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ON SOME INEQUALITIES INVOLVING BETA-LOGARITHMIC FUNCTION

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ABSTRACT. The Beta-logarithmic function, providing simultaneously a generalization of the logarithmic mean and the beta function, has been recently introduced by the authors. In this paper we deal with approximating this function when investing some advanced integral inequalities. Many old/new inequalities involving the standard beta function and the logarithmic mean are immediately deduced.

1. INTRODUCTION

The widely recognized beta function, commonly referred to as the first-order Euler's integral, is defined for any $x, y > 0$ by

$$B(x,y) := \int_0^1 u^{x-1}(1-u)^{y-1}\, du. \tag{1.1}$$

There is an extensive amount of literature that discusses properties of this standard function as well as its applications in various fields, for example [1, 2, 7]. Over the last years many generalizations have been given for this interesting function, as chronologically cited in what follows.

• In 1997, Chaudhry et al. [3] extended the beta function by setting for any $x, y > 0$ and $p \geq 0$,

$$B(x,y;p) := \int_0^1 u^{x-1}(1-u)^{y-1} \exp\left(\frac{-p}{u(1-u)}\right)\, du. \tag{1.2}$$

• In 2014, Choi et al. provided in [9] a double parameterized version of (1.1) defined for any $x, y > 0$ and $p, q \geq 0$ by the following formula:

$$\mathcal{B}(x,y;p,q) := \int_0^1 u^{x-1}(1-u)^{y-1} \exp\left(-\frac{p}{u} - \frac{q}{1-u}\right)\, du. \tag{1.3}$$

If $p = 0$, (1.2) is identical to (1.1), and if $p = q$, (1.3) is then reduced to (1.2). Many properties involving these generalized beta functions have been investigated in [16].

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• We have recently [15] defined the beta-logarithmic function, a new parametric extension of the beta function defined for $a, b; x, y > 0$ by,

$$\mathcal{BL}(a, b; x, y) := \int_0^1 a^{1-u} b^u u^{x-1} (1-u)^{y-1}\, du. \tag{1.4}$$

Many algebraic properties and applications of $\mathcal{BL}(a, b; x, y)$ were established in [15]. In particular, we pointed out that for all $a, b; x, y > 0$, we have the following expansion:

$$\mathcal{BL}(a, b; x, y) = \sum_{n,m=0}^{\infty} \frac{B(x+n, y+m)}{n!\, m!} \big(\log a\big)^m \big(\log b\big)^n. \tag{1.5}$$

Our generalization also covers the bivariate logarithmic mean $L(a, b)$. The following formulas clearly show this fact.

$$\mathcal{BL}(1, 1; x, y) = B(x, y) \quad \text{and} \quad \mathcal{BL}(a, b; 1, 1) = L(a, b), \tag{1.6}$$

where $L(a, b)$ is defined for two positive numbers as follows [14],

$$L(a, b) = \frac{a - b}{\log a - \log b} = \int_0^1 a^{1-u}\, b^u\, du, \;\; L(a, a) = a. \tag{1.7}$$

It is worthy to recall that a mean stands for any binary map m satisfying the following in-betweenness property

$$\min(a, b) \leq m(a, b) \leq \max(a, b), \;\text{ for any two positive numbers } a, b > 0. \tag{1.8}$$

Recently, an important study on means has been undertaken in [17] based on their representation by integrals. The fundamental target of the current work is to establish approximations for the beta-logarithmic function $\mathcal{BL}(a, b; x, y)$. Section 2 is devoted to providing inequalities of $\mathcal{BL}(a, b; x, y)$ when using Chebychev's inequality. In section 3, we state some other inequalities by employing various versions of Grüss type inequalities. In Section 4, we focus on obtaining further inequalities by applying some Ostrowski-type inequalities, particularly those that involve functions with bounded first and second derivatives.

2. Approximations via Chebychev's inequality

In this section, we outline some inequalities for $\mathcal{B}L(a, b; x, y)$ using Chebyshev's integral inequality. We first need to briefly define a class of functions that are of great importance in mathematical analysis.

Definition 2.1. Let ϕ and ψ be two real functions defined on a nonempty interval I of $\mathbb{R}$. ϕ and ψ are called synchronous (resp. asynchronous) on I if the following inequality

$$\big(\phi(u) - \phi(v)\big)\big(\psi(u) - \psi(v)\big) \geq (\leq)\; 0$$

holds for any $u, v \in I$.

This class of functions satisfies Chebyshev's inequality, which will be useful in what follows. It reads as follows [10, 11].

Lemma 2.2. *Consider three real functions ϕ, ψ, ω defined on a non-empty interval $I \subset \mathbb{R}$ such that $\omega(u) \geq 0$ for all $u \in I$ and $w, w\phi\psi, w\phi$ and $\omega\psi$ are integrable on I. If ϕ and ψ are synchronous (resp. asynchronous) on I, the following inequality holds:*

$$\int_I \omega(u)\, du \times \int_I \omega(u)\phi(u)\psi(u)\, du \geq (\leq) \int_I \omega(u)\phi(u)\, du \int_I \omega(u)\psi(u)\, du. \tag{2.1}$$

Using this lemma, we establish our first main result as indicated below.

