  $P_{-1} = 0$  and  $P_l \neq 0$  for all  $l \in \mathbb{N}$ .

(i) A sequence  $(P_m)$  of positive numbers is said to be regularly varying if for all  $\lambda > 0$

$$\lim_{m \rightarrow \infty} \frac{P_{\lambda m}}{P_m} = \varphi(\lambda) \text{ exists,}$$

where  $0 < \varphi(\lambda) < \infty$  (cf. [4]).

In spite of the fact that this definition has been used by many authors as a starting point for studies including regularly varying sequences, these sequences possess quite useful properties, the most important of which is probably the following characterization theorem.

**Theorem 2.2** [13] (Characterization Theorem) The following statements are equivalent:

- (i) A sequence  $(P_l)$  of positive numbers is a regularly varying sequence.
- (ii) There exists a real number  $\alpha > 0$  such that  $\varphi(\lambda) = \lambda^\alpha$  for all  $\lambda > 0$ .
- (iii) The sequence  $(P_l)$  has the form  $P_l = (l + 1)^\alpha L(l)$  for  $l \geq 0$  with constant  $\alpha \geq 0$  and slowly varying function  $L(\cdot)$  on  $(0, \infty)$ , i.e. the function  $L(\cdot)$  is positive, measurable, and satisfies

$$\lim_{t \rightarrow \infty} \frac{L(\lambda t)}{L(t)} = 1 \text{ for all } \lambda > 0.$$

To emphasize such  $\alpha$ , a sequence  $(P_l)$  is called a regularly varying sequence of positive index  $\alpha$ , as well. Note that a regularly varying sequence of index  $\alpha = 0$  corresponds to a slowly varying sequence.

The set of all sequences of positive numbers  $(p_l)$  with  $p_0 > 0$  satisfying (c) is denoted by  $SVA_{\text{reg}(\alpha)}$ .

Here, it is useful to give the following implication proved by Bojanic and Seneta [4].

**Lemma 2.3** [4] If a sequence  $P = (P_l)$  of positive numbers is regularly varying, then  $\frac{P_{l-1}}{P_l} \rightarrow 1$  as  $l \rightarrow \infty$

- (ii) A sequence  $(P_l)$  of positive numbers is said to be rapidly varying of index  $\infty$  if

$$\frac{P_{\lambda l}}{P_l} \rightarrow \begin{cases} 0 & \text{if } 0 < \lambda < 1, \\ 1 & \text{if } \lambda = 1, \\ \infty & \text{if } \lambda > 1 \end{cases} \text{ as } m \rightarrow \infty. \tag{2.7}$$

The set of all sequences of positive numbers  $(p_m)$  satisfying (2.7) is denoted by  $SVA_{\text{rap}}$ . In addition, it may be written conventionally as  $\lambda^\infty$  because the right hand side of (2.7) is the limit of  $\lambda^\alpha$  as  $\alpha \rightarrow \infty$ .

## 3. AUXILIARY RESULTS

In this section, we state and prove some auxiliary results to be benefited in the proofs of our main results. The next lemma presents two representations of difference between general terms of  $(z_{lmn})$  and  $(\sigma_{lmn}^{111})$  and it can be proved when it is make convenient modification in Lemma 1.2 proved by Fekete [?].

**Lemma 3.1:** Let  $z = (z_{lmn})$  be a triple sequence.

(i) For sufficiently large  $\mu > l$ ,  $\eta > m$  and  $\chi > n$ , we have

$$\begin{aligned} & z_{lmn} - \sigma_{lmn}^{111} \\ &= \frac{P_\mu Q_\eta R_\chi}{(P_\mu - P_l)(Q_\eta - Q_m)(R_\chi - R_n)} (\sigma_{\mu\eta\chi}^{111} - \sigma_{\mu mn}^{111} - \sigma_{l\eta n}^{111} + \sigma_{lmn}^{111}) \\ &+ \frac{P_\mu}{P_\mu - P_l} (\sigma_{\mu mn}^{111} - \sigma_{lmn}^{111}) + \frac{Q_\eta}{Q_\eta - Q_m} (\sigma_{l\eta n}^{111} - \sigma_{lmn}^{111}) + \frac{R_\chi}{Q_\chi - Q_n} (\sigma_{lm\chi}^{111} - \sigma_{lmn}^{111}) \\ &- \frac{1}{(P_\mu - P_l)(Q_\eta - Q_m)(R_\chi - Q_n)} \sum_{i=l+1}^{\mu} \sum_{j=m+1}^{\eta} \sum_{k=n+1}^{\chi} p_i q_j r_k (z_{ijk} - z_{lmn}). \end{aligned} \quad (3.1)$$

(ii) For sufficiently large  $\mu < l$ ,  $\eta < m$  and  $\chi < n$ , we have

$$\begin{aligned} & z_{lmn} - \sigma_{lmn}^{111} \\ &= \frac{P_\mu Q_\eta R_\chi}{(P_l - P_\mu)(Q_m - Q_\eta)(R_n - R_\chi)} (\sigma_{lmn}^{111} - \sigma_{\mu mn}^{111} - \sigma_{l\eta n}^{111} + \sigma_{\mu\eta\chi}^{111}) \\ &+ \frac{P_\mu}{P_l - P_\mu} (\sigma_{\mu mn}^{111} - \sigma_{lmn}^{111}) + \frac{Q_\eta}{Q_m - Q_\eta} (\sigma_{l\eta n}^{111} - \sigma_{lmn}^{111}) + \frac{R_\chi}{Q_n - Q_\chi} (\sigma_{lm\chi}^{111} - \sigma_{lmn}^{111}) \\ &- \frac{1}{(P_l - P_\mu)(Q_m - Q_\eta)(R_n - Q_\chi)} \sum_{i=\mu+1}^l \sum_{j=\eta+1}^m \sum_{k=\chi+1}^n p_i q_j r_k (z_{lmn} - z_{ijk}). \end{aligned} \quad (3.2)$$

Interpreted differently from the statement given in Lemma 3.1, the following lemma points out two representations of difference between the general terms of  $(z_{lmn})$  and  $(\sigma_{lmn}^{111})$  via the weighted de la Vallée Poussin means of  $(z_{lmn})$ .

**Lemma 3.2:** Let  $z = (z_{lmn})$  be a triple sequence.

(i) For sufficiently large  $\mu > l$ ,  $\eta > m$  and  $\chi > n$  we have

$$\begin{aligned} z_{lmn} - \sigma_{lmn}^{111} &= \frac{P_\mu Q_\eta R_\chi}{P_l Q_m R_n} (\sigma_{\mu\eta\chi}^{111} - \tau_{lmn}^{\mu\eta\chi}) - \frac{P_\mu}{P_l} (\sigma_{\mu mn}^{111} - \tau_{lmn}^{\mu\eta\chi}) \\ &- \frac{Q_\eta}{Q_m} (\sigma_{l\eta n}^{111} - \tau_{lmn}^{\mu\eta\chi}) - \frac{R_\chi}{Q_n} (\sigma_{lm\chi}^{111} - \tau_{lmn}^{\mu\eta\chi}) - (\tau_{lmn}^{\mu\eta\chi} - z_{lmn}). \end{aligned}$$

(ii) For sufficiently large  $\mu < l$ ,  $\eta < m$  and  $\chi < n$ , we have

$$\begin{aligned}
 z_{lmn} - \sigma_{lmn}^{111} &= -\frac{P_\mu Q_\eta R_\chi}{P_l Q_m R_n} (\tau_{\mu\eta\chi}^{lmn} - \sigma_{\mu\eta\chi}^{111}) + \frac{P_\mu}{P_l} (\tau_{\mu\eta\chi}^{lmn} - \sigma_{\mu mn}^{111}) \\
 &\quad + \frac{Q_\eta}{Q_m} (\tau_{\mu\eta\chi}^{lmn} - \sigma_{l\eta n}^{111}) + \frac{R_\eta}{R_n} (\tau_{\mu\eta\chi}^{lmn} - \sigma_{ln\chi}^{111}) + (z_{lmn} - \tau_{\mu\eta\chi}^{lmn}).
 \end{aligned}$$

**Proof:** (i) For sufficiently large  $\mu > l$ ,  $\eta > m$  and  $\chi > n$  we have from definition of the weighted de la Vallée Poussin means of  $(z_{lmn})$

$$\begin{aligned}
 &\tau_{lmn}^{\mu\eta\chi}(z) \\
 &= \frac{1}{(P_\mu - P_l)(Q_\eta - Q_m)(R_\chi - R_n)} \sum_{i=l+1}^{\mu} \sum_{j=m+1}^{\eta} \sum_{k=n+1}^{\chi} p_i q_j r_k z_{ijk} \\
 &= \frac{1}{(P_\mu - P_l)(Q_\eta - Q_m)(R_\chi - R_n)} \left[ \left( \sum_{i=0}^{\mu} - \sum_{i=0}^l \right) \left( \sum_{j=0}^{\eta} - \sum_{j=0}^m \right) \left( \sum_{k=0}^{\chi} - \sum_{k=0}^n \right) \right] p_i q_j r_k z_{ijk} \\
 &= \frac{1}{(P_\mu - P_l)(Q_\eta - Q_m)(R_\chi - R_n)} \left[ \sum_{i=0}^{\mu} \sum_{j=0}^{\eta} \sum_{k=0}^{\chi} - \sum_{i=0}^{\mu} \sum_{j=0}^{\eta} \sum_{k=0}^n - \sum_{i=0}^{\mu} \sum_{j=0}^m \sum_{k=0}^{\chi} + \sum_{i=0}^{\mu} \sum_{i=0}^m \sum_{j=0}^n \right. \\
 &\quad \left. - \sum_{i=0}^l \sum_{j=0}^{\eta} \sum_{k=0}^{\chi} + \sum_{i=0}^l \sum_{j=0}^{\eta} \sum_{k=0}^n + \sum_{i=0}^l \sum_{j=0}^m \sum_{k=0}^{\chi} - \sum_{i=0}^l \sum_{j=0}^m \sum_{k=0}^n \right] p_i q_j r_k z_{ijk} \\
 &= \frac{P_\mu Q_\eta R_\chi}{(P_\mu - P_l)(Q_\eta - Q_m)(R_\chi - R_n)} \sigma_{\mu\eta\chi}^{111} - \frac{P_n}{(P_\mu - P_l)(Q_\eta - Q_m)(R_\chi - R_n)} \sigma_{\mu\eta n}^{111} \\
 &\quad - \frac{P_l Q_m R_n}{(P_\mu - P_l)(Q_\eta - Q_m)(R_\chi - R_n)} \sigma_{l\eta n}^{111} + \frac{P_\mu}{(P_\mu - P_l)(Q_\eta - Q_m)(R_\chi - R_n)} \sigma_{l\mu n}^{111}. \tag{3.3}
 \end{aligned}$$

