Gulf Journal of Mathematics
Vol 16, Issue 1 (2024) 193-204

ISSN: 2309-4966
https://doi.org/10.56947/gjom.v16i1.1799

L^q INEQUALITIES FOR POLYNOMIALS WITH RESTRICTED ZEROS CONCERNING THE GROWTH

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ABSTRACT. If $\mathcal{P}_n$ denotes the class of polynomials of degree at most n, then for $P \in \mathcal{P}_n$, a consequence of Maximum Modulus theorem yields for $R > 1$,

$$\|P(R,\ .)\|_\infty \leq R^n \|P\|_\infty.$$

Various generalizations and refinements of this result are available in literature. In this paper, we consider a general class of polynomials $\mathcal{P}_{n,\mu}$ $1 \leq \mu \leq n$, with restriction on zeros in a specific way and obtain Zygmund-type inequalities concerning the growth of polynomials. Besides obtaining a refinement of a result due to Aziz and Shah, we improve a result of Aziz and Rather.

1. Introduction

Let $\mathcal{P}_{n,\mu}$, $1 \leq \mu \leq n$ be the class of polynomials

$$P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j, \tag{1.1}$$

of degree at most n. We choose this class of polynomials to obtain results that are more general and useful.
For $P \in \mathcal{P}_{n,\mu}$, define

$$\|P\|_q = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |P(e^{i\theta})|^q d\theta \right\}^{\frac{1}{q}},\ 1 \leq q < \infty \tag{1.2}$$

and

$$\|P\|_\infty = \max_{|z|=1} |P(z)|. \tag{1.3}$$

The study of Bernstein-type inequalities that relate the norm of a polynomial to its derivative and their various versions is a classical topic in Analysis. One of the basic and famous result in this regard due to Bernstein [7] read as:

Date: Received: May 30, 2023; Accepted: January 22, 2024.
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2010 *Mathematics Subject Classification.* 30A10; 30C10; 30C15.
Key words and phrases. Polynomial, Growth, Restricted zeros, Integral Mean, Inequalities.

If $P(z)$ is a polynomial of degree n, then

$$\|P'\|_\infty \leq n\|P\|_\infty. \tag{1.4}$$

Also concerning the maximum modulus of $P(z)$ on $|z| = R > 1$, we have

$$\|P(R,\ .)\|_\infty \leq R^n\|P\|_\infty. \tag{1.5}$$

Inequality (1.5) is a simple consequence of Maximum Modulus Principle.
Zygmund [19] proved an integral analog of inequality (1.4) in the following way

$$\|P'\|_q \leq n\|P\|_q,\ q \geq 1. \tag{1.6}$$

Whereas the following inequality which is extension of inequality (1.5), can be proved by using a result of Hardy [8],

$$\|P(R,\ .)\|_q \leq R^n\|P\|_q,\ R > 1, q > 0. \tag{1.7}$$

Arestov [1] proved that (1.6) remains true for $0 < q < 1$ as well. Aziz and Rather [2] obtained a generalization of inequalities (1.6) and (1.7). In fact, they proved that
If $P \in \mathcal{P}_n$, then for every $R > 1$ and $q \geq 1$,

$$\|P(Rz) - P(z)\|_q \leq (R^n - 1)\|P\|_q. \tag{1.8}$$

There exist number of results in the literature as refinements and generalizations of inequalities of (1.6) and (1.7) in various ways for references see [9, 11, 17, 18]. Aziz and Rather [3] considered a more general case for investigating the dependence of

$$\|P(Rz) - \beta P(rz)\|_q \text{ on } \|P\|_q$$

for every $\beta \in \mathbb{C}$ with $|\beta| \leq 1$, $R > r \geq 1$, $q > 0$ and extended inequality (1.8) for $0 < q < 1$ as following.