Theorem 2.3. *Let $a, b \geq 0$ and $x, y > 0$. Assume that $(a-1)(b-1) \leq (\geq) 0$ then there holds,*

$$B(x,y) \times \mathcal{BL}(a,b;x,y) \geq (\leq) \mathcal{BL}(1,b;x,y) \times \mathcal{BL}(a,1;x,y) \tag{2.2}$$

Proof. For fixed $a, b \geq 0$ and $x, y > 0$, we define on $(0,1)$ the following functions:

$$\phi(u) = a^{1-u}, \quad \psi(u) = b^u, \quad \text{and } \omega(u) = u^{x-1}(1-u)^{y-1}.$$

First, we note that ω is positive and integrable on $(0;1)$. On the other hand, we have for all $u \in (0,1)$

$$\phi'(u) = -(\log a)\, a^{1-u} \quad \text{and} \quad \psi'(u) = (\log b) b^u.$$

Consequently,

$$\begin{aligned} \phi \text{ and } \psi \text{ are synchronous (asynchronous)} \quad &\Leftrightarrow \quad \log a \log b \leq (\geq) 0 \\ &\Leftrightarrow \quad (a-1)(b-1) \leq (\geq) 0. \end{aligned}$$

Applying Chebychev's inequality (2.1), we obtain

$$\int_0^1 \omega(u)\, du \times \int_0^1 \omega(u)\phi(u)\psi(u)\, du \geq (\leq) \int_0^1 \omega(u)\phi(u)\, du \int_0^1 \omega(u)\psi(u)\, du.$$

Thus, substituting $\phi(u)$, $\psi(u)$, and $\omega(u)$ by their expressions, the following inequality holds

$$\begin{aligned} \int_0^1 u^{x-1}(1-u)^{y-1}\, du \times \int_0^1 u^{x-1}(1-u)^{y-1} a^{1-u} b^u\, du \geq (\leq) \\ \int_0^1 a^{1-u} u^{x-1}(1-u)^{y-1}\, du \times \int_0^1 b^u u^{x-1}(1-u)^{y-1}\, du. \end{aligned}$$

By taking into account the formulas (1.1) and (1.4), the result of the theorem is deduced. □

Using Lemma 2.2 again, we can state the following result.

Theorem 2.4. *For any positive real numbers $a, b, x,$ and $y > 0$, we have the following inequalities:*

(*i*) *If $(x-1)(b-a) \geq (\leq) 0$ then*

$$\mathcal{BL}(a,b;x,y) \geq (\leq)\, y\, B(x,y)\, \mathcal{BL}(a,b;1,y) \tag{2.3}$$

(*ii*) *If $(y-1)(a-b) \geq (\leq) 0$ then*

$$\mathcal{BL}(a,b;x,y) \geq (\leq)\, x\, B(x,y)\, \mathcal{BL}(a,b;x,1) \tag{2.4}$$

Proof. On $(0,1)$, we consider the following functions:

$$\phi(u)=a^{1-u}\,b^u,\quad \psi(u)=u^{x-1},\quad \text{and } \omega(u)=(1-u)^{y-1}.$$

For any $u\in(0,1)$ we have,

$$\phi'(u)=a^{1-u}\,b^u\,\log(b/a)\quad\text{and}\quad \psi'(u)=(x-1)\,u^{x-2}.$$

So, if the condition $(x-1)(b-a)\geq(\leq)0$ is fulfilled then f and g are synchronous (asynchronous) on $(0,1)$. In addition, the function ω is positive and integrable on $(0,1)$. Applying the Chebychev's inequality (2.1) we obtain,

$$\int_0^1(1-u)^{y-1}\,du\times\int_0^1 a^{1-u}\,b^u\,u^{x-1}\,(1-u)^{y-1}\,du$$
$$\geq(\leq)\int_0^1 a^{1-u}\,b^u\,(1-u)^{y-1}\,du\int_0^1 u^{x-1}\,(1-u)^{y-1}\,du.$$

That is,

$$B(1,y)\,\mathcal{BL}(a,b;x,y)\;\geq(\leq)\,\mathcal{BL}(a,b,1,y)\,B(x,y),$$

which is equivalent to (2.3).

We can proceed in the same way to prove the statement (ii) by making the following choices:

$$\phi(u)=a^{1-u}\,b^u,\quad \psi(u)=(1-u)^{y-1},\quad \text{and } \omega(u)=u^{x-1}.$$

The steps that follow these choices do not pose any complications and are therefore not repeated here. □

3. Approximations via Grüss type inequalities

We continue establishing inequalities involving the beta-logarithmic function in this section. This objective will be attained by investing some Grüss type inequalities. The first version proved by Grüss in 1935 [10, 11] is recited in the following lemma.

Lemma 3.1. *Let us consider two real functions ϕ and ψ integrable over (a,b) and satisfying for all $u\in(a,b)$*

$$m_1\leq\phi(u)\leq M_1\quad\textit{and}\quad m_2\leq\psi(u)\leq M_2,$$

where m_1,m_2,M_1 and M_2 are real constants. For any nonnegative real function ω integrable on (a,b), the following inequality holds

$$\left|\int_a^b\omega(u)\,du\times\int_a^b\phi(u)\psi(u)\omega(u)\,du-\int_a^b\phi(u)\omega(u)\,du\times\int_a^b\psi(u)\omega(u)\,du\right|$$
$$\leq\frac{1}{4}(M_1-m_1)(M_2-m_2)\Big(\int_a^b\omega(u)\,du\Big)^2.\quad(3.1)$$

To establish further findings via the Grüss inequality, we also require the following lemma, which can be shown with the classical tools of real analysis.

Lemma 3.2. *Let s and r be two positive real numbers. We have,*

$$\sup_{t\in(0,1)} t^s(1-t)^r = \frac{s^s r^r}{(s+r)^{s+r}}.$$

These last two lemmas allow us to state the following theorem.

Theorem 3.3. *The following inequality*

$$\Big|\mathcal{BL}(a,b;x,y) - L(a,b)\,B(x,y)\Big| \le \frac{|b-a|}{4}\,\frac{(x-1)^{x-1}\,(y-1)^{y-1}}{(x+y-2)^{x+y-2}} \tag{3.2}$$

holds for all $a,b>0$ and $x,y>1$.