It follows from Eq. (3.3) that

$$\begin{aligned}
 &-\sigma_{mn}^{11}(u) \\
 &= \frac{P_\mu Q_\eta R_\chi}{P_l Q_m R_n} \sigma_{\mu\eta\chi}^{11} - \frac{P_\mu}{P_l} \sigma_{\mu mn}^{111} - \frac{Q_\eta}{Q_m} \sigma_{l\eta n}^{111} - \frac{R_\eta}{R_n} \sigma_{ln\chi}^{111} - \frac{(P_\mu - P_l)(Q_\eta - Q_m)(R_\chi - R_n)}{P_l Q_m R_n} \tau_{lmn}^{\mu\eta\chi} \\
 &= \frac{P_\mu Q_\eta R_\chi}{P_l Q_m R_n} \sigma_{\mu\eta\chi}^{111} - \frac{P_\mu}{P_l} \sigma_{\mu mn}^{111} - \frac{Q_\eta}{Q_m} \sigma_{l\eta n}^{111} - \frac{R_\eta}{R_n} \sigma_{ln\chi}^{111} - \left[ \frac{P_\mu Q_\eta R_\chi}{P_l Q_m R_n} - \frac{P_\mu}{P_l} - \frac{Q_\eta}{Q_m} - \frac{R_\chi}{R_n} + 1 \right] \tau_{lmn}^{\mu\eta\chi} \\
 &= \frac{P_\mu Q_\eta R_\chi}{P_l Q_m R_n} (\sigma_{\mu\eta\chi}^{111} - \tau_{lmn}^{\mu\eta\chi}) - \frac{P_\mu}{P_l} (\sigma_{\mu mn}^{111} - \tau_{lmn}^{\mu\eta\chi}) - \frac{Q_\eta}{Q_m} (\sigma_{l\eta n}^{111} - \tau_{lmn}^{\mu\eta\chi}) \\
 &\quad - \frac{R_\chi}{R_n} (\sigma_{ln\chi}^{111} - \tau_{lmn}^{\mu\eta\chi}) - \tau_{lmn}^{\mu\eta\chi}. \tag{3.4}
 \end{aligned}$$

If we implicate in the term  $u_{mn}$  to both sides of equality (3.4), then we can observe that the proof of (i) is completed.

(ii) The proof is similar to that of part (i) of Lemma 3.2. So, we omit it.

To be also commented as a result of Lemma 3.1, the mentioned representations below give the difference between the weighted de la Vallée Poussin means and the weighted means of  $(u_{mn})$ .

**Lemma 3.3:** Let  $z = (z_{lmn})$  be a triple sequence.

(i) For sufficiently large  $\mu > l$ ,  $\eta > m$  and  $\chi > n$ , we have

$$\begin{aligned} & \tau_{lmn}^{\mu\eta\chi}(z) - \sigma_{lmn}^{111}(z) \\ &= \frac{P_\mu Q_\eta R_\chi}{(P_\mu - P_l)(Q_\eta - Q_m)(R_\chi - R_n)} (\sigma_{\mu\eta\chi}^{111} - \sigma_{\mu mn}^{111} - \sigma_{m\eta n}^{111} - \sigma_{lm\chi}^{111} + \sigma_{lmn}^{111}) \\ & \quad + \frac{P_\mu}{P_\mu - P_l} (\sigma_{\mu mn}^{111} - \sigma_{lmn}^{111}) + \frac{Q_\eta}{Q_\eta - Q_m} (\sigma_{l\eta n}^{111} - \sigma_{lmn}^{111}) \\ & \quad + \frac{R_\chi}{R_\chi - R_n} (\sigma_{lm\chi}^{111} - \sigma_{lmn}^{111}). \end{aligned}$$

(ii) For sufficiently large  $\mu < l$ ,  $\eta < m$  and  $\chi < n$ , we have

$$\begin{aligned} & \tau_{\mu\eta\chi}^{lmn}(z) - \sigma_{lmn}^{111}(z) \\ &= \frac{P_\mu Q_\eta R_\chi}{(P_\mu - P_l)(Q_\eta - Q_m)(R_\chi - R_n)} (\sigma_{lmn}^{111} - \sigma_{\mu mn}^{111} - \sigma_{l\eta n}^{111} - \sigma_{lm\chi}^{111} + \sigma_{\mu\eta\chi}^{111}) \\ & \quad + \frac{P_\mu}{P_m - P_\mu} (\sigma_{mn}^{111} - \sigma_{\mu mn}^{111}) + \frac{Q_\eta}{Q_m - Q_\eta} (\sigma_{lmn}^{111} - \sigma_{l\eta n}^{111}) \\ & \quad + \frac{R_\eta}{R_n - R_\chi} (\sigma_{lmn}^{111} - \sigma_{lm\chi}^{111}). \end{aligned}$$

In [7], Çanak pointed out that a generator sequence  $(V_m^{(0)}(\Delta z))$  converges under some proper conditions. Getting inspired the one for single sequences, we demonstrate under which conditions the weighted generator sequence of  $(z_{lmn})$  in sense  $(1, 1, 1)$  is  $P$ -convergent.

**Lemma 3.4:** For a sequence  $z = (z_{lmn}) \in w^3(\mathbb{R})$  and  $(p_l), (q_m), (r_n), \in SVA_{\text{reg}(\alpha)}$ , let the hypotheses

$$\begin{aligned} & \lim_{\lambda \rightarrow 1^+} \liminf_{l, m, n \rightarrow \infty} (\sigma_{\mu mn}^{111} - \tau_{lmn}^{\mu\eta\chi}) \geq 0, \quad \lim_{\kappa \rightarrow 1^+} \liminf_{l, m, n \rightarrow \infty} (\sigma_{l\eta n}^{111} - \tau_{lmn}^{\mu\eta\chi}) \geq 0, \\ & \lim_{\delta \rightarrow 1^+} \liminf_{l, m, n \rightarrow \infty} (\sigma_{lm\chi}^{111} - \tau_{lmn}^{\mu\eta\chi}) \geq 0, \\ & \lim_{\lambda, \kappa, \delta \rightarrow 1^+} \limsup_{l, m, n \rightarrow \infty} (\sigma_{\mu\eta\chi}^{111} - \tau_{lmn}^{\mu\eta\chi}) \leq 0 \end{aligned} \quad (3.5)$$

for  $\mu > l$ ,  $\eta > m$  and  $\chi > n$  and

$$\begin{aligned} & \lim_{\lambda \rightarrow 1^-} \limsup_{\mu, \eta, \chi \rightarrow \infty} (\sigma_{\mu mn}^{111} - \tau_{\mu\eta\chi}^{lmn}) \leq 0, \quad \lim_{\kappa \rightarrow 1^-} \limsup_{\mu, \eta, \chi \rightarrow \infty} (\sigma_{l\eta n}^{111} - \tau_{\mu\eta\chi}^{lmn}) \leq 0, \\ & \lim_{\lambda, \kappa \rightarrow 1^-} \liminf_{\mu, \eta, \chi \rightarrow \infty} (\sigma_{\mu\eta\chi}^{111} - \tau_{\mu\eta\chi}^{lmn}) \geq 0 \end{aligned} \quad (3.6)$$

$\mu < l, \eta < m$  and  $\chi < n$  hold.

If conditions

$$\lim_{\lambda, \kappa \rightarrow 1^+} \liminf_{l, m, n \rightarrow \infty} (\tau_{lmn}^{\mu\eta} - z_{lmn}) \geq 0 \tag{3.7}$$

and

$$\lim_{\lambda, \kappa \rightarrow 1^-} \liminf_{\mu, \eta, \chi \rightarrow \infty} (z_{lmn} - \tau_{\mu\eta\chi}^{lmn}) \geq 0 \tag{3.8}$$

are satisfied, then the weighted generator sequence  $(V_{lmn}^{111(0)}(\Delta_{111}u))$  is  $P$ -convergent to 0 .

**Proof:** Suppose that (3.5)-(3.8) are satisfied. To prove that

$$P - \lim_{l, m, n \rightarrow \infty} (V_{lmn}^{111(0)}(\Delta_{111}z)) = 0,$$

we investigate  $z_{lmn} - \sigma_{lmn}^{111}(z)$  in two cases  $\mu > l, \eta > m, \chi > n$  and  $\mu < l, \eta < m, \chi < n$ .