Theorem A. *If $P \in \mathcal{P}_n$, then for every $\beta \in \mathbb{C}$ with $|\beta| \leq 1$ and $R > r \geq 1$, $q > 0$,*

$$\|P(Rz) - \beta P(rz)\|_q \leq |R^n - \beta r^n|\|P\|_q. \tag{1.9}$$

Also for the class of polynomials not vanishing in $|z| < 1$, they proved:

Theorem B. *If $P \in \mathcal{P}_n$ and $P(z) \neq 0$ in $|z| < 1$, then for every $\beta \in \mathbb{C}$ with $|\beta| \leq 1$ and $R > r \geq 1$, $q > 0$,*

$$\|P(Rz) - \beta P(rz)\|_q \leq \frac{\|(R^n - \beta r^n)z + (1 - \beta)\|}{\|1 + z\|_q}\|P\|_q. \tag{1.10}$$

Also Aziz and Shah [6] proved a more general result in the following way.

Theorem C. *If $P \in P_{n,\mu}$ and $P(z) \neq 0$ in $|z| < k$, $k \geq 1$, then for $R \geq 1$ and $q > 0$,*

$$\|P(Rz) - P(z)\|_q \leq \frac{R^n - 1}{\|C_{\mu,k} + z\|_q}\|P\|_q. \tag{1.11}$$

Where

$$C_{\mu,k} = \frac{k^{\mu+1}\left\{\frac{R^{\mu}-1}{R^{n}-1}\left|\frac{a_{\mu}}{a_0}\right|k^{\mu-1}+1\right\}}{1+\frac{R^{\mu}-1}{R^{n}-1}\left|\frac{a_{\mu}}{a_0}\right|k^{\mu+1}} \tag{1.12}$$

The result is best possible in case $k = 1$ and equality holds for $P(z) = z^n + 1$.

In this paper, we consider a class of polynomials from $\mathcal{P}_{n,\mu}$ which does not vanish in $|z| < k$, $k \geq 1$ and obtain the following result which includes Theorem B as a special case and refines Theorem A. As a generalization of some already known results, we prove a result concerning the rate of growth of polynomials while putting restriction on the zeros.

2. Main Results

Theorem 1. *If $P \in \mathcal{P}_{n,\mu}$, with $P(z) \neq 0$ for $|z| < k$, $k \geq 1$, then for every complex number β with $|\beta| \leq 1$, $R > r \geq 1$, $1 \leq \mu \leq n$, and $q > 0$,*

$$\|P(Rz) - \beta P(rz)\|_q \leq \frac{|R^n - \beta r^n|}{\|c^*_{\mu,k} + z\|_q}\|P\|_q, \tag{2.1}$$

where

$$c^*_{\mu,k} = k^{\mu+1}\frac{\frac{R^{\mu}-r^{\mu}}{R^{n}-r^{n}}\left|\frac{a_{\mu}}{a_0}\right|k^{\mu-1}+1}{1+\frac{R^{\mu}-r^{\mu}}{R^{n}-r^{n}}\left|\frac{a_{\mu}}{a_0}\right|k^{\mu+1}}. \tag{2.2}$$

For $r = 1$, Theorem 1 gives

Corollary 1.1 *If $P \in \mathcal{P}_{n,\mu}$, with $P(z) \neq 0$ for $|z| < k$, $k \geq 1$, then for every complex number β with $|\beta| \leq 1$, $R > 1$ and $q > 0$,*

$$\|P(Rz) - \beta P(z)\|_q \leq \frac{|R^n - \beta|}{\|c^*_{\mu,k} + z\|_q}\|P\|_q, \tag{2.3}$$

where for $1 \leq \mu \leq n$,

$$c^*_{\mu,k} = k^{\mu+1}\frac{\frac{R^{\mu}-1}{R^{n}-1}\left|\frac{a_{\mu}}{a_0}\right|k^{\mu-1}+1}{1+\frac{R^{\mu}-1}{R^{n}-1}\left|\frac{a_{\mu}}{a_0}\right|k^{\mu+1}}. \tag{2.4}$$

Remark 1. Choosing $\beta = 1$ in (2.3), Corollary 1.1 reduces to a result due to Aziz and Shah [6].