Proof. Let us define on $(0,1)$ the functions,

$$\phi(u) = a^{1-u}\,b^u, \psi(u) = u^{x-1}\,(1-u)^{y-1}, \text{ and } \omega(u) = 1.$$

For every $u\in(0,1)$, by using successively (1.8) and Lemma 3.2 we get,

$$\min(a,b) \le \phi(u) \le \max(a,b) \quad \text{and} \quad 0 \le \psi(u) \le \frac{(x-1)^{x-1}\,(y-1)^{y-1}}{(x+y-2)^{x+y-2}}.$$

Using (3.1), we deduce

$$\left|\int_0^1 \phi(u)\psi(u)\,du - \int_0^1 \phi(u)\,du \times \int_0^1 \psi(u)\,du\right| \le \frac{|b-a|}{4}\,\frac{(x-1)^{x-1}\,(y-1)^{y-1}}{(x+y-2)^{x+y-2}}.$$

This last inequality combined with (1.1), (1.6) and (1.4) leads to (3.2). $\square$

The following inequalities can be stated as well.

Theorem 3.4. *Let $x,y\ge 1$ and $a,b\ge 0$, there hold*

$$\left|\mathcal{BL}(a,b;x,y) - \frac{\mathcal{BL}(a,b;1,y)}{x}\right| \le \frac{1}{4}\,\max(a,b) \tag{3.3}$$

and

$$\left|\mathcal{BL}(a,b;x,y) - \frac{\mathcal{BL}(a,b;x,1)}{y}\right| \le \frac{1}{4}\,\max(a,b). \tag{3.4}$$

Proof. Define the following auxiliary functions on $(0,1)$:

$$\phi(u) = u^{x-1}, \psi(t) = a^{1-u}\,b^u\,(1-u)^{y-1}, \text{ and } \omega(u) = 1.$$

If $x\ge 1$ and $y\ge 1$ then for all $t\in(0,1)$ we have,

$$0 \le \phi(u) \le 1 \text{ and } 0 \le \psi(u) \le \max(a,b).$$

By applying (3.1) to the functions ϕ and ψ, we find:

$$\left|\int_0^1 a^{1-u}\,b^u\,u^{x-1}\,(1-u)^{y-1}\,du - \int_0^1 u^{x-1}\,du \int_0^1 a^{1-u}\,b^u\,(1-u)^{y-1}\,du\right| \le \frac{1}{4}\,\max(a,b),$$

which is equivalent to (3.3).
To obtain (3.4), we perform the same steps considering the functions defined on $(0,1)$ by:

$$\phi(u) = (1-u)^{y-1} \quad \text{and} \quad \psi(u) = a^{1-u}\,b^u\,u^{x-1}.$$

What follows is straightforward. Thus, the proof is completed. $\square$

Corollary 3.5. *Let $x, y \geq 1$ and $a, b \geq 0$. Then*

$$\left|\frac{\mathcal{BL}(a,b;1,y)}{x} - \frac{\mathcal{BL}(a,b;x,1)}{y}\right| \leq \frac{1}{2}\max(a,b). \tag{3.5}$$

Proof. Combining (3.3) with (3.4) and using the standard triangular inequality we get (3.5). $\square$

Theorem 3.6. *The following inequalities*

$$\left|\mathcal{BL}(a,b;x,y) - L(1,a)\,\mathcal{BL}(1,b;x,y)\right| \leq \frac{|a-1|\,\max(1,b)}{4}\,\frac{(x-1)^{x-1}\,(y-1)^{y-1}}{(x+y-2)^{x+y-2}} \tag{3.6}$$

and

$$\left|\mathcal{BL}(a,b;x,y) - L(1,b)\,\mathcal{BL}(a,1;x,y)\right| \leq \frac{|b-1|\,\max(1,a)}{4}\,\frac{(x-1)^{x-1}\,(y-1)^{y-1}}{(x+y-2)^{x+y-2}} \tag{3.7}$$

are satisfied for all $x, y \geq 1$ and $a, b \geq 0$.

Proof. For $u \in (0,1)$, let us set

$$\phi(u) = a^{1-u}, \psi(u) = b^u\,u^{x-1}\,(1-u)^{y-1}, \text{ and } \omega(u) = 1.$$

For every $u \in (0,1)$, we have $\min(1,a) \leq \phi(u) \leq \max(1,a)$. Thanks to Lemma 3.2, we obtain

$$0 \leq \psi(u) \leq \max(1,b)\,\frac{(x-1)^{x-1}\,(y-1)^{y-1}}{(x+y-2)^{x+y-2}}.$$

Applying (3.1) to the functions ϕ and ψ, we get

$$\left|\mathcal{BL}(a,b;x,y) - \int_0^1 a^{1-u}\,du\,\mathcal{BL}(1,b,x,y)\right| \leq \frac{|a-1|\,\max(1,b)}{4}\,\frac{(x-1)^{x-1}\,(y-1)^{y-1}}{(x+y-2)^{x+y-2}}.$$

Noticing that $\int_0^1 a^{1-u}\,du = L(1,a)$, the inequality (3.6) is attained.

With similar steps, (3.7) can be proved when taking on $(0,1)$ the following functions,

$$\phi(u) = b^u \quad \text{and} \quad \psi(u) = a^{1-u}\,u^{x-1}\,(1-u)^{y-1}. \quad \square$$

Investing the inequality (3.1), we can state the following theorem.

