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COMPUTATION OF INTEGRAL TRANSFORMS IN TERMS OF DOUBLE HYPERGEOMETRIC SERIES ψ_2 WITH APPLICATIONS

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Abstract. The principal objective of the present work is to evaluate a recent integral transform involving the product of two Functions of Bessel of the first type stated in terms of the Humbert function ψ_2. We also derive some interesting special cases from the main result of this paper. The presented result will be applied to produce a few new laser beams called as the Humbert beams of type-II and study there propagation behaviour. From graphical simulations this type of wave seems interesting and useful in optical trapping which is used in some applications for pushing and moving particles such as: biological cells, neutral atoms, fluorescent, DNA molecules and trapping productivity on nano-sized spheres.

1. Introduction and preliminaries

The work of Belafhal and Hennani [2] is the pioneer of the present investigation. We have evaluated an expression in closed form of the integral $I^{\lambda}_{\mu,\nu}$ with an integrand that involves the product of two functions of Bessel of the first type with a quadratic argument. This integral has been derived regarding the Humbert's confluent hypergeometric function of two variables. The literature contains an incredibly high quantity of integral transforms incorporating a number of special functions. In 2016, Khan *et al.* [9] were able to derive an equation in terms of the Humbert function after studying the integral transform of the product of Bessel function and Whittaker function with a quadratic argument. In this reseach, we derived a novel closed form of the integral transform $I^{\lambda}_{\mu,\nu}$ whenever this improper converges.

Some important definitions were recalled to be used in this work as follows: We recall the generalized hypergeometric function ${}_aF_b$ defined as (see [7])

$$ {}_aF_b\left(\mu_1,\ldots,\mu_a;\nu_1,\ldots,\nu_b;z\right)=\sum_{p=0}^{\infty}\frac{\left(\mu_1\right)_p\ldots\left(\mu_a\right)_p}{\left(\nu_1\right)_p\ldots\left(\nu_b\right)_p}\frac{z^p}{p!}, \tag{1.1}$$

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where $(\beta)_s$ is the Pochhammer symbol given in connection with the familiar Gamma function, for $\beta \in \mathbb{C}$, by (see [14])

$$\begin{aligned}(\beta)_s &= \begin{cases} 1, & s=0 \\ \beta(\beta+1)\cdots(\beta+s-1), & s \in \mathbb{N}^* \end{cases} \\ &= \frac{\Gamma(\beta+s)}{\Gamma(\beta)}, \text{ for } \beta \in \mathbb{C} \setminus \mathbb{Z}_0^-, \end{aligned} \tag{1.2}$$

The Bessel function J_ν of the first type is explained by the series expansion (see [7])

$$J_\nu(x) = \sum_{k'=0}^{\infty} \frac{(-1)^{k'} (x/2)^{\nu+2k'}}{k'!\Gamma(\nu+k'+1)}, \tag{1.3}$$

with ν is the order and $x \in \mathbb{C} \setminus \{0\}$, $\nu \in \mathbb{C}$ and $\Re(\nu) > -1$.
The Humbert's confluent hypergeometric functions (see [14]) introduced by Humbert are defined as

$$\Psi_2\left[a'; b', c'; x, y\right] = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{(a')_{p+q}}{(b')_p (c')_q} \frac{x^p}{p!} \frac{y^q}{q!}, \tag{1.4}$$

and

$$\Phi_3\left[\alpha; \beta; x, y\right] = \sum_{a=0}^{\infty} \sum_{b=0}^{\infty} \frac{(\alpha)_a}{(\beta)_{a+b}} \frac{x^a}{a!} \frac{y^b}{b!}. \tag{1.5}$$

These functions converge absolutely $\forall x, y \in \mathbb{C}$.

We also recall the formula of the fourth Appell function with two variables (see [14]) expressed as

$$F_4\left[a_1, b_1; c_1, c'; x, y\right] = \sum_{l,s=0}^{\infty} \frac{(a_1)_{l+s} (b_1)_{l+s}}{(c_1)_l (c')_s} \frac{x^l}{l!} \frac{y^s}{s!}. \tag{1.6}$$

This function converges absolutely for $\sqrt{|x|} + \sqrt{|y|} < 1$.

2. Main results

In this part, we evaluate integral transform $I_{\mu,\nu}^{\lambda}$ with an integrand involving the product of two functions of Bessel of the first type with a quadratic argument. We have obtained an analytical formula regarding the Humbert's confluent hypergeometric function of two variables as an outcome of this integral transforms.

Theorem 2.1. *: The following integral transform holds true:*

$$\begin{aligned}\int_0^\infty x^{\lambda+1} e^{-\alpha x^2} J_\mu(\beta x) J_\nu(\gamma x)\, dx &= \frac{\left(\frac{\beta}{2}\right)^\mu \left(\frac{\gamma}{2}\right)^\nu \Gamma\left(\frac{\mu+\nu+\lambda}{2}+1\right)}{2\alpha^{\frac{\mu+\nu+\lambda}{2}+1} \mu! \nu!} \\ &\times \Psi_2\left[\frac{\mu+\nu+\lambda}{2}+1; \mu+1, \nu+1; -\frac{\beta^2}{4\alpha}, -\frac{\gamma^2}{4\alpha}\right],\end{aligned} \tag{2.1}$$

where $\Re(\alpha) > 0$, $\Re(\mu+\nu+\lambda) > -2$, $\beta > 0$ *and* $\gamma > 0$.