Firstly, we consider the case  $\mu > l, \eta > m$  and  $\chi > n$ . Putting

$$\mu = \operatorname{argmin} \{P_i \geq \lambda P_l\} = \min \{i > m : P_i \geq \lambda P_l\}$$

,

$$\eta = \operatorname{argmin} \{Q_j \geq \kappa Q_m\} = \min \{j > m : Q_j \geq \kappa Q_m\}$$

and

$$\chi = \operatorname{argmin} \{R_k \geq \delta R_n\} = \min \{k > n : R_k \geq \delta R_n\}$$

with  $\lambda, \kappa, \delta > 1$ , we observe by Lemma 2.3 that  $p_l/P_l \rightarrow 0$  ,  $q_m/Q_m$  and  $r_n/R_n \rightarrow 0$  as  $l, m, n \rightarrow \infty$  and hence

$$\frac{P_\mu}{P_l} \geq \frac{\lambda P_l}{P_l} = \lambda \quad \text{and} \quad \frac{P_\mu}{P_l} = \frac{P_{\mu-1}}{P_l} + \frac{p_\mu}{P_l} \leq \lambda + \frac{p_\mu}{P_\mu} \frac{P_\mu}{P_l} = \lambda(1 + o(1)), \tag{3.9}$$

$$\frac{Q_\eta}{Q_m} \geq \frac{\kappa Q_m}{Q_m} = \kappa \quad \text{and} \quad \frac{Q_\eta}{Q_m} = \frac{Q_{\eta-1}}{Q_m} + \frac{q_\eta}{Q_m} \leq \kappa + \frac{q_\eta}{Q_\eta} \frac{Q_\eta}{Q_m} = \kappa(1 + o(1)) \tag{3.10}$$

and

$$\frac{R_\chi}{R_n} \geq \frac{\delta R_n}{R_n} = \delta \quad \text{and} \quad \frac{R_\chi}{R_n} = \frac{R_{\chi-1}}{R_n} + \frac{q_\chi}{R_n} \leq \delta + \frac{q_\chi}{R_\chi} \frac{R_\chi}{R_n} = \delta(1 + o(1)) \tag{3.11}$$

which mean

$$\frac{P_\mu}{P_l} \rightarrow \lambda, \quad \frac{Q_\eta}{Q_m} \rightarrow \kappa \quad \text{and} \quad \frac{R_\chi}{R_n} \rightarrow \delta \quad \text{as } l, m, n \rightarrow \infty \quad (3.12)$$

respectively. If we get lim sup of both sides of identity (3.1) as  $l, m, n \rightarrow \infty$ , then we arrive by (3.12)

$$\begin{aligned} & \limsup_{l, m, n \rightarrow \infty} (z_{lmn} - \sigma_{lmn}^{111}) \\ & \leq \limsup_{l, m, n \rightarrow \infty} \frac{P_\mu Q_\eta R_\chi}{P_m Q_m} \limsup_{l, m, n \rightarrow \infty} (\sigma_{\mu\eta\chi}^{111} - \tau_{lmn}^{\mu\eta\chi}) + \limsup_{l \rightarrow \infty} \frac{P_\mu}{P_l} \limsup_{l, m, n \rightarrow \infty} (- (\sigma_{\mu mn}^{111} - \tau_{lmn}^{\mu\eta\chi})) \\ & \quad + \limsup_{m \rightarrow \infty} \frac{Q_\eta}{Q_m} \limsup_{l, m, n \rightarrow \infty} (- (\sigma_{l\eta n}^{111} - \tau_{lmn}^{\mu\eta\chi})) + \limsup_{l, m, n \rightarrow \infty} (- (\tau_{lmn}^{\mu\eta\chi} - z_{lmn})) \\ & = \lambda \kappa \delta \limsup_{l, m, n \rightarrow \infty} (\sigma_{\mu\eta\chi}^{111} - \tau_{lmn}^{\mu\eta\chi}) - \lambda \liminf_{l, m, n \rightarrow \infty} (\sigma_{\mu mn}^{111} - \tau_{lmn}^{\mu\eta\chi}) - \kappa \liminf_{l, m, n \rightarrow \infty} (\sigma_{l\eta n}^{111} - \tau_{lmn}^{\mu\eta\chi}) \\ & \quad - \delta \liminf_{l, m, n \rightarrow \infty} (\sigma_{lm\chi}^{111} - \tau_{lmn}^{\mu\eta\chi}) - \liminf_{l, m, n \rightarrow \infty} (\tau_{lmn}^{\mu\eta\chi} - z_{lmn}). \end{aligned}$$

If we get limit on both sides of last inequality as  $\lambda, \kappa, \delta \rightarrow 1^+$ , we obtain

$$\begin{aligned} \limsup_{l, m, n \rightarrow \infty} (z_{lmn} - \sigma_{lmn}^{111}) & \leq \lim_{\lambda, \kappa, \delta \rightarrow 1^+} \limsup_{l, m, n \rightarrow \infty} (\sigma_{\mu\eta\chi}^{111} - \tau_{lmn}^{\mu\eta\chi}) - \lim_{\lambda \rightarrow 1^+} \liminf_{l, m, n \rightarrow \infty} (\sigma_{\mu mn}^{111} - \tau_{lmn}^{\mu\eta\chi}) \\ & \quad - \lim_{\kappa \rightarrow 1^+} \liminf_{l, m, n \rightarrow \infty} (\sigma_{l\eta n}^{111} - \tau_{lmn}^{\mu\eta\chi}) - \lim_{\delta \rightarrow 1^+} \liminf_{l, m, n \rightarrow \infty} (\sigma_{lm\chi}^{111} - \tau_{lmn}^{\mu\eta\chi}) \\ & \quad - \lim_{\lambda, \kappa, \delta \rightarrow 1^+} \liminf_{l, m, n \rightarrow \infty} (\tau_{lmn}^{\mu\eta\chi} - z_{lmn}). \end{aligned}$$

From hypotheses in (3.5) and (3.7), it follows that

$$\limsup_{l, m, n \rightarrow \infty} (z_{lmn} - \sigma_{lmn}^{111}) \leq 0. \quad (3.13)$$

On the other hand, we consider the case  $\tilde{\mu} < l$ ,  $\tilde{\eta} < m$  and  $\tilde{\chi} < n$ . Putting

$$\tilde{\mu} = \operatorname{argmax} \{P_l \geq \lambda P_i\} = \max \{l > i : P_m \geq \lambda P_i\},$$

$$\tilde{\eta} = \operatorname{argmax} \{Q_m \geq \kappa Q_j\} = \max \{m > j : Q_m \geq \kappa Q_j\}$$

and

$$\tilde{\chi} = \operatorname{argmax} \{R_n \geq \delta R_k\} = \max \{n > k : R_n \geq \delta R_k\}$$

with  $\lambda, \kappa, \delta > 1$ , we observe by Lemma 2.3 that  $p_{\tilde{\mu}}/P_{\tilde{\mu}} \rightarrow 0$ ,  $q_{\tilde{\eta}}/Q_{\tilde{\eta}} \rightarrow 0$  and  $r_{\tilde{\chi}}/R_{\tilde{\chi}} \rightarrow 0$  as  $\tilde{\mu}, \tilde{\eta}, \tilde{\chi} \rightarrow \infty$  and hence

$$\begin{aligned} \frac{P_{\tilde{\mu}}}{P_l} & \leq \frac{1}{\lambda} \quad \text{and} \quad \frac{P_{\tilde{\mu}}}{P_l} = \frac{P_{\tilde{\mu}+1}}{P_m} - \frac{p_{\tilde{\mu}+1}}{P_l} > \frac{1}{\lambda} - \frac{p_{\tilde{\mu}+1}}{P_{\tilde{\mu}+1}} \frac{P_{\tilde{\mu}+1}}{P_{\tilde{\mu}}} \frac{P_{\tilde{\mu}}}{P_l} \\ & = \frac{1}{\lambda} - \frac{1}{\lambda} o(1)(1 + o(1)) = \frac{1}{\lambda} (1 - o(1)), \end{aligned} \quad (3.14)$$

$$\begin{aligned} \frac{Q_{\tilde{\eta}}}{Q_m} &\leq \frac{1}{\kappa} \text{ and } \frac{Q_{\tilde{\eta}}}{Q_m} = \frac{Q_{\tilde{\eta}+1}}{Q_m} - \frac{q_{\tilde{\eta}+1}}{Q_m} > \frac{1}{\kappa} - \frac{q_{\tilde{\eta}+1}}{Q_{\tilde{\eta}+1}} \frac{Q_{\tilde{\eta}+1}}{Q_{\tilde{\eta}}} \frac{Q_{\tilde{\eta}}}{Q_m} \\ &= \frac{1}{\kappa} - \frac{1}{\kappa} o(1)(1 + o(1)) = \frac{1}{\kappa} (1 - o(1)) \end{aligned} \quad (3.15)$$

and

$$\begin{aligned} \frac{R_{\tilde{\chi}}}{R_n} &\leq \frac{1}{\delta} \text{ and } \frac{R_{\tilde{\chi}}}{R_n} = \frac{R_{\tilde{\chi}+1}}{R_n} - \frac{q_{\tilde{\chi}+1}}{R_n} > \frac{1}{\delta} - \frac{q_{\tilde{\chi}+1}}{R_{\tilde{\chi}+1}} \frac{R_{\tilde{\chi}+1}}{R_{\tilde{\chi}}} \frac{R_{\tilde{\chi}}}{R_n} \\ &= \frac{1}{\delta} - \frac{1}{\delta} o(1)(1 + o(1)) = \frac{1}{\delta} (1 - o(1)) \end{aligned} \quad (3.16)$$

which mean

$$\frac{P_{\tilde{\mu}}}{P_l} \rightarrow \frac{1}{\lambda}, \quad \frac{Q_{\tilde{\eta}}}{Q_m} \rightarrow \frac{1}{\kappa} \text{ and } \frac{R_{\tilde{\chi}}}{R_n} \rightarrow \frac{1}{\delta} \quad \text{as } \tilde{\mu}, \tilde{\eta}, \tilde{\chi} \rightarrow \infty,$$

respectively. If we get  $\liminf$  of both sides of identity (3.2) as  $\tilde{\mu}, \tilde{\eta}, \tilde{\chi} \rightarrow \infty$ , then we arrive by (3.12)