Theorem 3.7. *Let $a, b, c, d > 0$ and $x, y, z, t > 1$. The following inequality holds*

$$\left|\mathcal{BL}(a\,c, b\,d; x+z-1, y+t-1) - \mathcal{BL}(a,b;x,y) \times \mathcal{BL}(c,d;z,t)\right| \leq \frac{\max(a,b)\,\max(c,d)}{4}\Psi_{x,y;z,t}. \tag{3.8}$$

where,

$$\Psi_{x,y,z,t} := \frac{(x-1)^{x-1}\,(y-1)^{y-1}}{(x+y-2)^{x+y-2}} \times \frac{(z-1)^{z-1}\,(t-1)^{t-1}}{(z+t-2)^{z+t-2}}.$$

Proof. We define on $(0,1)$, the following functions:

$$\phi(u) = a^{1-u}\, b^u\, u^{x-1}\,(1-u)^{y-1}, \psi(u) = c^{1-u}\, d^u\, u^{z-1}\,(1-u)^{t-1}, \text{ and } \omega(u) = 1.$$

We have the following,

$$0 \le \phi(u) \le \max(a,b)\, \frac{(x-1)^{x-1}\,(y-1)^{y-1}}{(x+y-2)^{x+y-2}}$$

and

$$0 \le \psi(u) \le \max(c,d)\, \frac{(z-1)^{z-1}\,(t-1)^{t-1}}{(z+t-2)^{z+t-2}}.$$

Applying (3.1) to the functions ϕ and ψ, we get (3.8). □

Corollary 3.8. *Let $a,b>0$ and $x,y>1$. We have the following inequality*

$$\left|\mathcal{BL}(a^2,b^2,2\,x-1,2\,y-1) - (\mathcal{BL}(a,b;x,y))^2\right| \le \frac{\big(\max(a,b)\big)^2}{4}\, \frac{(x-1)^{2x-2}\,(y-1)^{2y-2}}{(x+y-2)^{2x+2y-4}}. \quad (3.9)$$

Proof. Substituting in (3.9) z,t,c and d by x,y,a and b respectively, we get the required result. □

Theorem 3.9. *The following inequality*

$$\Big|B(x,y)\,\mathcal{BL}(a,b;x,y) - \mathcal{BL}(a,1,x,y)\,\mathcal{BL}(1,b,x,y)\Big| \le \frac{|a-1|\,|b-1|}{4}\,\big(B(x,y)\big)^2, \quad (3.10)$$

holds for all positive numbers a,b,x and y.

Proof. Let us put for every $u \in (0,1)$,

$$\phi(u) = a^{1-u}, \quad \psi(u) = b^u \quad \text{and} \quad \omega(u) = u^{x-1}\,(1-u)^{y-1}.$$

We have for $u \in (0,1)$, $\min(1,a) \le \phi(u) \le \max(1,a)$ and $\min(1,b) \le \psi(u) \le \max(1,b)$. Using (3.1), we get

$$\Big|\int_0^1 \omega(u)\,du \times \int_0^1 \phi(u)\psi(u)\omega(u)\,du - \int_0^1 \phi(u)\omega(u)\,du \times \int_0^1 \psi(u)\omega(u)\,du\Big| \le \frac{|a-1|\,|b-1|}{4}\Big(\int_0^1 \omega(u)\,du\Big)^2.$$

Replacing in this last inequality $\phi(u), \psi(u)$ and $\omega(u)$ by their expressions, we get the inequality (3.10). □

Another important result will be stated in the following theorem.

Theorem 3.10. *Let $a,b>0$ and $x,y \ge 1$. Then there holds*

$$\Big|L(a,b)\,\mathcal{BL}(a,b;x,y) - \mathcal{BL}(a,b,x,1)\,\mathcal{BL}(a,b,1,y)\Big| \le \frac{1}{4}\big(L(a,b)\big)^2. \quad (3.11)$$

Proof. Let us define on $(0,1)$ the functions,

$$\phi(u) = u^{x-1}, \quad \psi(u) = (1-u)^{y-1} \quad \text{and} \quad \omega(u) = a^{1-u}\, b^{u}.$$

We have for $u \in (0,1)$, $0 \le \phi(u) \le 1$ and $0 \le \psi(u) \le 1$. Applying (3.1), we get

$$\Big| \int_0^1 \omega(u)\, du \times \int_0^1 \phi(u)\psi(u)\omega(u)\, du - \int_0^1 \phi(u)\omega(u)\, du \times \int_0^1 \psi(u)\omega(u)\, du \Big| \le \frac{1}{4}\Big(\int_0^1 \omega(u)\, du \Big)^2.$$

Taking into account the integral representation in (1.7) and substituting $\phi(u), \psi(u)$ and $\omega(u)$ by their expressions, we deduce the inequality (3.11). ☐

We end this section by yielding further inequalities for $\mathcal{BL}(a,b;x,y)$, making use of the following result, recognized by Grüss's premature inequality [5, Theorem 55].

Lemma 3.11. *Let ϕ and ψ be two integrable functions defined on $[0,1]$. Assume that, for all $u \in [0,1]$, we have $m \le \psi(u) \le M$ for some constants $m, M \in \mathbb{R}$. We have*

$$\Big| \int_0^1 \phi(u)\psi(u)\, du - \int_0^1 \phi(u)\, du \times \int_0^1 \psi(u)\, du \Big| \le \frac{M-m}{2} \left(\int_0^1 \phi^2(u)\, du - \Big(\int_0^1 \phi(u)\, du \Big)^2 \right)^{1/2}. \quad (3.12)$$

Moreover, $1/2$ is the best possible constant.