Proof. Using the following identity (see [11])

$$J_\nu = \frac{(z/2\ )^\nu}{\nu!}\ {}_0F_1\left(-;\nu+1;-\frac{z^2}{4}\right), \tag{2.2}$$

and presenting the expansion of J_μ introduced by (1.3), we obtain

$$I^\lambda_{\mu,\nu} = \frac{\left(\frac{\beta}{2}\right)^\mu\left(\frac{\gamma}{2}\right)^\nu}{\nu!}\sum_{m=0}^{\infty}\frac{\left(\frac{-\beta^2}{4}\right)^m}{m!\ \Gamma\left(\mu+m+1\right)}I_m, \tag{2.3}$$

where I_m is given by

$$I_m = \int\limits_0^\infty x^{\mu+\nu+\lambda+2m+1}e^{-\alpha x^2}{}_0F_1\left(-;\nu+1;-\frac{\gamma^2}{4}x^2\right)dx. \tag{2.4}$$

By using (1.1), and replacing the variable x^2 by t, (2.4) can be written as

$$I_m = \frac{1}{2}\sum_{k=0}^{\infty}\frac{\left(\frac{-\gamma^2}{4}\right)^k}{(\nu+1)_k k!}\int\limits_0^\infty t^{\frac{\mu+\nu+\lambda}{2}+m+k}e^{-\alpha t}dt. \tag{2.5}$$

The undermentioned identities are used to evaluate this integral (see [7, 14])

$$\int\limits_0^\infty t^{\sigma-1}e^{-\alpha t}dt = \frac{\Gamma(\sigma)}{\alpha^\sigma}, \tag{2.6}$$

and

$$(b)_{p'+q'} = (b)_{p'}(b+p')_{q'}, \tag{2.7}$$

which facilitates rearranging (2.5) to indicate

$$I_m = \frac{\Gamma\left(\frac{\mu+\nu+\lambda}{2}+m+1\right)}{2\alpha^{\frac{\mu+\nu+\lambda}{2}+m+1}}{}_1F_1\left(\frac{\mu+\nu+\lambda}{2}+m+1;\nu+1;-\frac{\gamma^2}{4\alpha}\right). \tag{2.8}$$

Now in view of (2.8), (2.3) becomes

$$\begin{aligned} I^\lambda_{\mu,\nu} =& \frac{(\beta/2)^\mu(\gamma/2)^\nu\Gamma\left(\frac{\mu+\nu+\lambda}{2}+1\right)}{2\mu!\nu!\alpha^{\frac{\mu+\nu+\lambda}{2}+1}}\sum_{m=0}^{\infty}\frac{\left(-\beta^2/4\alpha\ \right)^m}{m!}\frac{\left(\frac{\mu+\nu+\lambda}{2}+1\right)_m}{(\mu+1)_m} \\ &\times{}_1F_1\left(\frac{\mu+\nu+\lambda}{2}+m+1;\nu+1;\frac{-\gamma^2}{4\alpha}\right). \end{aligned} \tag{2.9}$$

Evaluating the previously mentioned equation using the expression of Ψ_2 shown as: [14],

$$\Psi_2\left[a';b',c';x,y\right] = \sum_{l=0}^{\infty}\frac{(a')_l}{(b')_l}\frac{x^l}{l!}\ {}_1F_1\left(a'+l;c';y\right). \tag{2.10}$$

our main result (2.1) is easily reached. This completes the evidence. □

Remark 2.2. It is easy to show that the result in (2.1) is equivalent to (see Eq. 6.633.1 of [7]). For that, we use a result (see [10, 13]) given by

$$\Psi_2\left[a';b',c';x,y\right]=\sum_{l=0}^{\infty}\frac{(a')_l}{(b')_l}\frac{x^l}{l!}\ {}_2F_1\left(-l,1-b'-l;c';\frac{y}{x}\right), \tag{2.11}$$

for $c'\neq 0,-1,-2,\ldots$ and $x\neq 0$. Consequently, (2.1) can be written as

$$\begin{aligned} I_{\mu,\nu}^{\lambda}=&\frac{(\beta/2)^{\mu}(\gamma/2)^{\nu}\Gamma\left(\frac{\mu+\nu+\lambda}{2}+1\right)}{2\mu!\nu!\alpha^{\frac{\mu+\nu+\lambda}{2}+1}}\sum_{m=0}^{\infty}\frac{(-\beta^2/4\alpha)^m}{m!}\frac{\left(\frac{\mu+\nu+\lambda}{2}+1\right)_m}{(\mu+1)_m}\\ &\times{}_2F_1\left(-m,-m-\mu;\nu+1;\frac{\gamma^2}{\beta^2}\right). \end{aligned} \tag{2.12}$$

Thus, our main result permit us to write (see Eq. 6.633.1 of [7]) which is an infinite series of the hypergeometric function ${}_2F_1$, in terms of the Humbert function ψ_2.

3. Particular cases

In this part, as particular cases of our primary outcome, we present some new interesting integral transforms by specializing certain parameters.