$$\begin{aligned} &\liminf_{l,m,n \rightarrow \infty} (z_{lmn} - \sigma_{lmn}^{111}) \\ &\geq \liminf_{\tilde{\mu}, \tilde{\eta}, \tilde{\chi} \rightarrow \infty} \frac{P_{\tilde{\mu}} Q_{\tilde{\eta}} R_{\tilde{\chi}}}{P_l Q_m R_n} \liminf_{\tilde{\mu}, \tilde{\eta}, \tilde{\chi} \rightarrow \infty} (\sigma_{\tilde{\mu}\tilde{\eta}}^{111} - \tau_{\mu\eta}^{mn}) + \liminf_{\tilde{\mu} \rightarrow \infty} \frac{P_{\tilde{\mu}}}{P_l} \liminf_{\tilde{\mu}, \tilde{\eta}, \tilde{\chi} \rightarrow \infty} (- (\sigma_{\tilde{\mu}mn}^{111} - \tau_{\mu\eta\chi}^{lmn})) \\ &+ \liminf_{\tilde{\eta} \rightarrow \infty} \frac{Q_{\tilde{\eta}}}{Q_m} \liminf_{\tilde{\mu}, \tilde{\eta}, \tilde{\chi} \rightarrow \infty} (- (\sigma_{l\tilde{\eta}n}^{111} - \tau_{\mu\eta\chi}^{lmn})) + \liminf_{\tilde{\chi} \rightarrow \infty} \frac{R_{\tilde{\chi}}}{R_n} \liminf_{\tilde{\mu}, \tilde{\eta}, \tilde{\chi} \rightarrow \infty} (- (\sigma_{lm\tilde{\chi}}^{111} - \tau_{\mu\eta\chi}^{lmn})) \\ &+ \liminf_{l,m,n \rightarrow \infty} (z_{lmn} - \tau_{\mu\eta\chi}^{lmn}) \\ &= \frac{1}{\lambda\kappa\delta} \liminf_{\tilde{\mu}, \tilde{\eta}, \tilde{\chi} \rightarrow \infty} (\sigma_{\tilde{\mu}\tilde{\eta}\tilde{\chi}}^{111} - \tau_{\mu\eta\chi}^{lmn}) - \frac{1}{\lambda} \limsup_{\tilde{\mu}, \tilde{\eta}, \tilde{\chi} \rightarrow \infty} (\sigma_{\tilde{\mu}mn}^{111} - \tau_{\mu\eta\chi}^{lmn}) - \frac{1}{\kappa} \limsup_{\tilde{\mu}, \tilde{\eta}, \tilde{\chi} \rightarrow \infty} (\sigma_{l\tilde{\eta},n}^{111} - \tau_{\mu\eta\chi}^{lmn}) \\ &- \frac{1}{\delta} \limsup_{\tilde{\mu}, \tilde{\eta}, \tilde{\chi} \rightarrow \infty} (\sigma_{lm\tilde{\chi}}^{111} - \tau_{\mu\eta\chi}^{lmn}) + \liminf_{l,m,n \rightarrow \infty} (z_{lmn} - \tau_{\mu\eta\chi}^{lmn}). \end{aligned}$$

If we get limit on both sides of last inequality as  $\lambda, \kappa, \delta \rightarrow 1^-$ , we obtain

$$\begin{aligned} &\liminf_{l,m,n \rightarrow \infty} (z_{lmn} - \sigma_{lmn}^{111}) \\ &\geq \lim_{\lambda, \kappa, \delta \rightarrow 1^-} \liminf_{\tilde{\mu}, \tilde{\eta}, \tilde{\chi} \rightarrow \infty} (\sigma_{\tilde{\mu}\tilde{\eta}\tilde{\chi}}^{111} - \tau_{\mu\eta\chi}^{lmn}) - \lim_{\lambda \rightarrow 1^-} \limsup_{\tilde{\mu}, \tilde{\eta}, \tilde{\chi} \rightarrow \infty} (\sigma_{\tilde{\mu}mn}^{111} - \tau_{\mu\eta\chi}^{lmn}) \\ &- \lim_{\kappa \rightarrow 1^-} \limsup_{\tilde{\mu}, \tilde{\eta}, \tilde{\chi} \rightarrow \infty} (\sigma_{l\tilde{\eta}n}^{111} - \tau_{\mu\eta\chi}^{lmn}) + \liminf_{l,m,n \rightarrow \infty} (z_{lmn} - \tau_{\mu\eta\chi}^{lmn}). \end{aligned}$$

From hypotheses in (3.6) and (3.8), it follows that

$$\liminf_{l,m,n \rightarrow \infty} (z_{lmn} - \sigma_{lmn}^{111}(z)) \geq 0. \quad (3.17)$$

If we combine inequalities (3.13) with (3.17), we reach



$$\lim_{l,m,n \rightarrow \infty} (z_{lmn} - \sigma_{lmn}^{111}(z)) = 0$$

which means by the triple weighted Kronecker identity that  $\left( V_{lmn}^{111(0)}(\Delta_{111}z) \right)$  is  $P$ -convergent to 0 .

#### 4. MAIN RESULTS FOR THE $(\bar{N}, p, q, r)$ SUMMABLE TRIPLE SEQUENCES

In this section, we introduce several Tauberian theorems concerning triple sequences where the concept of  $P$ -convergence can be derived from the summability condition of  $(\bar{N}, p, q, r)$ , given certain requirements on the weighted generator sequence  $\left( V_{lmn}^{111(0)}(\Delta_{111}z) \right)$  in terms of slow decrease or slow oscillation. Additionally, certain conditions are imposed on the sequences  $(p_l)$ ,  $(q_m)$ , and  $(r_n)$ . Subsequently, we provide some related corollaries based on these outcomes.

**Theorem 4.1** Let  $(z_{lmn}) \in \ell_{\infty}^3(\mathbb{R})$  and  $(p_l), (q_m), (r_n) \in SVA_{\text{reg}(\alpha)}$ . If a sequence  $(z_{lmn})$  is  $(\bar{N}, p, q, r)$  summable to a number  $\ell$  and

$$\begin{aligned} & \left( V_{lmn}^{111(0)}(\Delta_{111}z) \right) \in \mathfrak{SD}_{(P)} \cap \mathfrak{SD}_{(Q)} \cap \mathfrak{SD}_{(R)} \cap \mathfrak{SDST}_{(Q)} \\ & \text{or} \\ & \left( V_{lmn}^{111(0)}(\Delta_{111}z) \right) \in \mathfrak{S}_{(P)} \cap \mathfrak{SD}_{(Q)} \cap \mathfrak{SD}_{(R)} \cap \mathfrak{SDST}_{(P)}, \\ & \text{or} \\ & \left( V_{lmn}^{111(0)}(\Delta_{111}z) \right) \in \mathfrak{S}_{(P)} \cap \mathfrak{SD}_{(Q)} \cap \mathfrak{SD}_{(R)} \cap \mathfrak{SDST}_{(R)}, \end{aligned} \quad (4.1)$$

then  $(z_{lmn})$  is  $P$ -convergent to  $\ell$ .

**Proof:** Without loss of generality, assume that  $(z_{lmn}) \in \ell_{\infty}^3(\mathbb{R})$  is  $(\bar{N}, p, q, r)$  summable to  $\ell$  and  $\left( V_{lmn}^{111(0)}(\Delta_{111}z) \right) \in \mathfrak{SD}_{(P)} \cap \mathfrak{SD}_{(Q)} \cap \mathfrak{SD}_{(R)} \cap \mathfrak{SDST}_{(P)}$ . To prove that  $(z_{lmn})$  is  $P$ -convergent to  $\ell$ , we demonstrate that  $\left( V_{lmn}^{111(0)}(\Delta_{111}z) \right)$  is  $P$ -convergent to 0 . Because  $(\sigma_{lmn}^{111}(z))$  is  $P$ -convergent to  $\ell$  and the  $(\bar{N}, p, q, r)$  method is regular under the boundedness condition of  $(z_{lmn})$ , we attain that  $\left( \sigma_{lmn}^{111(2)}(z) \right)$  is also  $P$ -convergent to the same number. It follows from the triple weighted Kronecker identity that  $\left( V_{lmn}^{111(1)}(\Delta_{111}z) \right)$  is  $P$ -convergent to 0 . For  $\mu > l, \eta >$  and  $\chi > n$  , if we replace  $z_{lmn}$  by  $V_{lmn}^{111(0)}(\Delta_{111}z)$  in Lemma 3.1(i), we obtain