Theorem 3.12. *For any $x, y \ge 1$ and $a, b > 0$, we have*

$$\Big| \mathcal{BL}(a,b;x,y) - L(a,b) \times B(x,y) \Big| \le \frac{(x-1)^{x-1}\,(y-1)^{y-1}}{2\,(x+y-2)^{x+y-2}} \Big[L(a^2,b^2) \; - \; L^2(a,b) \Big]^{1/2}. \quad (3.13)$$

Proof. Let us consider $x, y \ge 1$ and $a, b \ge 0$. We define on $[0,1]$ the following functions,

$$\phi(u) = a^{1-u}\, b^{u} \quad \text{and} \quad \psi(u) = u^{x-1}\,(1-u)^{y-1}.$$

For all $u \in [0,1]$ we have from Lemma 3.2, $0 \le \psi(u) \le \frac{(x-1)^{x-1}\,(y-1)^{y-1}}{(x+y-2)^{x+y-2}}$. Using (3.12), we obtain

$$\Big| \int_0^1 \phi(u)\psi(u)\; du - \int_0^1 \phi(u)\; du \times \int_0^1 \psi(u)\; du \Big| \le \frac{(x-1)^{x-1}\,(y-1)^{y-1}}{2\,(x+y-2)^{x+y-2}} \Big(\int_0^1 \phi^2(u)\; du - \Big(\int_0^1 \phi(u)\; du \Big)^2 \Big)^{1/2}.$$

Substituting in this last inequality by the formulas in (1.1), (1.4) and (1.6), we find (3.13). ☐

At the end of this section, we present the following statement.

Theorem 3.13. *For all $x, y \geq 1$ and $a, b > 0$, the following inequality holds,*

$$\Big|\mathcal{BL}(a,b;x,y) - L(a,b) \times B(x,y)\Big| \leq \frac{|b-a|}{2}\Big[B(2x-1,2y-1) - B^2(x,y)\Big]^{1/2}. \quad (3.14)$$

Proof. Let us take $x, y \geq 1$ and $a, b \geq 0$. We define the following two functions on $[0,1]$,

$$\phi(u) = u^{x-1}(1-u)^{y-1} \quad \text{and} \quad \psi(u) = a^{1-u}\, b^u.$$

For all $u \in [0,1]$, we have from Lemma 3.2, $\min(a,b) \leq \psi(u) \leq \max(a,b)$. Using (3.12) again, we get

$$\Big|\int_0^1 \phi(u)\psi(u)\, du - \int_0^1 \phi(u)\, du \times \int_0^1 \psi(u)\, du\Big| \leq \frac{|b-a|}{2}\Big(\int_0^1 \phi^2(u)\, du - \Big(\int_0^1 \phi(u)\, du\Big)^2\Big)^{1/2}.$$

By the formulas in (1.1), (1.6) and (1.4), the inequality (3.14) is deduced. □

4. Approximations via Ostrowski type inequalities

In this section, we will use certain Ostrowski-type inequalities to establish other inequalities involving the beta-logarithmic function. We begin by recalling the following, [5, Theorem 52].

Lemma 4.1. *Let ϕ be a bounded measurable function on $I = (a,b)$ such that $c_1 \leq \phi(u) \leq c_2$, for all $u \in I$. Assume that $\psi'(u)$ exists and is bounded on I. Then,*

$$\Big|\frac{1}{b-a}\int_a^b \phi(u)\psi(u)\, du - \frac{1}{(b-a)^2}\int_a^b \phi(u)\, du \times \int_a^b \psi(u)\, du\Big| \leq \frac{b-a}{8}(c_2 - c_1)\sup_{u\in I}|\psi'(u)|. \quad (4.1)$$

Further, $1/8$ is the best possible constant.

Theorem 4.2. *Let $a, b > 0$ and $x, y > 1$. We have the following inequality,*

$$\Big|\mathcal{BL}(a,b;x,y) - L(a,b)\, B(x,y)\Big| \leq \frac{(x-1)^{x-1}\,(y-1)^{y-1}}{8\,(x+y-2)^{x+y-2}}\big|\log(b/a)\big|\, \max(a,b). \quad (4.2)$$

Proof. Consider the functions defined on $[0,1]$ by $\phi(u) = u^{x-1}(1-u)^{y-1}$ and $\psi(u) = a^{1-u}\, b^u$. On one hand, we have

$$0 \leq \psi(u) \leq \frac{(x-1)^{x-1}\,(y-1)^{y-1}}{(x+y-2)^{x+y-2}}.$$

On the other, $|\psi'(u)| = |\log(b/a)|\, a^{1-u}\, b^u$, and so,

$$\sup_{u\in[0,1]} |\psi'(u)| \le |\log(b/a)| \max(a,b).$$

By virtue of (4.1), we have

$$\Big|\int_0^1 \phi(t)\psi(u)\, du - \int_0^1 \phi(u)\, du \times \int_0^1 \psi(u)\, du\Big| \le \frac{(x-1)^{x-1}\,(y-1)^{y-1}}{8\,(x+y-2)^{x+y-2}}\Big|\log(b/a)\Big| \max(a,b).$$

This leads straightforward to (4.2) by the use of the explicit expressions of ϕ,ψ and ω and the formulas (1.1), (1.4) and (1.6). ☐

Theorem 4.3. *Let $a,b>0$ and $x,y>2$. The following inequality holds*

$$\Big|\mathcal{BL}(a,b;x,y) - L(a,b)\,B(x,y)\Big| \le \frac{|b-a|}{8}\frac{(x-2)^{x-2}\,(y-2)^{y-2}}{(x+y-4)^{x+y-4}} \max(x-1,y-1). \quad (4.3)$$

Proof. Let us take the following two functions defined on $[0,1]$ by

$$\phi(u) = a^{1-u}\, b^u \text{ and } \psi(u) = u^{x-1}\,(1-u)^{y-1}.$$

We know that, $\min(a,b) \le \phi(u) \le \max(a,b)$. Moreover, we have

$$\begin{aligned} |\psi'(u)| &= u^{x-2}\,(1-u)^{y-2}\,\big[(x-1)(1-u) - u(y-1)\big] \\ &\le \frac{(x-2)^{x-2}\,(y-2)^{y-2}}{(x+y-4)^{x+y-4}} \max(x-1,y-1). \end{aligned}$$

Now, applying (4.1) we deduce that,

$$\Big|\int_0^1 \phi(u)\psi(u)\, du - \int_0^1 \phi(u)\, du \times \int_0^1 \psi(u)\, du\Big| \le \frac{|b-a|}{8}\frac{(x-2)^{x-2}\,(y-2)^{y-2}}{(x+y-4)^{x+y-4}} \max(x-1,y-1).$$

Utilizing (1.1), (1.4) and (1.6), (4.3) is established. ☐

In order to determine more inequalities for the beta-logarithmic function, we recall the following important integral inequality established by Ostrowski in 1938 [13].