Corollary 3.1. *Let $\Re(p)>-1$, $|\arg\sqrt{\alpha}|<\frac{\pi}{4}$, $\beta>0$ and $\gamma>0$. Then*

$$\int_0^{\infty} xe^{-\alpha x^2}J_p(\beta x)\,J_p(\gamma x)\,dx=\frac{1}{2\alpha}e^{-\frac{\beta^2+\gamma^2}{4\alpha}}I_p\left(\frac{\beta\gamma}{2\alpha}\right). \tag{3.1}$$

Proof. By substituting $\lambda=0$ and $\mu=\nu=p$, (2.1) becomes

$$I_{p,p}^{0}=\frac{(\beta\gamma/4\alpha)^p}{2\alpha p!}\ \Psi_2\left[p+1;p+1,p+1;-\frac{\beta^2}{4\alpha},-\frac{\gamma^2}{4\alpha}\right]. \tag{3.2}$$

By using the following relationships (see [11])

$$\Psi_2\left[b;b,b;x,y\right]=e^{(x+y)}\ {}_0F_1\left(-;b;xy\right), \tag{3.3}$$

and

$$I_p(z)=\frac{(z/2)^p}{p!}\ {}_0F_1\left(-;p+1,\frac{z^2}{4}\right), \tag{3.4}$$

one can write the identity

$$\Psi_2\left[p+1;p+1,p+1;-\frac{\beta^2}{4\alpha},-\frac{\gamma^2}{4\alpha}\right]=p!\left(\frac{4\alpha}{\beta\gamma}\right)^p e^{-\frac{\beta^2+\gamma^2}{4\alpha}}I_p\left(\frac{\beta\gamma}{2\alpha}\right). \tag{3.5}$$

Thus, we get the desired assertion (3.1) which is analogous to the result in (see Eq. 6.633.2 of [7]). This completes the proof of corollary (3.1). □

Note that if we evaluate $I^0_{p,p}$ involving the product of $I_p(\beta x)$ with $J_p(\gamma x)$, we find the following result

$$\int_0^\infty xe^{-\alpha x^2} I_p(\beta x) J_p(\gamma x)\, dx = \frac{1}{2\alpha} e^{\frac{\beta^2-\gamma^2}{4\alpha}} J_p\left(\frac{\beta\gamma}{2\alpha}\right), \tag{3.6}$$

is equivalent to (see Eq. 6.633.4 of [7]).

Corollary 3.2. *Let $\beta > 0$, $\Re(\mu+\nu+\lambda) > 2$ and $\Re(\alpha) > -2$. Then*

$$\int_0^\infty x^{\lambda+1} e^{-\alpha x^2} J_\mu(\beta x)\, J_\nu(\beta x)\, dx = \frac{(\beta/2\,)^{(\mu+\nu)}\,\Gamma\left(\frac{\mu+\nu+\lambda}{2}+1\right)}{2\mu!\,\nu!\,\alpha^{\frac{\mu+\nu+\lambda}{2}+1}}$$
$$\times\,{}_3F_3\left(\frac{\mu+\nu+\lambda}{2}+1, \frac{\mu+\nu+1}{2}, \frac{\mu+\nu}{2}+1; \mu+1, \nu+1, \mu+\nu+1; -\frac{\beta^2}{\alpha}\right). \tag{3.7}$$

Proof. The following representation (see [10])

$$\Psi_2[a; b, c; x, x] = {}_3F_3\left(a, \frac{c+b-1}{2}, \frac{c+b}{2}; b, c, c+b-1; 4x\right) \tag{3.8}$$

allows us to easily obtain (3.7).
Note here that, in the case of $\lambda = \chi - 2$, (3.7) becomes

$$I^{\chi-2}_{\mu,\nu} = \int_0^\infty x^{\chi-1} e^{-\alpha x^2} J_\mu(\beta x)\, J_\nu(\beta x)\, dx = \frac{(\beta/2\,)^{\mu+\nu}}{2\mu!\nu!} \frac{\Gamma\left(\frac{\mu+\nu+\chi}{2}\right)}{\alpha^{\frac{\mu+\nu+\chi}{2}}}$$
$$\times\,{}_3F_3\left(\frac{\mu+\nu+\chi}{2}, \frac{\mu+\nu+1}{2}, \frac{\mu+\nu}{2}+1; \mu+1, \nu+1, \mu+\nu+1; -\frac{\beta^2}{\alpha}\right). \tag{3.9}$$

This last equation is analogous to (see Eq. 6.633.5 of [7]). □

Corollary 3.3. *Let $\beta > 0$, $\gamma > 0$, $\Re(\alpha) > 0$ and $\Re(\mu) > -1$. Then*

$$\int_0^\infty x^{\lambda+1} e^{-\alpha x^2} J_\mu(\beta x)\, J_{\mu-\lambda}(\gamma x)\, dx$$
$$= \frac{(\beta\gamma/4\alpha\,)^\mu}{2\alpha\,(\mu-\lambda)!(\gamma/2\,)^\lambda}\; e^{-\frac{\beta^2+\gamma^2}{4\alpha}}\; \Phi_3\left(-\lambda; \mu-\lambda+1; \frac{\gamma^2}{4\alpha}, \frac{\beta^2\gamma^2}{16\alpha^2}\right). \tag{3.10}$$