Taking $q \to \infty$ in (2.1), we get

Corollary 1.2 *If $P \in \mathcal{P}_{n,\mu}$, with $P(z) \neq 0$ for $|z| < k$, $k \geq 1$, then for every complex number β with $|\beta| \leq 1$, and $R > r \geq 1$*

$$\|P(Rz) - \beta P(rz)\|_\infty \leq \frac{|R^n - \beta r^n|}{\|c^*_{\mu,k} + z\|_\infty}\|P\|_\infty, \tag{2.5}$$

where for $1 \leq \mu \leq n$, $c^*_{\mu,k}$ is same as defined in (2.2).

Theorem 2. *If $P \in \mathcal{P}_{n,\mu}$ and $P(z) \neq 0$ in $|z| < k$, $k \geq 1$, then for every complex number β with $|\beta| \leq 1$, $R > r \geq 1$, $q > 0$ and real α,*

$$\|P(Rz) - \beta P(rz)\|_q \leq \frac{|(R^n - \beta r^n)z + e^{i\alpha}(1-\beta)|}{\|c^*_{\mu,k} + z\|_q}\|P\|_q, \tag{2.6}$$

where $c^*_{\mu,k}$ is same as defined in (2.2).
Equality in (2.6) holds for polynomial $P(z) = az^n + b$, $|a| = |b| = 1$.

For $k = 1$ in Theorem 2, a result due to Aziz and Rather [3] is obtained and on choosing $r = 1$, we get the following result.

Corollary 2.1 *If $P \in \mathcal{P}_{n,\mu}$ and $P(z) \neq 0$ in $|z| < k$, $k \geq 1$, then for every complex number β with $|\beta| \leq 1$, $R > 1$ $q > 0$ and real α,*

$$\|P(Rz) - \beta P(z)\|_q \leq \frac{|(R^n - \beta)z + e^{i\alpha}(1-\beta)|}{\|c^*_{\mu,k} + z\|_q}\|P\|_q, \tag{2.7}$$

where $c^*_{\mu,k}$ is same as defined in (2.4).
Choosing $\beta = 1$ in Corollary 2.1, we get Theorem C.
Taking $q \to \infty$ in (2.6), we get the following result.

Corollary 2.2 *If $P \in \mathcal{P}_{n,\mu}$ and $P(z) \neq 0$ in $|z| < 1$, then for every complex number β with $|\beta| \leq 1$, $R > r \geq 1$ and real α,*

$$\|P(Rz) - \beta P(rz)\|_\infty \leq \frac{|(R^n - \beta r^n)z + e^{i\alpha}(1-\beta)|}{\|c^*_{\mu,k} + z\|_\infty}\|P\|_\infty, \tag{2.8}$$

where $c^*_{\mu,k}$ is same as defined in (2.2).

Corollary 2.3 *If $P \in \mathcal{P}_{n,\mu}$ and $P(z) \neq 0$ in $|z| < k$, $k \geq 1$, then for every complex number β with $|\beta| \leq 1$, $R > r \geq 1$, $q \geq 1$ and real α,*

$$\|P(Rz)\|_q \leq \left\{|\beta|r^n + \frac{|(R^n - \beta r^n)z + e^{i\alpha}(1-\beta)|}{\|c^*_{\mu,k} + z\|_q}\right\}\|P\|_q. \tag{2.9}$$

where $c^*_{\mu,k}$ is same as defined in (2.2).