Lemma 4.4. *Consider a differentiable ϕ function on (a,b) such that for all $u\in(a,b)$, we have $|\phi'(u)| \le M$ for some fixed real number $M>0$. Then, for any $u\in(a,b)$, we have*

$$\Big|\phi(u) - \frac{1}{b-a}\int_a^b \phi(u)\, du\Big| \le \Big(\frac{1}{4} + \frac{\big(u-\frac{a+b}{2}\big)^2}{(b-a)^2}\Big)(b-a)M. \quad (4.4)$$

Further, the constant $1/4$ is the best possible.

Theorem 4.5. *Consider four positive numbers $a, b, x,$ and y with $a \neq b$ and $x, y > 2$. For every $s \in (0,1)$, we have*

$$\Big|\mathcal{BL}(a,b;x,y) - a^{1-s}\, b^s\, s^{x-1}\,(1-s)^{y-1}\Big| \leq \lambda_{a,b}^{x,y}\Big(\frac{1}{4} + \big(s - \frac{1}{2}\big)^2\Big), \tag{4.5}$$

with,

$$\lambda_{a,b}^{x,y} := \max(a,b)\,\frac{(x-2)^{x-2}\,(y-2)^{y-2}}{(x+y-4)^{x+y-4}} \times \max\Big(x-1, y-1, \Big|\frac{(\log(b/a) - x - y + 2)^2}{4\log(b/a)} + x - 1\Big|\Big).$$

Proof. For any $u \in (0,1)$, we set $\phi(u) = a^{1-u}\, b^u\, u^{x-1}\,(1-u)^{y-1}$. For all $u \in (0,1)$ we have,

$$\begin{aligned} |\phi'(u)| &= a^{1-u}\, b^u\, u^{x-2}\,(1-u)^{y-2}\,|\omega(u)| \\ &\leq \max(a,b)\,\frac{(x-2)^{x-2}\,(y-2)^{y-2}}{(x+y-4)^{x+y-4}} \sup_{u\in(0,1)} |\omega(u)|. \end{aligned}$$

where $\omega(u) = -u^2 \log(b/a) + (\log(b/a) - (x+y-2))\,u + x - 1$. If $x, y \geq 2$ then, it is easy to check that the function ω attains its maximum point for a certain value $u \in \{0, 1, u_0\}$ with $u_0 = \frac{1}{2}\Big[1 - \frac{x+y-2}{\log(b/a)}\Big]$. This implies that,

$$\begin{aligned} \sup_{u\in(0,1)} |\omega(u)| &= \max(|\omega(0)|, |\omega(1)|, |\omega(u_0)|) \\ &= \max\left(x-1, y-1, \left|\frac{(\log(b/a) - x - y + 2)^2}{4\log(b/a)} + x - 1\right|\right). \end{aligned} \tag{4.6}$$

Now if we apply (4.4), we get, for all $s \in (0,1)$,

$$\Big|\phi(s) - \int_0^1 \phi(u)\, du\Big| \leq \lambda_{a,b}^{x,y}\Big(\frac{1}{4} + \big(s - \frac{1}{2}\big)^2\Big).$$

This, when added to the expression of the function φ, achieves the proof. □

To state another result, we will use a particular version of Ostrowski's inequality [7, Corollary 11] recalled in the following lemma. This will enable us to establish another upper bound for the left term of (4.5).

Lemma 4.6. *Let φ be a differentiable and integrable function on (a,b) such that φ' is continuous on (a,b) with $\|\varphi'\|_1 =: \int_a^b |\varphi'(u)|du < \infty$. We have*

$$\Big|\int_a^b \varphi(u)du - \varphi(x)\big(b-a\big)\Big| \leq \left(\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right)\|\varphi'\|_1. \tag{4.7}$$

Theorem 4.7. *Consider four positive numbers a, b, x and y with $a \neq b$ and $x, y > 2$. For all $s \in (0,1)$ there holds,*

$$\Big|\mathcal{BL}(a,b;x,y) - a^{1-s}\, b^s\, s^{x-1}\,(1-s)^{y-1}\Big| \leq \beta_{a,b}^{x,y}\left(\frac{1}{2} + \left|s - \frac{1}{2}\right|\right)\mathcal{BL}(a,b,x-1,y-1), \tag{4.8}$$

where

$$\beta_{a,b}^{x,y} = \max\Big(x-1, y-1, \big|\frac{(\log(b/a)-x-y+2)^2}{4\log(b/a)}+x-1\big|\Big).$$

Proof. We consider again the function $\phi(u) = a^{1-u}\, b^u\, u^{x-1}\,(1-u)^{y-1}$, defined for $u \in [0,1]$. We have,

$$|\phi'(u)| = a^{1-u}\, b^u\, u^{x-2}\,(1-u)^{y-2} \times |-u^2 \log(b/a) + (\log(b/a)-(x+y-2))\, u + x - 1|.$$