Proof. If we take $\nu = \mu - \lambda$ and apply the following identity (see [10])

$$\Psi_2[b; b, c; x, y] = e^{(x+y)} \Phi_3(c-b; c; -y, xy), \tag{3.11}$$

where Φ_3 is given by (1.5), we get the desired assertion (3.10). □

Lemma 3.4. *Let $\Re(\alpha) > 0$, $\beta > 0$, and $\gamma > 0$. Then*

$$\Phi_3\left[-; p+1; \frac{\gamma^2}{4\alpha}, \frac{\beta^2\gamma^2}{16\alpha^2}\right] = p!\left(\frac{4\alpha}{\beta\gamma}\right)^p I_p\left(\frac{\beta\gamma}{2\alpha}\right). \tag{3.12}$$

Proof. If $\lambda = 0$ and $\mu = p$ (3.10) becomes

$$I^0_{p,p} = \frac{(\beta\gamma/4\alpha\ )^p}{2\alpha p!}\ e^{-\frac{\beta^2+\gamma^2}{4\alpha}}\ \Phi_3\left(-;p+1;\frac{\gamma^2}{4\alpha},\frac{\beta^2\gamma^2}{16\alpha^2}\right). \tag{3.13}$$

With the help of (3.1), one can deduce a particular expression of the Humbert function Φ_3 given by (3.12). □

Corollary 3.5. *Let $\Re(\alpha) > 0$, $\beta > 0$, $\gamma > 0$ and $\Re(\nu) > \Re(\mu)$. Then*

$$\begin{aligned}&\int_0^\infty x^{\nu-3\mu-2}e^{-\alpha x^2}J\mu\left(\beta x\right)J_\nu\left(\gamma x\right)dx\\&=\left(\frac{\alpha\beta}{2}\right)^\mu\left(\frac{\gamma}{2\alpha}\right)^\nu\frac{\Gamma\left(\nu-\mu\right)}{2\mu!\nu!}e^{-\gamma^2/4\alpha}\ {}_1F_1\left(\nu-\mu;\nu+1;\frac{\gamma^2-\beta^2}{4\alpha}\right).\end{aligned} \tag{3.14}$$

Proof. If we take $\lambda = \nu - 3\mu - 2$ and by using the identity (see [11])

$$\Psi_2\left[b;b',b+b';x,y\right] = e^y{}_1F_1\left(b;b+b';x-y\right), \tag{3.15}$$

we get (3.14). □

Corollary 3.6. *Let $\Re(\alpha) > 0$, $\beta > 0$ and $\Re\left(\mu\right) > \frac{1}{2}$. Then*

$$\begin{aligned}&\int_0^\infty x^{-\mu}e^{-\alpha x^2}J_\mu\left(\beta x\right)J_{2\mu+1}\left(\beta x\right)dx\\&=\frac{\beta}{4\alpha\left(2\mu+1\right)!}\left(\frac{\beta^3}{8\alpha}\right)^\mu{}_2F_2\left(\frac{3\left(\mu+1\right)}{2},\frac{3\mu}{2}+1;2\left(\mu+1\right),3\mu+2;-\frac{\beta^2}{\alpha}\right).\end{aligned} \tag{3.16}$$

Proof. It's easy to prove (3.16) by taking $\lambda = -(\mu+1)$, $\nu = 2\mu+1$ and by using the following identity (see [13])

$$\Psi_2\left[b;b,2b;x,x\right] = {}_2F_2\left(\frac{3b}{2},\frac{3b-1}{2};2b,3b-1;4x\right). \tag{3.17}$$

□

Corollary 3.7. *Let $\Re(\alpha) > 0$, $\beta > 0$ and $\Re(2\mu+\lambda) > -2$. Then*

$$\begin{aligned}&\int_0^\infty x^{\lambda+1}e^{-\alpha x^2}J_\mu\left(\beta x\right)I_\mu\left(\beta x\right)dx = \frac{\Gamma\left(\lambda/2\ +\mu+1\right)}{2(\mu!)^2\alpha^{\lambda/2\ +1}}\left(\frac{\beta}{4\alpha}\right)^\mu\\&\times{}_2F_3\left(\frac{\lambda+2\left(\mu+1\right)}{4},\frac{\lambda+2\left(\mu+2\right)}{4};\mu+1,\frac{\mu+1}{2},\frac{\mu}{2}+1;-\frac{\beta^4}{16\alpha^2}\right)\end{aligned} \tag{3.18}$$

Proof. If $\mu = \nu$, $\gamma = i\beta$ and by using the identites (see [7], [10])

$$I_\nu\left(z\right) = i^{-\gamma}J_\nu\left(iz\right), \tag{3.19}$$

and

$$\Psi_2\left[a_1;c_1,c_1;x,-x\right] = {}_2F_3\left(\frac{a_1}{2},\frac{a_1+1}{2};c_1,\frac{c_1}{2},\frac{c_1+1}{2};-x^2\right), \tag{3.20}$$