Proof. We have by Minkowski inequality,

$$\begin{aligned}\|P(Rz)\|_q &= \|P(Rz) - \beta P(rz) + \beta P(rz)\|_q \\ &\leq \|P(Rz) - \beta P(rz)\|_q + |\beta|\|P(rz)\|_q.\end{aligned}$$

Using (2.6), we obtain

$$\begin{aligned}\|P(Rz)\|_q &\leq \frac{|(R^n - \beta r^n)z + e^{i\alpha}(1-\beta)|}{\|c^*_{\mu,k} + z\|_q}\|P\|_q + |\beta|\|P(rz)\|_q \\ &\leq \left\{\frac{|(R^n - \beta r^n)z + e^{i\alpha}(1-\beta)|}{\|c^*_{\mu,k} + z\|_q} + |\beta|r^n\right\}\|P(z)\|_q.\end{aligned}$$

Remark 2. For $\beta = 1$ and $r = 1$, Corollary 2.3 reduces to a result of Aziz and Shah [6].
Before proceeding to our next result, we define the following:
If $P(z)$ is a polynomial of degree n, then its polar derivative with respect to α written as $D_\alpha P(z)$ is defined by

$$D_\alpha P(z) = nP(z) + (\alpha - z)P'(z).$$

$D_\alpha P(z)$ is a polynomial of degree at most $n-1$ and generalizes the ordinary derivative in the sense

$$P'(z) = \lim_{\alpha \to \infty} D_\alpha P(z).$$

Concerning the inequalities for $|D_\alpha P(z)|$ in terms of $|P(z)|$ an interested reader may go through [9, 11], [14]-[17]. Now, the following result is regarding the growth of polar derivative of a polynomial.
Theorem 3. *If $P \in \mathcal{P}_{n,\mu}$ and $P(z)$ have no zeros in $|z| < k$, $k \geq 1$ except $s-$fold zeros at a point z_0 with $|z_0| < 1$, then $0 \leq r \leq R \leq k$ and every complex number α with $|\alpha| \geq R$,*

$$\begin{aligned}\max_{|z|=R} |D_\alpha P(z)| \leq & \left(\frac{R + |z_0|}{r - |z_0|}\right)^s \Delta_1 \delta_{\mu,k} \max_{|z|=r} |P(z)| \\ & - \left(\frac{R + |z_0|}{k + |z_0|}\right)^s \{\delta_{\mu,k}(\Delta_1 - 1) + \delta_{\mu,R}\} \min_{|z|=k} |P(z)| \\ & + \frac{s(|\alpha| + R)R^n}{(R + |z_0|)(R - |z_0|)^s} \max_{|z|=1} |P(z)|, \end{aligned} \tag{2.10}$$

where

$$\delta_{\mu,k} = (n - s)\left(\frac{R^{(\mu-1)}|\alpha| + k^\mu}{R^\mu + k^\mu}\right), \tag{2.11}$$

and

$$\Delta_1 = \left(\frac{R^\mu + k^\mu}{r^\mu + k^\mu}\right)^{\frac{n-s}{\mu}} \tag{2.12}$$

Equality holds for the polynomial $P(z) = (z^\mu + k^\mu)^{\frac{n}{\mu}}$.

3. Preliminary observations and lemmas

Lemma 1 *If $P \in \mathcal{P}_n$ and $P(z)$ has no zero in $|z| < k$ where $k \geq 1$, then for every $R \geq r > 1$ and $|z| = 1$,*

$$|P(rz)| \leq |P(Rz)|. \tag{3.1}$$

Lemma 1 can be obtained from a result of Aziz and Rather [4].

Lemma 2 [6] *If $P \in \mathcal{P}_{n,\mu}$, $1 \le \mu \le n$ having no zeros in $|z| < k$ $k \ge 1$, then for every complex number β with $|\beta| \le 1$, $R \ge r > 1$ and $|z| = 1$,*

$$|P(Rz) - \beta P(rz)| \le \frac{1}{k^{\mu+1}} \left[\frac{1 + \frac{R^\mu - r^\mu}{R^n - r^n}\left|\frac{a_\mu}{a_0}\right| k^{\mu+1}}{\frac{R^\mu - r^\mu}{R^n - r^n}\left|\frac{a_\mu}{a_0}\right| k^{\mu-1} + 1} \right] |Q(Rz) - \beta Q(rz)|, \tag{3.2}$$

where $Q(z) = z^n \overline{P(\frac{1}{\overline{z}})}$.