So,

$$\begin{aligned}\parallel \phi' \parallel_1 &= \int_0^1 |\phi'(u)|\, du \\ &\le \sup_{u\in(0,1)} |-u^2 \log(b/a) + (\log(b/a)-(x+y-2))\, u + x - 1| \times \\ &\quad \int_0^1 a^{1-u}\, b^u\, u^{x-2}\,(1-u)^{y-2}\, du \le \mathcal{BL}(a,b,x-1,y-1) \times \\ &\qquad \max\Big(x-1, y-1, \big|\frac{(\log(b/a)-x-y+2)^2}{4\log(b/a)}+x-1\big|\Big).\end{aligned}$$

According to (4.7), we get for every $s \in [0,1]$,

$$\Big|\int_0^1 \phi(u)\, du - \phi(s)\Big| \le \Big(\frac{1}{2} + \big|s - \frac{1}{2}\big|\Big) \parallel \phi' \parallel_1 .$$

Whence, inequality (4.8) is deduced. □

Now, we will use another integral inequality known in the literature as Ostrowski-Grüss inequality stated by Dragomir and Wang in [8, Corollary 2.2].

Lemma 4.8. *Let $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable mapping on a nonempty interval $I \subset \mathbb{R}$, and let $a, b \in I$ with $a < b$. Suppose that $f \in L^1\big([a,b]\big)$ and for all $u \in [a,b]$, we have $|f'(u)| \le M$ for some $M > 0$. The inequality*

$$\Big|f(x) - \frac{1}{b-a}\int_a^b f(u)\, du - \frac{f(b)-f(a)}{b-a}\big(x - \frac{a+b}{2}\big)\Big| \le \frac{1}{2}(b-a)\, M, \tag{4.9}$$

holds for all $x \in [a,b]$.

Theorem 4.9. *Consider four positive numbers a, b, x and y with $a \neq b$ and $x, y > 2$. For every $s \in (0,1)$, we have*

$$\begin{aligned}\Big|\mathcal{BL}(a,b;x,y) - a^{1-s}\, b^s\, s^{x-1}\,(1-s)^{y-1}\Big| &\le \frac{1}{2}\,\frac{(x-2)^{x-2}\,(y-2)^{y-2}}{(x+y-4)^{x+y-4}} \times \max(a,b) \\ &\times \max\left(x-1, y-1, \left|\frac{(\log(b/a)-x-y+2)^2}{4\log(b/a)}+x-1\right|\right).\end{aligned} \tag{4.10}$$

Proof. Let the function φ be defined on $(0,1)$ by $\varphi(u) = a^{1-u}\, b^u\, u^{x-1}\,(1-u)^{y-1}$. We have for all $u \in (0,1)$,

$$|\varphi'(u)| \le \max(a,b)\, \frac{(x-2)^{x-2}\,(y-2)^{y-2}}{(x+y-4)^{x+y-4}}\, \max(x-1, y-1).$$

Employing (4.9) we obtain, for each $s \in (0,1)$,

$$\Big|\varphi(s) - \int_0^1 \varphi(u)\, du\Big| \leq \frac{1}{2}\,\frac{(x-2)^{x-2}\,(y-2)^{y-2}}{(x+y-4)^{x+y-4}}\,\max(a,b)$$
$$\times \max\Big(x-1, y-1, \Big|\frac{(\log(b/a)-x-y+2)^2}{4\log(b/a)} + x - 1\Big|\Big).$$

This added to the expression of the function φ concludes the proof. □

To establish more inequalities for the beta-logarithmic function we will use some variants of Ostrowski type inequalities which involves the second derivatives. We the recall the following result due to Milovanović and Pečarić [12].

Lemma 4.10. *Let $f : [a,b] \longrightarrow \mathbb{R}$ be a twice differentiable mapping such that $\| f'' \|_\infty := \sup\limits_{u\in(a,b)} |f''(u)| < \infty$. Then the inequality*

$$\Big|\frac{1}{2}\big[f(x) + \frac{(x-a)f(a)+(b-x)f(b)}{b-a}\big] - \frac{1}{b-a}\int_a^b f(t)\,dt\Big| \leq$$
$$\frac{\| f'' \|_\infty}{4}(b-a)^2\Big[\frac{1}{12} + \frac{\big(x-\frac{a+b}{2}\big)^2}{(b-a)^2}\Big], \quad (4.11)$$

holds for all $x \in (a,b)$.

Employing this lemma, we point out the following theorem.

Theorem 4.11. *Let $a, b, x,$ and y be four positive numbers with $a \neq b$ and $x, y > 3$. For all $s \in (0,1)$, we have*

$$\Big|\mathcal{BL}(a,b;x,y) - \frac{1}{2}a^{1-s}\,b^s\,s^{x-1}\,(1-s)^{y-1}\Big| \leq \frac{K_1+K_2}{4}\Big[\frac{1}{12} + \big(s-\frac{1}{2}\big)^2\Big], \quad (4.12)$$

where,

$$K_1 := \max\left(x-2, y-2, \left|\frac{(\log(b/a)-x-y+4)^2}{4\log(b/a)} + x - 2\right|\right) \times$$
$$\max\left(x-1, y-1, \left|\frac{(\log(b/a)-x-y+2)^2}{4\log(b/a)} + x - 1\right|\right) \times$$
$$\max(a,b)\frac{(x-3)^{x-3}\,(y-3)^{y-3}}{(x+y-6)^{x+y-6}}$$

and

$$K_2 := \max(a,b)\frac{(x-2)^{x-2}\,(y-2)^{y-2}}{(x+y-4)^{x+y-4}}\times$$
$$\max(|\log(b/a)-x-y+2|, |\log(b/a)+x+y-2|).$$