(3.18) is deduced. □

Corollary 3.8. *Let* $\Re(\alpha) > 0$, $\beta > 0$ *and* $\Re(\mu) > \frac{1}{2}$. *Then*

$$\begin{aligned}\int_0^\infty e^{-\alpha x^2} J_\mu(\beta x)\, I_\mu(\beta x) dx &= \left(\frac{\beta^2}{4\alpha}\right)^\mu \frac{\Gamma(\mu + 1/2)}{2\sqrt{\alpha}(\mu!)^2} \\ &\times {}_1F_1\left(\frac{2\mu+1}{4}; \mu+1; -\frac{i\beta^2}{2\alpha}\right) {}_1F_1\left(\frac{2\mu+1}{4}; \mu+1; \frac{i\beta^2}{2\alpha}\right).\end{aligned} \tag{3.21}$$

Proof. This case is obtained by taking $\lambda = -1$, $\mu = \nu$, $\gamma = i\beta$ and using the identity (see [10])

$$\Psi_2\left[2a; 2a+\frac{1}{2}, 2a+\frac{1}{2}; x, -x\right] = {}_1F_1\left(a; 2a+\frac{1}{2}; 2ix\right) {}_1F_1\left(a; 2a+\frac{1}{2}; -2ix\right). \tag{3.22}$$

So, our main finding given by (2.1). The evidence of the corollary is thus completed. □

Corollary 3.9. *Let* $\Re(\alpha) > 0$, $\beta > 0$, $\gamma > 0$ *and* $\Re(\lambda + \mu + \gamma) > -2$. *Then*

$$\begin{aligned}\int_0^\infty x^{\lambda+1} e^{-\alpha x^2} J_{-\frac{1}{2}}(\beta x) J_{-\frac{1}{2}}(\gamma x)\, dx &= \frac{\Gamma((\lambda+1)/2)\,\Gamma(\lambda/2+1)\, 2^{\frac{\lambda-3}{2}}}{\sqrt{\pi\beta\gamma}(-^1/_2!)^2 \alpha^{\frac{\lambda+1}{2}}} e^{-\frac{\beta^2+\gamma^2}{8\alpha}} \\ \times \Bigg\{ e^{-\frac{\beta\gamma}{4\alpha}} &\left[D_{-(\lambda+1)}\left(-\frac{i}{\sqrt{2\alpha}}(\beta+\gamma)\right) + D_{-(\lambda+1)}\left(\frac{i}{\sqrt{2\alpha}}(\beta+\gamma)\right)\right] \\ + e^{\frac{\beta\gamma}{4\alpha}} &\left[D_{-(\lambda+1)}\left(-\frac{i}{\sqrt{2\alpha}}(\beta-\gamma)\right) + D_{-(\lambda+1)}\left(\frac{i}{\sqrt{2\alpha}}(\beta-\gamma)\right)\right] \Bigg\}.\end{aligned} \tag{3.23}$$

Proof. If we take $\mu = \nu = -\frac{1}{2}$, (2.1) can be written as

$$I^{\lambda}_{-^1/_2 -^1/_2} = \frac{\Gamma\left(\frac{\lambda+1}{2}\right)}{2\left(\frac{\beta\gamma}{4}\right)^{1/2}\left(-\frac{1}{2}!\right)^2 \alpha^{\left(\frac{\lambda+1}{2}\right)}} \Psi_2\left[\frac{\lambda+1}{2}; \frac{1}{2}, \frac{1}{2}; -\frac{\beta^2}{4\alpha}, -\frac{\gamma^2}{4\alpha}\right]. \tag{3.24}$$

We know that (see [4])

$$\begin{aligned}\Psi_2\left[a; \frac{1}{2}, \frac{1}{2}; x, y\right] &= \frac{2^{a-2}}{\sqrt{\pi}} e^{(x+y)/2}\, \Gamma\left(a+\frac{1}{2}\right) \\ &\times \Big\{ e^{\sqrt{xy}}\left[D_{-2a}(-\sqrt{2}(\sqrt{x}+\sqrt{y})) + D_{-2a}(\sqrt{2}(\sqrt{x}+\sqrt{y}))\right] \\ &+ e^{-\sqrt{xy}}\left[D_{-2a}(-\sqrt{2}(\sqrt{x}-\sqrt{y})) + D_{-2a}(\sqrt{2}(\sqrt{x}-\sqrt{y}))\right] \Big\},\end{aligned} \tag{3.25}$$

where D_ν is the parabolic cylinder function. With this representation and by taking $a = \frac{\lambda+1}{2}$, $x = -\frac{\beta^2}{4\alpha}$ and $y = -\frac{\gamma^2}{4\alpha}$, the corollary is proved. □

Corollary 3.10. *Let $\Re(\alpha) > 0$, $\beta > 0$, $\gamma > 0$, $\Re(\lambda) > -2$ and $\nu = -\mu = \frac{1}{2}$. Then*

$$\begin{aligned}
&\int_0^\infty x^{\lambda+1} e^{-\alpha x^2} J_{-\frac{1}{2}}(\beta x)\, J_{\frac{1}{2}}(\gamma x)\, dx \\
&= \sqrt{\frac{\gamma}{\pi\beta\alpha^\lambda}} \frac{\Gamma(\lambda/2+1)}{(-^1/_2!)\,(^1/_2!)} \frac{2^{\frac{\lambda-5}{2}}}{i\gamma} \Gamma\left(\frac{\lambda+1}{2}\right) e^{-\frac{\beta^2+\gamma^2}{8\alpha}} \\
&\times \left\{ e^{-\frac{\beta\gamma}{4\alpha}} \left[D_{-(\lambda+1)}\left(-\frac{i}{\sqrt{2\alpha}}(\beta+\gamma)\right) - D_{-(\lambda+1)}\left(\frac{i}{\sqrt{2\alpha}}(\beta+\gamma)\right)\right]\right. \\
&\left. - e^{\frac{\beta\gamma}{4\alpha}} \left[D_{-(\lambda+1)}\left(-\frac{i}{\sqrt{2\alpha}}(\beta-\gamma)\right) - D_{-(\lambda+1)}\left(\frac{i}{\sqrt{2\alpha}}(\beta-\gamma)\right)\right]\right\}.
\end{aligned} \tag{3.26}$$