Lemma 3 [3] *If $P \in \mathcal{P}_n$, then for every complex number β with $|\beta| \le 1$, $R > r \ge 1$, $q > 0$ and real α,*

$$\int_0^{2\pi} \left| \big(P(Re^{i\theta}) - \beta P(re^{i\theta})\big) + e^{i\alpha}\big(R^n P(e^{i\theta}/R) - \beta r^n P(e^{i\theta}/r)\big) \right|^q d\theta$$
$$\le \left| R^n - \beta r^n \right|^q \int_0^{2\pi} |P(e^{i\theta})|^q d\theta. \tag{3.3}$$

Lemma 4 [3, Lemma 5] *If $P \in \mathcal{P}_n$, then for every real or complex number β with $|\beta| \le 1$, $R > r \ge 1$, $q > 0$ and real α,*

$$\int_0^{2\pi} \left| \big(P(Re^{i\theta}) - \beta P(re^{i\theta})\big) + e^{i\alpha}\big(R^n P(e^{i\theta}/R) - \overline{\beta} r^n P(e^{i\theta}/r)\big) \right|^q d\theta$$
$$\le \left| (R^n - \beta r^n) + e^{i\alpha}(1 - \overline{\beta}) \right|^q \int_0^{2\pi} |P(e^{i\theta})|^q d\theta. \tag{3.4}$$

Lemma 5. [14, Theorem 2] *If $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$, $1 \le \mu \le n$, be a polynomial of degree n having no zeros in $|z| < k$, $k \ge 1$ for $0 \le r \le R \le k$, then for every complex number α with $|\alpha| \ge R$,*

$$\max_{|z|=R} |D_\alpha P(z)| \le \frac{n(R^{\mu-1}|\alpha| + k^\mu)}{R^\mu + k^\mu} \left[\left(\frac{R^\mu + k^\mu}{r^\mu + k^\mu} \right)^{\frac{n}{\mu}} \max_{|z|=r} |P(z)| \right.$$
$$\left. - \left\{ \left(\frac{R^\mu + k^\mu}{r^\mu + k^\mu} \right)^{\frac{n}{\mu}} - 1 \right\} \min_{|z|=k} |P(z)| \right]$$
$$- \frac{n(R^{\mu-1}|\alpha| - R^\mu)}{R^\mu + k^\mu} \min_{|z|=k} |P(z)|. \tag{3.5}$$

Equality in (3.5) holds for $P(z) = (z^\mu + k^\mu)^{\frac{n}{\mu}}$.

Lemma 6. [12] *If $P \in \mathcal{P}_n$ then $R > 1$,*

$$\max_{|z|=R} |P(z)| \le R^n \max_{|z|=1} |P(z)|. \tag{3.6}$$

The estimate is best possible and equality holds for polynomial $P(z) = az^n$.

4. Proofs of Theorems

Proof of Theorem 1. Let $F(\theta) = P(Re^{i\theta}) - \beta P(re^{i\theta})$ and $G(\theta) = R^n P(e^{i\theta}/R) - \beta r^n P(e^{i\theta}/r)$, then by Lemma 3, for every real α, $R > r \geq 1$ and $q > 0$, we get

$$\int_0^{2\pi} |F(\theta) + e^{i\alpha}G(\theta)|^q d\theta \leq |R^n - \beta r^n|^q \int_0^{2\pi} |P(e^{i\theta})|^q d\theta. \tag{4.1}$$

Inequality (4.1) in particular gives for $q > 0$,

$$\int_0^{2\pi}\int_0^{2\pi} |F(\theta) + e^{i\alpha}G(\theta)|^q d\theta d\alpha \leq 2\pi|R^n - \beta r^n|^q \int_0^{2\pi} |P(e^{i\theta})|^q d\theta. \tag{4.2}$$