$$\begin{aligned}
 & V_{lmn}^{111(0)} - V_{lmn}^{111(1)} \\
 &= \frac{P_\mu Q_\eta R_\chi}{(P_\mu - P_l)(Q_\eta - Q_m)(R_\chi - R_n)} \left( V_{\mu\eta\chi}^{111(1)} - V_{\mu mn}^{111(1)} - V_{l\eta n}^{111(1)} - V_{lm\chi}^{111(1)} + V_{lmn}^{111(1)} \right) \\
 &+ \frac{P_\mu}{P_\mu - P_l} \left( V_{\mu mn}^{111(1)} - V_{lmn}^{111(1)} \right) + \frac{Q_\eta}{Q_\eta - Q_m} \left( V_{l\eta n}^{111(1)} - V_{lmn}^{111(1)} \right) \\
 &+ \frac{R_\chi}{R_\chi - R_m} \left( V_{lm\chi}^{111(1)} - V_{lmn}^{111(1)} \right) \\
 &- \frac{1}{(P_\mu - P_l)(Q_\eta - Q_m)(R_\chi - R_n)} \sum_{i=l+1}^{\mu} \sum_{j=m+1}^{\eta} \sum_{k=n+1}^{\chi} p_i q_j r_k \left( V_{ijk}^{111(0)} - V_{lmn}^{111(0)} \right) \\
 &= \frac{P_\mu Q_\eta R_\chi}{(P_\mu - P_l)(Q_\eta - Q_m)(R_\chi - R_n)} \left( V_{\mu\eta\chi}^{111(1)} - V_{\mu mn}^{111(1)} - V_{l\eta n}^{111(1)} - V_{lm\chi}^{111(1)} + V_{lmn}^{111(1)} \right) \\
 &+ \frac{P_\mu}{P_\mu - P_l} \left( V_{\mu mn}^{111(1)} - V_{lmn}^{111(1)} \right) + \frac{Q_\eta}{Q_\eta - Q_m} \left( V_{l\eta n}^{111(1)} - V_{lmn}^{111(1)} \right) \\
 &+ \frac{R_\chi}{R_\chi - R_m} \left( V_{lm\chi}^{111(1)} - V_{lmn}^{111(1)} \right) \\
 &- \frac{1}{(P_\mu - P_l)(Q_\eta - Q_m)(R_\chi - R_n)} \sum_{i=l+1}^{\mu} \sum_{j=m+1}^{\eta} \sum_{k=n+1}^{\chi} p_i q_j r_k \left( V_{ijk}^{111(0)} - V_{ljk}^{11(0)} \right) \\
 &- \frac{1}{(P_\mu - P_l)(Q_\eta - Q_m)(R_\chi - R_n)} \sum_{i=l+1}^{\mu} \sum_{j=m+1}^{\eta} \sum_{k=n+1}^{\chi} p_i q_j r_k \left( V_{ijk}^{111(0)} - V_{lmn}^{111(0)} \right) \\
 &\leq \frac{P_\mu Q_\eta R_\chi}{(P_\mu - P_l)(Q_\eta - Q_m)(R_\chi - R_n)} \left( V_{\mu\eta\chi}^{111(1)} - V_{\mu mn}^{111(1)} - V_{l\eta n}^{111(1)} - V_{lm\chi}^{111(1)} + V_{lmn}^{111(1)} \right) \\
 &+ \frac{P_\mu}{P_\mu - P_l} \left( V_{\mu mn}^{111(1)} - V_{lmn}^{111(1)} \right) + \frac{Q_\eta}{Q_\eta - Q_m} \left( V_{l\eta n}^{111(1)} - V_{lmn}^{111(1)} \right) \\
 &+ \frac{R_\chi}{R_\chi - R_m} \left( V_{lm\chi}^{111(1)} - V_{lmn}^{111(1)} \right) \\
 &- \min_{\substack{m \leq i \leq \mu \\ n \leq j \leq \eta}} \left( V_{ij}^{11(0)} - V_{mj}^{11(0)} \right) - \min_{n \leq j \leq \eta} \left( V_{mj}^{11(0)} - V_{mn}^{11(0)} \right). \tag{4.2}
 \end{aligned}$$

Putting  $\mu = \operatorname{argmin} \{P_i \geq \lambda P_l\}$ ,  $\eta = \operatorname{argmin} \{Q_j \geq \kappa Q_m\}$  and  $\chi = \operatorname{argmin} \{Q_k \geq \delta R_n\}$  with  $\lambda, \kappa, \delta > 1$ , we can observe inequalities (3.9), (3.10) and, accordingly, (3.11).

Grounding that  $V_{lmn}^{111(1)}(\Delta_{111}u) \rightarrow 0$  as  $l, m, n \rightarrow \infty$ , we could remark  $V_{ijk}^{111(1)}(\Delta_{111}z) - V_{lmn}^{111(1)}(\Delta_{111}z) \rightarrow 0$  for  $i = l$  or  $\mu$ ,  $j = m$  or  $\eta$  and  $k = n$  or  $\chi$  as  $l, m, n \rightarrow \infty$ . Since the sequences  $(P_l)$ ,  $(Q_m)$  and  $(R_n)$  are strictly increasing sequences, we attain from (4.2) that

$$\begin{aligned}
& V_{lmn}^{111(0)} - V_{lmn}^{111(1)} \\
& \leq \frac{P_\mu Q_\eta R_\chi}{(P_\mu - P_m)(Q_\eta - Q_n)} \left( V_{\mu\eta\chi}^{111(1)} - V_{\mu mn}^{111(1)} - V_{l\eta n}^{111(1)} - V_{lm\chi}^{111(1)} + V_{lmn}^{111(1)} \right) \\
& + \frac{P_\mu}{P_\mu - P_l} \left( V_{\mu mn}^{111(1)} - V_{lmn}^{111(1)} \right) + \frac{Q_\eta}{Q_\eta - Q_m} \left( V_{l\eta n}^{111(1)} - V_{lmn}^{111(1)} \right) \\
& + \frac{R_\chi}{R_\chi - R_m} \left( V_{lm\chi}^{111(1)} - V_{lmn}^{111(1)} \right) \\
& - \min_{\substack{P_m \leq P_i \leq \lambda P_m \\ Q_n \leq Q_j \leq \kappa Q_n}} \left( V_{ij}^{11(0)} - V_{mj}^{11(0)} \right) - \min_{Q_n \leq Q_j \leq \kappa Q_n} \left( V_{mj}^{11(0)} - V_{mn}^{11(0)} \right) \\
& \leq \frac{\lambda\kappa\delta + \lambda(\kappa - 1) + (\lambda - 1)\kappa + \lambda(\delta - 1) + (\lambda - 1)\delta + \delta(\kappa - 1) + (\delta - 1)\kappa}{(\lambda - 1)(\kappa - 1)(\delta - 1)} (1 + o(1))o(1) \\
& \tag{4.3}
\end{aligned}$$

$$\begin{aligned}
& - \min_{\substack{P_m \leq P_i \leq \lambda P_l \\ Q_m \leq Q_j \leq \kappa Q_m \\ R_n \leq R_j \leq \kappa R_n}} \left( V_{ijk}^{111(0)} - V_{ljn}^{111(0)} \right) - \min_{R_n \leq Q_j \leq \kappa Q_m} \left( V_{ljm}^{111(0)} - V_{lmn}^{111(0)} \right)
\end{aligned}$$

where

$$\begin{aligned}
\frac{P_\mu}{P_\mu - P_l} &= \frac{P_\mu/P_l}{P_\mu/P_l - 1} \leq \frac{\lambda}{(\lambda - 1)}(1 + o(1)), \\
\frac{Q_\eta}{Q_\eta - Q_m} &= \frac{Q_\eta/Q_m}{Q_\eta/Q_m - 1} \leq \frac{\kappa}{(\kappa - 1)}(1 + o(1))
\end{aligned}$$

and

$$\frac{R_\chi}{R_\chi - R_n} = \frac{R_\chi/R_n}{R_\chi/R_n - 1} \leq \frac{\delta}{(\delta - 1)}(1 + o(1)).$$

If we get  $\limsup$  on both sides of inequality (4.3) as  $l, m, n \rightarrow \infty$ , then we reach for any  $\lambda, \kappa, \delta > 1$

$$\begin{aligned}
\limsup_{l, m, n \rightarrow \infty} \left( V_{lmn}^{111(0)} - V_{lmn}^{111(1)} \right) &\leq - \liminf_{l, m, n \rightarrow \infty} \min_{\substack{P_l \leq P_i \leq \lambda P_l \\ Q_m \leq Q_j \leq \kappa Q_m \\ R_n \leq R_k \leq \kappa R_n}} \left( V_{ijk}^{111(0)} - V_{ljk}^{111(0)} \right) \\
&\quad - \liminf_{l, m, n \rightarrow \infty} \min_{Q_m \leq Q_j \leq \kappa Q_m} \left( V_{ljk}^{111(0)} - V_{lmn}^{111(0)} \right).
\end{aligned}$$

If we get limit on both sides of last inequality as  $\lambda, \kappa, \delta \rightarrow 1^+$ , then we find

$$\limsup_{l, m, n \rightarrow \infty} \left( V_{lmn}^{111(0)} - V_{lmn}^{111(1)} \right) \leq 0 \tag{4.4}$$

due to  $\left(V_{lmn}^{111(0)}(\Delta_{111}z)\right) \in \mathfrak{SD}_{(P)} \cap \mathfrak{SD}_{(Q)} \cap \mathfrak{SD}_{(R)} \cap \mathfrak{SD}\mathfrak{S}\mathfrak{T}_{(P)}$ . Following a similar procedure to above for  $\tilde{\mu} < l$ ,  $\tilde{\eta} < m$  and  $\tilde{\chi} < n$ , if we replace  $z_{lmn}$  by  $V_{lmn}^{111(0)}(\Delta_{111}z)$  in Lemma 3.1(ii), we obtain