Proof. It is easy to prove (3.26) by introducing in (3.12) the identity (see [4])

$$\begin{aligned}
&\Psi_2\left[a; \frac{1}{2}, \frac{3}{2}; x, y\right] = \frac{2^{a-^7/_2}}{\sqrt{\pi y}}\; e^{(x+y)/_2}\; \Gamma\left(a-\frac{1}{2}\right) \\
&\times \left\{ e^{\sqrt{xy}} \left[D_{1-2a}(-\sqrt{2}(\sqrt{x}+\sqrt{y})) - D_{1-2a}(\sqrt{2}(\sqrt{x}+\sqrt{y}))\right]\right. \\
&\left. - e^{-\sqrt{xy}} \left[D_{1-2a}(-\sqrt{2}(\sqrt{x}-\sqrt{y})) - D_{1-2a}(\sqrt{2}(\sqrt{x}+\sqrt{y}))\right]\right\}.
\end{aligned} \tag{3.27}$$

□

4. APPLICATION: A HOLLOW HUMBERT BEAM OF TYPE-II GENERATION

This section investigates the generation of novel beams family called Humbert beams of type-II by a spiral phase plate (SPP).

For this, we consider the propagation of a Bessel-Gauss beam via this element which its field is given at $z' = 0$ in the cylindrical coordinates (ρ', ϕ', z') by (see [6])

$$E_m(\rho', \phi', z' = 0) = E_f e^{im\phi'} J_m(\alpha\rho')\, e^{-\rho'^2/\omega_0^2}, \tag{4.1}$$

where $\alpha = k \sin\theta$, m is the order of beam, w_0 is the Gaussian waist envelope, E_f is the constant parameter and J_m is the order of mth Bessel function of the first type. When $\alpha \to \infty$, the considered field becomes a Gaussian beam. However, for $m \neq 0$ to a higher order Gauss-Laguerre beam. We obtain a nondiffracting beam referred to as Bessel beam, when $\omega_0 \to \infty$.

The conversion of the field given by (4.1) is studied by a SPP with negligible thickness in order to create a novel hollow family beam spreading via an ABCD optical system (see Fig. 1). The expression $e^{i\chi\phi'}$ is azimuth dependent phase radiation which is enforced by SPP, where $\chi = \frac{h.\Delta n}{\lambda}$ is the topological charge, with λ is the wavelength of the incident field, h is the step height and Δn is the difference of refractive index between the SPP and its surrounding.

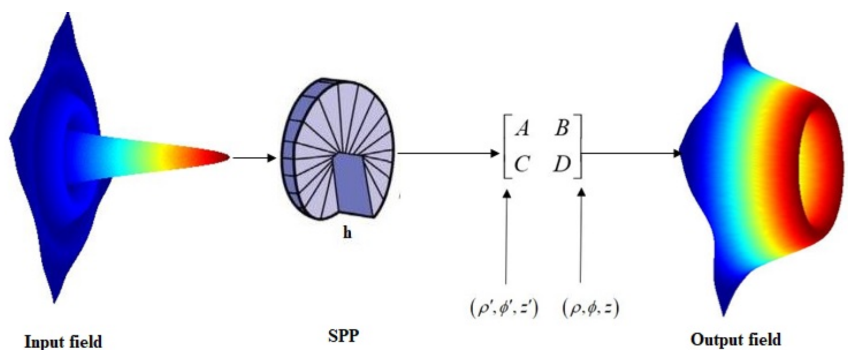

FIGURE 1. Production of a donut Humbert beam of type-II by propagating a pseudo-nondiffracting beam via a SPP.

Next, the Collins-Huygens integral is applied (see [5]) on the incident beam centred positioned z' on the SPP, which can be defined as:

$$E_m^{\chi}(\rho,\phi,z) = -\frac{i}{\lambda B}e^{ikz}\int_0^{+\infty}\int_0^{2\pi} E_m(\rho',\phi',z')exp\left\{\frac{ik}{2B}\left(A\rho'^2 + D\rho^2\right)\right\} \times \exp\left[-i\frac{k\rho\rho'\cos\left(\phi-\phi'\right)}{B}\right]\rho'.d\rho'.d\phi', \tag{4.2}$$

where

$$E_m^{\chi}\left(\rho',\phi',z'\right) = E_f\frac{\omega_0}{\omega\left(z'\right)}J_m\left(\alpha'\rho'\right)e^{-\beta\rho'^2}e^{ikz'}e^{im\phi'}e^{i\chi\phi'}, \tag{4.3}$$

with

$$\alpha' = \frac{\alpha}{1+{}^{iz'}/_{z_R}},\ \beta = \frac{1}{\omega^2\left(z'\right)} - i\frac{k}{2R\left(z'\right)},\ \omega\left(z'\right) = \omega_0\sqrt{1+{}^{z'^2}/_{z_R^2}},$$
$$R\left(z'\right) = \frac{z_R^2}{z'} + z' \text{ and } \phi'\left(z'\right) = tg^{-1}\left(\frac{z'}{z_R}\right). \tag{4.4}$$

In the above equations, $k = {}^{2\pi}/_{\lambda}$ is the wave number and z_R is the Rayleigh range. For the topological charge, we take $\chi = l$ with l is an integer (see [3]).