Proof. For the function $\varphi(u) = a^{1-u}\, b^u\, u^{x-1}\,(1-u)^{y-1}$ defined for $u \in [0,1]$, we have,

$$\begin{aligned}\varphi''(u) = a^{1-u}\, b^u\, u^{x-3}\,(1-u)^{y-3}(-u^2\,\log(b/a)+(\log(b/a)-(x+y-4))\,u+x-2)\times \\ (-u^2\,\log(b/a)+(\log(b/a)-(x+y-2))\,u+x-1)+ \\ a^{1-u}\, b^u\, u^{x-2}\,(1-u)^{y-2}[-2u\log(b/a)+\log(b/a)-x-y+2].\end{aligned}$$

On one hand, we know from (4.6) that

$$\sup_{u\in(0,1)} |-u^2\,\log(b/a)+(\log(b/a)-(x+y-2))\,u+x-1| \le \\ \max\left(x-1, y-1, \left|\frac{(\log(b/a)-x-y+2)^2}{4\log(b/a)}+x-1\right|\right).$$

We can also prove that

$$\sup_{u\in(0,1)} |-u^2\,\log(b/a)+(\log(b/a)-(x+y-4))\,u+x-2| \le \\ \max\left(x-2, y-2, \left|\frac{(\log(b/a)-x-y+4)^2}{4\log(b/a)}+x-2\right|\right).$$

On the other hand, we have

$$\sup_{u\in(0,1)} |-2u\log(b/a)+\log(b/a)-x-y+2| \le \\ \max(|\log(b/a)-x-y+2|, |\log(b/a)+x+y-2|).$$

Using Lemma 3.2, we get

$$\sup_{u\in(0,1)} a^{1-u}\, b^u\, u^{x-3}\,(1-u)^{y-3} \le \max(a,b)\frac{(x-3)^{x-3}\,(y-3)^{y-3}}{(x+y-6)^{x+y-6}}$$

and

$$\sup_{u\in(0,1)} a^{1-u}\, b^u\, u^{x-2}\,(1-u)^{y-2} \le \max(a,b)\frac{(x-2)^{x-2}\,(y-2)^{y-2}}{(x+y-4)^{x+y-4}}.$$

Consequently, φ'' is bounded on $(0,1)$ and we have

$$\parallel \varphi'' \parallel_\infty \le K_1 + K_2.$$

By using (4.11), we get (4.12). $\square$

At the end of this article, we will employ the subsequent Ostrowski-type inequality established by Cerone et al. in [4].

Lemma 4.12. *Let $f : [a,b] \longrightarrow \mathbb{R}$ be a twice differentiable mapping on (a,b) such that $\parallel f'' \parallel_\infty < \infty$. Then we have the inequality,*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt - \left(x-\frac{a+b}{2}\right)f'(x)\right| \le \\ \left[\frac{1}{24}(b-a)^2 + \frac{1}{2}\left(x-\frac{a+b}{2}\right)^2\right] \parallel f'' \parallel_\infty \le \frac{(b-a)2}{6} \parallel f'' \parallel_\infty \quad (4.13)$$

for all $x \in [a, b]$.

By this lemma, we obtain the following theorem.

Theorem 4.13. *Let* $a, b, x,$ *and* y *be with* $a \neq b$ *and* $x, y > 3$. *For all* $s \in [0, 1]$ *we have,*

$$\Big|\mathcal{BL}(a, b; x, y) + \left(s - \frac{1}{2}\right) a^{1-s}\, b^s\, s^{x-2}\, (1-s)^{y-2} \times$$
$$\big(-s^2 \log(b/a) + (\log(b/a) - (x+y-2))\, s + x - 1\big) - a^{1-s}\, b^s\, s^{x-1}\, (1-s)^{y-1}\Big| \leq$$
$$\left[\frac{1}{24} + \frac{1}{2}\left(s - \frac{1}{2}\right)^2\right] (K_1 + K_2) \leq \frac{K_1 + K_2}{6}. \quad (4.14)$$

Proof. Let us take once again the function defined on $[0, 1]$ by $\varphi(u) = a^{1-u}\, b^u\, u^{x-1}\, (1-u)^{y-1}$. Using the same notations as in the previous theorem we have, $\| \varphi'' \|_\infty \leq K_1 + K_2$. So, applying (4.13) we get for all $s \in [0, 1]$,

$$\Big|a^{1-s}\, b^s\, s^{x-1}\, (1-s)^{y-1} - \mathcal{BL}(a, b; x, y) -$$
$$\left(s - \frac{1}{2}\right) a^{1-s}\, b^s\, s^{x-2}\, (1-s)^{y-2}\, (-s^2 \log(b/a) + (\log(b/a) - (x+y-2))\, s + x - 1)\Big| \leq$$
$$\left[\frac{1}{24} + \frac{1}{2}\left(s - \frac{1}{2}\right)^2\right] (K_1 + K_2) \leq \frac{K_1 + K_2}{6},$$

which is exactly (4.14). □

Taking in (4.14) $s = 1/2$ we obtain the next approximation.

Corollary 4.14. *Let* a, b, x *and* y *be four positive numbers with* $a \neq b$ *and* $x, y > 3$. *We have,*

$$\Big|\mathcal{BL}(a, b; x, y) - \sqrt{a\, b}\, (1/2)^{x+y-2}\Big| \leq \frac{K_1 + K_2}{24}. \quad (4.15)$$

If we set $s = 0$ or $s = 1$ in (4.14) we get the following corollary.

Corollary 4.15. *Let* a, b, x *and* y *be four positive numbers with* $a \neq b$ *and* $x, y > 3$. *We have,*

$$\mathcal{BL}(a, b; x, y) \leq \frac{K_1 + K_2}{6}. \quad (4.16)$$

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