$$\begin{aligned}
 & V_{lmn}^{111(0)} - V_{lmn}^{111(1)} \\
 &= \frac{P_{\tilde{\mu}}Q_{\tilde{\eta}}R_{\tilde{\chi}}}{(P_l - P_{\tilde{\mu}})(Q_m - Q_{\tilde{\eta}})(R_n - P_{\tilde{\chi}})} \left( V_{lmn}^{111(1)} - V_{\tilde{\mu}mn}^{111(1)} - V_{l\tilde{\eta}n}^{111(1)} - V_{lm\tilde{\chi}}^{111(1)} + V_{\tilde{\mu}\tilde{\eta}\tilde{\chi}}^{111(1)} \right) \\
 & \quad + \frac{P_{\tilde{\mu}}}{P_l - P_{\tilde{\mu}}} \left( V_{lmn}^{111(1)} - V_{\tilde{\mu}mn}^{111(1)} \right) + \frac{Q_{\tilde{\eta}}}{Q_m - Q_{\tilde{\eta}}} \left( V_{lmn}^{111(1)} - V_{l\tilde{\eta}n}^{111(1)} \right) \\
 & \quad + \frac{R_{\tilde{\chi}}}{R_n - R_{\tilde{\chi}}} \left( V_{lmn}^{111(1)} - V_{lm\tilde{\chi}}^{111(1)} \right) \\
 & \quad + \frac{1}{(P_l - P_{\tilde{\mu}})(Q_m - Q_{\tilde{\eta}})(R_n - P_{\tilde{\chi}})} \sum_{i=\tilde{\mu}+1}^l \sum_{j=\tilde{\eta}+1}^m \sum_{k=\tilde{\chi}+1}^n p_i q_j r_k \left( V_{lmn}^{111(0)} - V_{ijk}^{111(0)} \right) \\
 &= \frac{P_{\tilde{\mu}}Q_{\tilde{\eta}}R_{\tilde{\chi}}}{(P_l - P_{\tilde{\mu}})(Q_m - Q_{\tilde{\eta}})(R_n - P_{\tilde{\chi}})} \left( V_{lmn}^{111(1)} - V_{\tilde{\mu}mn}^{111(1)} - V_{l\tilde{\eta}n}^{111(1)} - V_{lm\tilde{\chi}}^{111(1)} + V_{\tilde{\mu}\tilde{\eta}\tilde{\chi}}^{111(1)} \right) \\
 & \quad + \frac{P_{\tilde{\mu}}}{P_l - P_{\tilde{\mu}}} \left( V_{lmn}^{111(1)} - V_{\tilde{\mu}mn}^{111(1)} \right) + \frac{Q_{\tilde{\eta}}}{Q_m - Q_{\tilde{\eta}}} \left( V_{lmn}^{111(1)} - V_{l\tilde{\eta}n}^{111(1)} \right) \\
 & \quad + \frac{R_{\tilde{\chi}}}{R_n - R_{\tilde{\chi}}} \left( V_{lmn}^{111(1)} - V_{lm\tilde{\chi}}^{111(1)} \right) \\
 & \quad + \frac{m}{(P_l - P_{\tilde{\mu}})(Q_m - Q_{\tilde{\eta}})(R_n - P_{\tilde{\chi}})} \sum_{i=\tilde{\mu}+1}^l \sum_{j=\tilde{\eta}+1}^m \sum_{k=\tilde{\chi}+1}^n p_i q_j r_k \left( V_{lmn}^{111(0)} - V_{ljn}^{111(0)} \right) \\
 & \quad + \frac{1}{(P_l - P_{\tilde{\mu}})(Q_m - Q_{\tilde{\eta}})(R_n - P_{\tilde{\chi}})} \sum_{i=\tilde{\mu}+1}^l \sum_{j=\tilde{\eta}+1}^m \sum_{k=\tilde{\chi}+1}^n p_i q_j r_k \left( V_{ljk}^{111(0)} - V_{ijk}^{111(0)} \right) \\
 & \geq \frac{P_{\tilde{\mu}}Q_{\tilde{\eta}}R_{\tilde{\chi}}}{(P_l - P_{\tilde{\mu}})(Q_m - Q_{\tilde{\eta}})(R_n - P_{\tilde{\chi}})} \left( V_{lmn}^{111(1)} - V_{\tilde{\mu}mn}^{111(1)} - V_{l\tilde{\eta}n}^{111(1)} - V_{lm\tilde{\chi}}^{111(1)} + V_{\tilde{\mu}\tilde{\eta}\tilde{\chi}}^{111(1)} \right) \\
 & \quad + \frac{P_{\tilde{\mu}}}{P_l - P_{\tilde{\mu}}} \left( V_{lmn}^{111(1)} - V_{\tilde{\mu}mn}^{111(1)} \right) + \frac{Q_{\tilde{\eta}}}{Q_m - Q_{\tilde{\eta}}} \left( V_{lmn}^{111(1)} - V_{l\tilde{\eta}n}^{111(1)} \right) \\
 & \quad + \frac{R_{\tilde{\chi}}}{R_n - R_{\tilde{\chi}}} \left( V_{lmn}^{111(1)} - V_{lm\tilde{\chi}}^{111(1)} \right) \\
 & \quad + \min_{\tilde{\eta} \leq j \leq m} \left( V_{lmn}^{111(0)} - V_{mj}^{111(0)} \right) + \min_{\substack{\tilde{\mu} \leq i \leq l \\ \tilde{\eta} \leq j \leq m \\ \tilde{\chi} \leq k \leq n}} \left( V_{ljk}^{111(0)} - V_{ijk}^{111(0)} \right). \tag{4.5}
 \end{aligned}$$

Putting  $\tilde{\mu} = \operatorname{argmax} \{P_l \geq \lambda P_i\}$ ,  $\tilde{\eta} = \operatorname{argmax} \{Q_m \geq \kappa Q_j\}$  and  $\tilde{\chi} = \operatorname{argmax} \{R_n \geq \delta R_k\}$  with  $\lambda, \kappa, \delta > 1$ , we can observe (3.13), (3.14) and, accordingly, (3.11). Grounding that  $V_{lmn}^{111(1)}(\Delta_{111}z) \rightarrow 0$  as  $l, m, n \rightarrow \infty$ , we could remark  $V_{lmn}^{111(1)}(\Delta_{111}z) - V_{ij}^{111(1)}(\Delta_{111}z) \rightarrow 0$  for  $i = l$  or  $\tilde{\mu}$ ,  $j = m$  or  $\tilde{\eta}$  and  $k = n$  or  $\tilde{\chi}$  as  $\tilde{\mu}, \tilde{\eta}, \tilde{\chi} \rightarrow \infty$ .

Since the sequences  $(P_l)$ ,  $(Q_m)$  and  $(R_n)$  are strictly increasing sequences, we attain from (4.5) that

$$\begin{aligned}
& V_{lmn}^{111(0)} - V_{lmn}^{111(1)} \\
& \geq \frac{P_{\tilde{\mu}} Q_{\tilde{\eta}} R_{\tilde{\chi}}}{(P_l - P_{\tilde{\mu}})(Q_m - Q_{\tilde{\eta}})(R_n - P_{\tilde{\chi}})} \left( V_{lmn}^{111(1)} - V_{\tilde{\mu}mn}^{111(1)} - V_{l\tilde{\eta}n}^{111(1)} - V_{lm\tilde{\chi}}^{111(1)} + V_{\tilde{\mu}\tilde{\eta}\tilde{\chi}}^{111(1)} \right) \\
& + \frac{P_{\tilde{\mu}}}{P_l - P_{\tilde{\mu}}} \left( V_{lmn}^{111(1)} - V_{\tilde{\mu}mn}^{111(1)} \right) + \frac{Q_{\tilde{\eta}}}{Q_m - Q_{\tilde{\eta}}} \left( V_{lmn}^{111(1)} - V_{l\tilde{\eta}n}^{111(1)} \right) \\
& + \frac{R_{\tilde{\chi}}}{R_n - R_{\tilde{\chi}}} \left( V_{lmn}^{111(1)} - V_{lm\tilde{\chi}}^{111(1)} \right) \tag{4.6} \\
& \geq \frac{\tilde{\lambda} + \tilde{\kappa} + \tilde{\delta} - \tilde{\lambda}\tilde{\kappa}\tilde{\delta}}{(1 - \tilde{\lambda})(1 - \tilde{\kappa})(1 - \tilde{\delta})} (1 - o(1))o(1) + \min_{\tilde{\kappa}Q_m < Q_j \leq Q_n} \left( V_{lmn}^{111(0)} - V_{ljk}^{111(0)} \right) \\
& + \min_{\substack{\tilde{\lambda}P_l \leq P_i \leq P_l \\ \tilde{\kappa}Q_m \leq Q_j \leq Q_m \\ \tilde{\delta}R_n \leq R_k \leq R_n}} \left( V_{ijk}^{111(0)} - V_{ijk}^{111(1)} \right)
\end{aligned}$$

where

$$\begin{aligned}
\frac{P_{\tilde{\mu}}}{P_l - P_{\tilde{\mu}}} &= \frac{P_{\tilde{\mu}}/P_l}{1 - P_{\tilde{\mu}}/P_l} \geq \frac{\tilde{\lambda}(1 - o(1))}{1 - \tilde{\lambda}}, \\
\frac{Q_{\tilde{\eta}}}{Q_m - Q_{\tilde{\eta}}} &= \frac{Q_{\tilde{\eta}}/Q_m}{1 - Q_{\tilde{\eta}}/Q_m} \geq \frac{\tilde{\kappa}(1 - o(1))}{1 - \tilde{\kappa}},
\end{aligned}$$

and

$$\frac{R_{\tilde{\chi}}}{R_m - R_{\tilde{\chi}}} = \frac{R_{\tilde{\chi}}/R_n}{1 - R_{\tilde{\chi}}/R_n} \geq \frac{\tilde{\delta}(1 - o(1))}{1 - \tilde{\delta}},$$

for  $1/\lambda = \tilde{\lambda}$ ,  $1/\kappa = \tilde{\kappa}$ ,  $1/\delta = \tilde{\delta}$ , and  $0 < \tilde{\lambda}, \tilde{\kappa}, \tilde{\delta} < 1$ . If we get  $\liminf$  of both sides of inequality (4.6) as  $\tilde{\mu}, \tilde{\eta}, \tilde{\chi} \rightarrow \infty$ , then we reach for any  $0 < \tilde{\lambda}, \tilde{\kappa}, \tilde{\delta} < 1$

$$\begin{aligned}
& \liminf_{l,m,n \rightarrow \infty} \left( V_{lmn}^{111(0)} - V_{lmn}^{111(1)} \right) \geq \liminf_{l,m,n \rightarrow \infty} \min_{\tilde{\kappa}Q_m < Q_j \leq Q_m} \left( V_{lmn}^{111(0)} - V_{ljk}^{111(0)} \right) \\
& + \liminf_{l,m,n \rightarrow \infty} \min_{\substack{\tilde{\lambda}P_l \leq P_i \leq P_l \\ \tilde{\kappa}Q_m \leq Q_j \leq Q_m \\ \tilde{\delta}R_n \leq R_k \leq R_n}} \left( V_{ijk}^{111(0)} - V_{ijk}^{111(1)} \right).
\end{aligned}$$

If we get limit on both sides of last inequality as  $\tilde{\lambda}, \tilde{\kappa}, \tilde{\delta} \rightarrow 1^-$ , then we arrive

$$\liminf_{l,m,n \rightarrow \infty} \left( V_{lmn}^{111(0)} - V_{lmn}^{111(1)} \right) \geq 0 \tag{4.7}$$

due to  $\left( V_{lmn}^{111(0)}(\Delta_{111z}) \right) \in \mathfrak{SD}_{(P)} \cap \mathfrak{SD}_{(Q)} \cap \mathfrak{SD}_{(R)} \cap \mathfrak{SDST}_{(P)}$ . If we combine inequalities (4.4) with (4.7), we reach

$$\lim_{l,m,n \rightarrow \infty} V_{lmn}^{111(0)} = \lim_{l,m,n \rightarrow \infty} V_{lmn}^{111(1)}$$

which means that  $(V_{lmn}^{111(0)}(\Delta_{111}z))$  is  $P$ -convergent to 0. Therefore, we conclude from the triple weighted Kronecker identity that  $(z_{lmn})$  is  $P$ -convergent to  $\ell$ .