By substituting (4.3) in (4.2) and with the help of the following identity (see [7])

$$\int_0^{2\pi} e^{i(m+l)\phi'}e^{-i\frac{k\rho\rho'\cos(\phi-\phi')}{B}}d\phi' = 2\pi i^{(m+l)}e^{i(m+l)\phi}J_{(m+l)}\left(\frac{k\rho}{B}\rho'\right), \tag{4.5}$$

and after tedious algebraic calculations, we obtain

$$E_m^l\left(\rho,\phi,z\right) = -2\pi\frac{E_f\omega_0}{\lambda B\omega\left(z'\right)}i^{m+l+1}e^{ik\left(z+z'+{}^{D\rho^2}/_{2B}\right)}e^{i(m+l)\phi}.K_m^l, \tag{4.6}$$

where

$$K_m^l = \int_0^{+\infty} e^{-\left(\beta - i\frac{kA}{2B}\right)\rho'^2} J_m\left(\alpha'\rho'\right) J_{m+l}\left(\frac{k\rho}{B}\rho'\right)\rho' d\rho'. \tag{4.7}$$

This last equation can be evaluated by using our Theorem 2.1, and the result can be rearranged as

$$E_m^l\left(\rho,\phi,z\right) = A_m^l \times \Psi_2\left[m+\frac{l}{2}+1; m+1, m+l+1; -\frac{\alpha'^2}{4\xi}, -\frac{k^2}{4\xi B^2}\rho^2\right], \tag{4.8}$$

where

$$\xi = \beta - i\frac{kA}{2B},$$

and

$$A_m^l = -\frac{\pi E_f}{\lambda B\xi}\frac{\omega_0}{\omega\left(z'\right)} i^{m+l+1}\left(\frac{k\alpha'}{4B\xi}\right)^m\left(\frac{k}{2B\sqrt{\xi}}\right)^l\frac{\Gamma\left(m+\frac{l}{2}+1\right)}{m!\left(m+l\right)!}$$
$$\times e^{ik\left(z+z'+D\rho^2/2B\right)} e^{i(m+l)\phi}\rho^{(m+l)}.$$

The expression in (4.8) is the output beam generated behind the SPP stated as function of Humbert's confluent hypergeometric function Ψ_2 defined by (1.4). For this reason, this beams family is known as: Donut Humbert beam of type-II. Note that the term $\rho^{(m+l)}$ in (4.8) yields a donut field amplitude propagating through any ABCD optical system. These type of beams are very useful and suitable to trap or manipulate the nano-particles (see [1]).

Our new field is a generation of some used beams. Then, by using the following relation

$${}_1F_1\left(a+n; 2\mu+1, y\right) = \frac{e^{y/2}}{y^{\mu+1/2}} M_{\mu+\frac{1}{2}-a-n,\mu}\left(y\right), \tag{4.9}$$

where M is the Whittaker function, (2.10) can be expressed as

$$\Psi_2\left[a_1; b_1, c_1; x', y'\right] = \frac{e^{y'/2}}{y'^{c_1/2}}\sum_{s=0}^{\infty}\frac{\left(a_1\right)_s}{\left(b_1\right)_s}\frac{x'^s}{s!} M_{\frac{c_1}{2}-a_1-s,\frac{c_1-1}{2}}\left(y'\right). \tag{4.10}$$

Replacing (4.10) into (4.8), we obtain

$$E_m^l\left(\rho,\phi,z\right) = A\frac{\exp\left(-k^2\rho^2/8B^2\xi\right)}{\left(-k^2\rho^2/4B^2\xi\right)^{\frac{m+l+1}{2}}}\sum_{n=0}^{\infty}\frac{\left(m+\frac{l}{2}+1\right)_n}{\left(m+1\right)_n}\frac{\left(-\alpha'^2/4\xi\right)^n}{n!}$$
$$\times M_{-\frac{(m+1)}{2}-n,\frac{m+l}{2}}\left(-k^2\rho^2/4B^2\xi\right). \tag{4.11}$$

This last output field is a superposition of Whittaker beams and a generalization of hypergeometric fields proposed by Karimi *et al.* [8].

One can note that, for some special values of the index of the Whittaker function, we can prove that the donut Humbert beam of type-II is a superposition of Bessel-modulated Gaussian, Laguerre-Gaussian and Hermite-Gaussian lights.

In the following, some numerical simulations are depicted to study the propagation through an ABCD optical system of this new beam. The calculation parameters are: $z = 3000mm$, $z' = 300mm$, $\omega_0 = 1mm$, $\lambda = 632.8nm$ and $l = 1$. Fig. (2) presents the intensity distribution for Humbert beams of type-II for two values of α. We observe from this curve, that when $m = 0$ and by increasing the parameter α, the intensity increases.

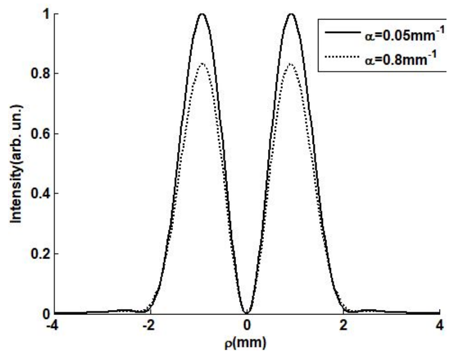

FIGURE 2. Intensity profile of Humbert beams of type-II with $m = 0$ and two values of α.

The 3D simulation and its corresponding contour graph is given in Fig. 3, to show the influence of the beam order m on the evolution of the donut Humbert beam of type-II propagating in a free-space.

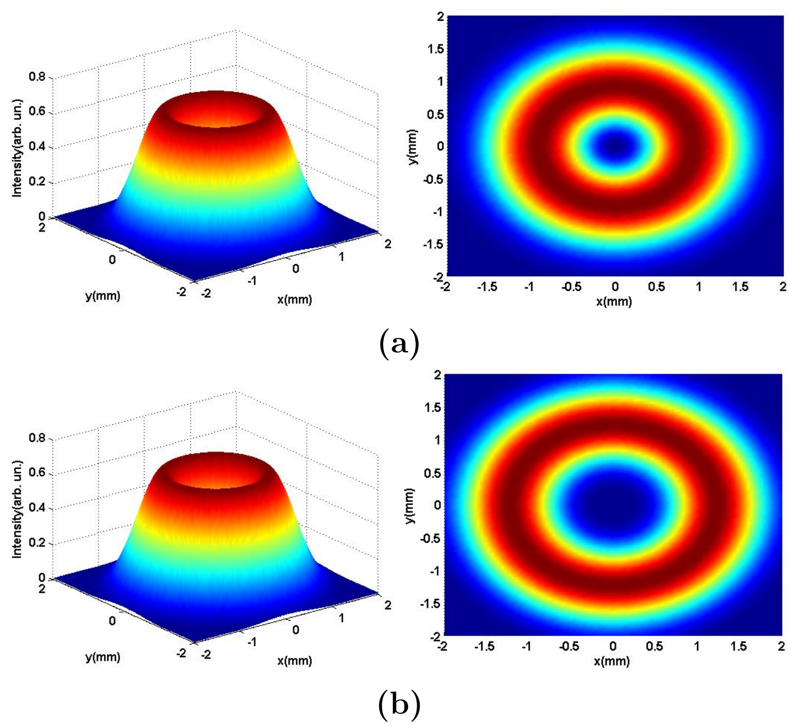

FIGURE 3. Intensity profile of Humbert beams of type-II in 3D and 2D, with **(a)** m=0 and **(b)** m=1, for $\alpha = 0.8mm^{-1}$.

These plots show that the considered beam has darkness in the center, which increases as a result of the increment of the beam order m.

Figure. 4 shows the impact of the variation of the parameter ω_0 on the propagation for Humbert beams of type-II with $l = 2$, $m = 0$, $\alpha = 0.8mm^{-1}$ and the other parameters are the same of those taken in Figs. (2) and (3). As shown in the below figure, the maximum of the intensity of the main lobe and also the side lobes increase when ω_0 increases.

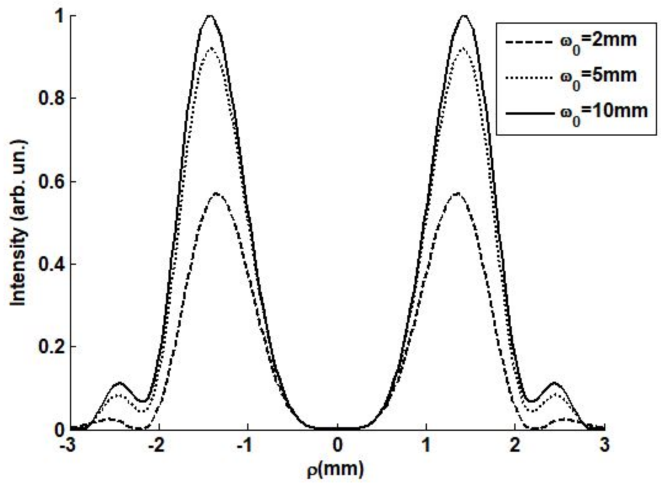

FIGURE 4. Intensity profile of Humbert beams of type-II with l=2, m=0 and for three values of ω_0.

5. Conclusion

The integral transforms containing the product of Bessel functions is computed regarding the Humbert Function ψ_2. This integral has a distinct advantage in generating a family of new beams oftenly encountered in the problems of physics. Some interesting cases have also been investigated as special case of our main result. As an application of our main findings, we generate a new kind of donut waves called Humbert beams of type-II. Several graphical representations are also established to show the influence of some parameters on the behaviour of this new beams family.

Conflict of Interest: The authors declare that there is no conflict of interest.

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