In regard to Theorem 4.1, we can point out the following theorem.

**Theorem 4.2:** Let  $(z_{lmn}) \in \ell_\infty^3(\mathbb{R})$  and  $(p_l), (q_m) \in SVA_{\text{reg}(\alpha)}$ . If a sequence  $(z_{lmn})$  is  $(\bar{N}, p, q, r)$  summable to a number  $\ell$  and conditions

$$\begin{aligned} \frac{P_l}{p_l} \Delta_{100} V_{lmn}^{111(0)}(\Delta_{111}z) = O_L(1), \quad \frac{Q_m}{q_m} \Delta_{010} V_{lmn}^{111(0)}(\Delta_{111}z) = O_L(1) \\ \text{and } \frac{R_n}{r_n} \Delta_{001} V_{lmn}^{111(0)}(\Delta_{111}z) = O_L(1) \end{aligned} \tag{4.8}$$

are satisfied, then  $(z_{lmn})$  is  $P$ -convergent to  $\ell$ .

**Proof:** Assume that  $(z_{lmn})$  is  $(\bar{N}, p, q, r)$  summable to  $\ell$  and conditions (4.8) are satisfied. If we indicate that conditions (4.8) imply one of the conditions (4.1), shall we say,  $(V_{lmn}^{111(0)}(\Delta_{111}z)) \in \mathfrak{S}_{\mathfrak{D}(P)} \cap \mathfrak{S}_{\mathfrak{D}(Q)} \cap \mathfrak{S}_{\mathfrak{D}(R)} \cap \mathfrak{S}_{\mathfrak{D}\mathfrak{S}\mathfrak{T}(Q)}$ , then we prove this theorem with the help of Theorem 4.1. Put  $\mu = \text{argmin}\{P_i \geq \lambda P_l\}$ ,  $\eta = \text{argmin}\{Q_j \geq \kappa Q_m\}$  and  $\chi = \text{argmin}\{R_k \geq \delta R_n\}$  with  $\lambda, \kappa, \delta > 1$ . Then, we have for  $n_0 \leq m \leq i \leq \mu$  and  $n_0 \leq n$

$$\begin{aligned} V_{lmn}^{111(0)} - V_{lmn}^{111(0)} &= \sum_{j=m+1}^i \Delta_{10} V_{kn}^{11(0)} \geq -M_1 \sum_{k=m+1}^i \frac{p_k}{P_k} \\ &\geq -M_1 \left( \frac{P_\mu}{P_m} - 1 \right) \geq -M_1(\lambda - 1 + \lambda o(1)) \end{aligned}$$

for any constant  $M_1 > 0$ . If we get  $\liminf$  and limit of both sides of last inequality as  $m, n \rightarrow \infty$  and  $\lambda \rightarrow 1^+$  respectively, then we reach

$$\lim_{\lambda \rightarrow 1^+} \liminf_{m,n \rightarrow \infty} \min_{P_m \leq P_i \leq \lambda P_m} (V_{imn}^{111(0)} - V_{lmn}^{111(0)}) \geq 0$$

which means that  $(V_{lmn}^{111(0)}(\Delta_{111}u)) \in \mathfrak{S}_{\mathfrak{D}(P)}$ . On the other hand, we obtain for  $n_0 \leq m, n$  and  $n_0 \leq m, n \leq j, k \leq \eta, \chi$

$$\begin{aligned} V_{ijn}^{111(0)} - V_{lmn}^{111(0)} &= \sum_{r=n+1}^j \Delta_{01} V_{mr}^{111(0)} \geq -M_2 \sum_{r=n+1}^j \frac{q_r}{Q_r} \\ &\geq -M_2 \left( \frac{Q_\eta}{Q_n} - 1 \right) \geq -M_2(\kappa - 1 + \lambda o(1)) \end{aligned}$$

for any constant  $M_2 > 0$ . If we get  $\liminf$  and limit of both sides of last inequality as  $l, m, n \rightarrow \infty$  and  $\kappa \rightarrow 1^+$  respectively, then we reach

$$\lim_{\kappa \rightarrow 1^+} \liminf_{l, m, n \rightarrow \infty} \min_{Q_m \leq Q_j \leq \kappa Q_m} \left( V_{ljn}^{111(0)} - V_{lmn}^{111(0)} \right) \geq 0$$

which means that  $\left( V_{lmn}^{111(0)} (\Delta_{111}z) \right) \in \mathfrak{SD}_{(Q)}$ . Similarly, we can easily observe that  $\left( V_{lmn}^{111(0)} (\Delta_{111}z) \right) \in \mathfrak{SDST}(Q)$  is verified. Therefore, we conclude with the help of Theorem 4.1 that  $(z_{lmn})$  is  $P$ -convergent to  $\ell$ .

Analogous results for triple sequences of complex numbers can be formulated as follows.

**Theorem 4.3:** Let  $(z_{lmn}) \in \ell_{\infty}^3(\mathbb{R})$  and  $(p_i), (q_m), (r_n) \in SVA_{\text{reg}(\alpha)}$ . If a sequence  $(z_{lmn})$  is  $(\bar{N}, p, q, r)$  summable to a number  $\ell$  and

$$\left( V_{lmn}^{111(0)} (\Delta_{111}z) \right) \in \mathfrak{SD}_{(P)} \cap \mathfrak{SD}_{(Q)} \cap \mathfrak{SD}_{(R)} \cap \mathfrak{SDST}(Q)$$

or

$$\left( V_{lmn}^{111(0)} (\Delta_{111}z) \right) \in \mathfrak{SD}_{(P)} \cap \mathfrak{SD}_{(Q)} \cap \mathfrak{SD}_{(R)} \cap \mathfrak{SDST}(P)$$

or

$$\left( V_{lmn}^{111(0)} (\Delta_{111}z) \right) \in \mathfrak{SD}_{(P)} \cap \mathfrak{SD}_{(Q)} \cap \mathfrak{SD}_{(R)} \cap \mathfrak{SDST}(R)$$

then  $(z_{lmn})$  is  $P$ -convergent to  $\ell$ .

In regard to Theorem 4.3, we can point out the following theorem.

**Theorem 4.4:** Let  $(z_{lmn}) \in \ell_{\infty}^3(\mathbb{R})$  and  $(p_l), (q_m), (r_n) \in SVA_{\text{reg}(\alpha)}$ . If a sequence  $(z_{lmn})$  is  $(\bar{N}, p, q, r)$  summable to a number  $\ell$  and conditions

$$\frac{P_l}{p_l} \Delta_{100} V_{lmn}^{111(0)} (\Delta_{111}z) = O(1), \quad \frac{Q_m}{q_m} \Delta_{010} V_{lmn}^{111(0)} (\Delta_{111}z) = O(1)$$

$$\text{and } \frac{R_n}{r_n} \Delta_{001} V_{lmn}^{111(0)} (\Delta_{111}z) = O(1)$$

are satisfied, then  $(z_{lmn})$  is  $P$ -convergent to  $\ell$ . Before finishing this section, we discuss some conditions needed for  $(\bar{N}, p, q, r)$  summable triple sequences to be convergent.

**Theorem 4.5:** Let  $(z_{lmn}) \in \ell_{\infty}^3(\mathbb{R})$  and  $(p_i), (q_m), (r_n) \in SVA_{\text{reg}(\alpha)}$ . If a sequence  $(z_{lmn})$  is  $(\bar{N}, p, q, r)$  summable to a number  $\ell$  and conditions

$$\lim_{\lambda, \kappa, \delta \rightarrow 1^+} \liminf_{l, m, n \rightarrow \infty} (\tau_{lmn}^{\mu\eta\chi} - z_{lmn}) \geq 0 \quad (4.9)$$

and

$$\lim_{\lambda, \kappa, \delta \rightarrow 1^-} \liminf_{l, m, n \rightarrow \infty} (z_{lmn} - \tau_{\mu\eta\chi}^{lmn}) \geq 0 \quad (4.10)$$

are satisfied, then  $(z_{lmn})$  is  $P$ -convergent to  $\ell$ .

**Proof:** Assume that  $(z_{lmn})$  is  $(\bar{N}, p, q, r)$  summable to  $\ell$  and conditions (4.9) and (4.10) are satisfied. To prove that  $(z_{lmn})$  is  $P$ -convergent to the same number, it is enough to prove that conditions in (3.5) and (3.6) are verified. For  $\mu > l$ ,  $\eta > m$  and  $\chi > n$ , we have from Lemma 3.3(i)

$$\begin{aligned} \sigma_{\mu mn}^{111}(z) - \tau_{lmn}^{\mu\eta\chi}(z) = & \\ & \frac{P_\mu Q_\eta R_\chi}{(P_\mu - P_l)(Q_\eta - Q_m)(R_\eta - R_n)} (\sigma_{\mu mn}^{111} - \sigma_{\mu\eta\chi}^{111} - \sigma_{lmn}^{111} + \sigma_{l\eta}^{111}) \\ & + \frac{P_l}{P_\mu - P_l} (\sigma_{lmn}^{111} - \sigma_{\mu mn}^{111}) + \frac{Q_\eta}{Q_\eta - Q_m} (\sigma_{lmn}^{111} - \sigma_{l\eta n}^{111}) + \frac{R_\chi}{R_\chi - R_m} (\sigma_{lmn}^{111} - \sigma_{lm\chi}^{111}). \end{aligned} \quad (4.11)$$

Putting  $\mu = \operatorname{argmin} \{P_i \geq \lambda P_l\}$ ,  $\eta = \operatorname{argmin} \{Q_j \geq \kappa Q_m\}$  and  $\chi = \operatorname{argmin} \{R_k \geq \delta R_n\}$  with  $\lambda, \kappa, \delta > 1$ , we can observe inequalities (3.9), (3.10) and, accordingly, (3.11). Grounding that  $\sigma_{mn}^{11}(u) \rightarrow \ell$  as  $l, m, n \rightarrow \infty$ , we could remark  $\sigma_{lmn}^{111} - \sigma_{ijk}^{111} \rightarrow 0$  for  $i = l$  or  $\mu$ ,  $j = m$  or  $\eta$  and  $k = n$  or  $\chi$  as  $l, m, n \rightarrow \infty$ . If we get  $\liminf$  of both sides of equality (4.11) as  $l, m, n \rightarrow \infty$ , then we obtain

$$\begin{aligned} & \liminf_{l, m, n \rightarrow \infty} (\sigma_{\mu mn}^{111} - \tau_{lmn}^{\mu\eta\chi}) \\ & \geq \left( \frac{\lambda}{\lambda - 1} \right) \left( \frac{\kappa}{\kappa - 1} \right) \left( \frac{\delta}{\delta - 1} \right) \liminf_{l, m, n \rightarrow \infty} (\sigma_{\mu mn}^{11} - \sigma_{\mu\eta\chi}^{11} - \sigma_{lmn}^{11} + \sigma_{l\eta n}^{11} + \sigma_{lm\chi}^{11}) \\ & \quad + \frac{1}{\lambda - 1} \liminf_{l, m, n \rightarrow \infty} (\sigma_{lmn}^{11} - \sigma_{\mu mn}^{11}) + \frac{1}{\kappa - 1} \liminf_{l, m, n \rightarrow \infty} (\sigma_{lmn}^{11} - \sigma_{l\eta n}^{11}) \\ & \quad + \frac{1}{\delta - 1} \liminf_{l, m, n \rightarrow \infty} (\sigma_{lmn}^{11} - \sigma_{lm\chi}^{11}) \end{aligned}$$

where

$$\begin{aligned} \frac{P_\mu}{P_\mu - P_l} &= \frac{P_\mu/P_l}{P_\mu/P_m - 1} \leq \frac{\lambda}{(\lambda - 1)}(1 + o(1)), \\ \frac{Q_\eta}{Q_\eta - Q_m} &= \frac{Q_\eta/Q_m}{Q_\eta/Q_m - 1} \leq \frac{\kappa}{(\kappa - 1)}(1 + o(1)) \end{aligned}$$

and

$$\frac{R_\chi}{R_\chi - R_n} = \frac{R_\chi/R_n}{R_\chi/R_n - 1} \leq \frac{\delta}{(\delta - 1)}(1 + o(1)).$$

If we get limit of both sides of last inequality as  $\lambda, \kappa, \delta \rightarrow 1^+$ , then we arrive



$$\lim_{\lambda, \kappa, \delta \rightarrow 1^+} \liminf_{l, m, n \rightarrow \infty} (\sigma_{\mu mn}^{111} - \tau_{lmn}^{\mu\eta\chi}) \geq 0.$$

In the same vein, for  $\mu < l$ ,  $\eta < m$  and  $\chi < n$ , we have from Lemma 3.3(i)

$$\begin{aligned} & \sigma_{l\eta n}^{111}(z) - \tau_{lmn}^{\mu\eta\chi}(z) \\ &= \frac{P_\mu Q_\eta}{(P_\mu - P_l)(Q_\eta - Q_n)} (\sigma_{\mu mn}^{111} - \sigma_{\mu\eta\chi}^{111} - \sigma_{lmn}^{111} + \sigma_{l\eta n}^{111}) \\ &+ \frac{P_\mu}{P_\mu - P_l} (\sigma_{lmn}^{111} - \sigma_{\mu mn}^{111}) + \frac{Q_m}{Q_\eta - Q_m} (\sigma_{lmn}^{111} - \sigma_{l\eta n}^{111}) + \frac{R_n}{R_\eta - R_n} (\sigma_{lmn}^{111} - \sigma_{lm\chi}^{111}). \end{aligned} \quad (4.12)$$

If we get  $\liminf$  on both sides of equality (4.12) as  $l, m, n \rightarrow \infty$ , then we obtain

$$\begin{aligned} & \liminf_{l, m, n \rightarrow \infty} (\sigma_{l\eta n}^{111}(z) - \tau_{lmn}^{\mu\eta\chi}(z)) \\ & \geq \left( \frac{\lambda}{\lambda - 1} \right) \left( \frac{\kappa}{\kappa - 1} \right) \left( \frac{\delta}{\delta - 1} \right) (\sigma_{\mu mn}^{111} - \sigma_{\mu\eta\chi}^{111} - \sigma_{lmn}^{111} + \sigma_{l\eta n}^{111} + \sigma_{lm\chi}^{111}) \\ & \quad + \frac{\lambda}{\lambda - 1} \liminf_{l, m, n \rightarrow \infty} (\sigma_{lmn}^{111} - \sigma_{\mu mn}^{111}) + \frac{\kappa}{\kappa - 1} \liminf_{l, m, n \rightarrow \infty} (\sigma_{lmn}^{111} - \sigma_{l\eta n}^{111}) \\ & \quad + \frac{\delta}{\delta - 1} \liminf_{l, m, n \rightarrow \infty} (\sigma_{lmn}^{111} - \sigma_{lm\chi}^{111}). \end{aligned}$$

If we get limit of both sides of last inequality as  $\lambda, \kappa, \delta \rightarrow 1^+$ , then we arrive

$$\lim_{\lambda, \kappa, \delta \rightarrow 1^+} \liminf_{l, m, n \rightarrow \infty} (\sigma_{l\eta n}^{111}(z) - \tau_{lmn}^{\mu\eta\chi}(z)) \geq 0.$$

In addition to what is attained above, for  $\mu < l$ ,  $\eta < m$  and  $\chi < n$ , we have from Lemma 3.3(i)

$$\begin{aligned} & \sigma_{\mu\eta\chi}^{111} - \tau_{lmn}^{\mu\eta\chi} \\ &= \frac{P_l Q_\eta R_\chi}{(P_\mu - P_l)(Q_\eta - Q_m)(R_\chi - R_n)} (\sigma_{l\eta n}^{111} - \sigma_{\mu\eta\chi}^{111}) \\ &+ \frac{P_\mu Q_m R_n}{(P_\mu - P_l)(Q_\eta - Q_m)(R_\chi - R_n)} (\sigma_{\mu mn}^{111} - \sigma_{lmn}^{111}) + \frac{Q_m}{Q_\eta - Q_m} (\sigma_{lmn}^{111} - \sigma_{\mu\eta\chi}^{111}) \\ &+ \frac{R_n}{R_\chi - R_n} (\sigma_{lmn}^{111} - \sigma_{\mu\eta\chi}^{111}). \end{aligned} \quad (4.13)$$

If we get  $\limsup$  on both sides of equality (4.13) as  $l, m, n \rightarrow \infty$ , then we obtain

$$\begin{aligned}
& \limsup_{l,m,n \rightarrow \infty} (\sigma_{\mu\eta\chi}^{111} - \tau_{lmn}^{\mu\eta\chi}) \\
& \leq \frac{\kappa}{(\lambda-1)(\kappa-1)(\delta-1)} \limsup_{l,m,n \rightarrow \infty} (\sigma_{l\eta n}^{111} - \sigma_{\mu\eta\chi}^{111}) + \frac{\lambda}{(\lambda-1)(\kappa-1)(\delta-1)} \\
& \times \limsup_{l,m,n \rightarrow \infty} (\sigma_{\mu mn}^{111} - \sigma_{lmn}^{111}) + \frac{\delta}{(\lambda-1)(\kappa-1)(\delta-1)} \limsup_{l,m,n \rightarrow \infty} (\sigma_{\mu mn}^{111} - \sigma_{lmn}^{111}) \\
& + \frac{1}{\kappa-1} \limsup_{l,m,n \rightarrow \infty} (\sigma_{lmn}^{111} - \sigma_{\mu\eta\chi}^{111}).
\end{aligned}$$

If we get limit on both sides of last inequality as  $\lambda, \kappa, \delta \rightarrow 1^+$ , then we arrive

$$\lim_{\lambda, \kappa, \delta \rightarrow 1^+} \limsup_{l,m,n \rightarrow \infty} (\sigma_{\mu\eta\chi}^{111} - \tau_{lmn}^{\mu\eta\chi}) \leq 0.$$

Hence, we can state that conditions in (3.5) are verified. Following a similar procedure to above for  $\mu < l, \eta < m$  and  $\chi < n$ , we can behold that conditions in (3.6) are also verified. In that case, we reach from Lemma 3.4 that  $\left( V_{lmn}^{111(0)}(\Delta_{111}z) \right)$  is  $P$ -convergent to 0. Therefore, we conclude from the triple weighted Kronecker identity that  $(z_{lmn})$  is  $P$ -convergent to  $\ell$ .

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