Now for every real α and $A \geq B \geq 1$, we have

$$|A + e^{i\alpha}|^q \geq |B + e^{i\alpha}|^q.$$

This gives for every $q > 0$,

$$\int_0^{2\pi} |A + e^{i\alpha}|^q d\alpha \geq \int_0^{2\pi} |B + e^{i\alpha}|^q d\alpha. \tag{4.3}$$

If $F(\theta) \neq 0$, we take $A = \left|\frac{G(\theta)}{F(\theta)}\right|$ and

$$B = c^*_{\mu,k} = k^{\mu+1}\frac{\frac{R^\mu - r^\mu}{R^n - r^n}\left|\frac{a_\mu}{a_0}\right|k^{\mu-1} + 1}{1 + \frac{R^\mu - r^\mu}{R^n - r^n}\left|\frac{a_\mu}{a_0}\right|k^{\mu+1}} \quad 1 \leq \mu \leq n.$$

Clearly by (3.2), we have $A \geq B \geq 1$ and therefore by making use of inequality (4.3), we get

$$\begin{aligned}\int_0^{2\pi} |F(\theta) + e^{i\alpha}G(\theta)|^q d\alpha &= |F(\theta)|^q \int_0^{2\pi} \left|1 + e^{i\alpha}\left|\frac{G(\theta)}{F(\theta)}\right|\right|^q d\alpha \\ &= |F(\theta)|^q \int_0^{2\pi} \left|\left|\frac{G(\theta)}{F(\theta)}\right| + e^{i\alpha}\right|^q d\alpha \\ &= |F(\theta)|^q \int_0^{2\pi} |A + e^{i\alpha}|^q d\alpha \\ &\geq |F(\theta)|^q \int_0^{2\pi} |B + e^{i\alpha}|^q d\alpha \\ &= |F(\theta)|^q \int_0^{2\pi} |c^*_{\mu,k} + e^{i\alpha}|^q d\alpha. \end{aligned} \tag{4.4}$$

This gives

$$|P(Re^{i\theta}) - \beta P(re^{i\theta})|^q \int_0^{2\pi} |c^*_{\mu,k} + e^{i\alpha}|^q d\alpha \leq \int_0^{2\pi} |F(\theta) + e^{i\alpha}G(\theta)|^q d\alpha.$$

Implies

$$\int_0^{2\pi} |P(Re^{i\theta}) - \beta P(re^{i\theta})|^q d\theta \int_0^{2\pi} |c^*_{\mu,k} + e^{i\alpha}|^q d\alpha \le \int_0^{2\pi}\int_0^{2\pi} |F(\theta) + e^{i\alpha}G(\theta)|^q d\alpha d\theta. \tag{4.5}$$

For $F(\theta) = 0$, inequality (4.4) is trivially true. Combining Inequalities (4.5) and (4.2), we get

$$\int_0^{2\pi} |P(Re^{i\theta}) - \beta P(re^{i\theta})|^q d\theta \int_0^{2\pi} |c^*_{\mu,k} + e^{i\alpha}|^q d\alpha \le 2\pi |R^n - \beta r^n|^q \int_0^{2\pi} |P(e^{i\theta})|^q d\theta.$$

Therefore for every $q > 0$, we get

$$\int_0^{2\pi} |P(Re^{i\theta}) - \beta P(re^{i\theta})|^q d\theta \le \frac{2\pi |R^n - \beta r^n|^q}{\int_0^{2\pi} |c^*_{\mu,k} + e^{i\alpha}|^q d\alpha} \int_0^{2\pi} |P(e^{i\theta})|^q d\theta.$$

Hence

$$\|P(Rz) - \beta P(rz)\|_q \le \frac{|R^n - \beta r^n|}{\|c^*_{\mu,k} + z\|_q} \|P\|_q.$$

This completes the proof of Theorem 1.

Proof of Theorem 2. Take $F(\theta) = P(Re^{i\theta}) - \beta P(re^{i\theta})$ and $G(\theta) = R^n P(e^{i\theta}/R) - \beta r^n P(e^{i\theta}/r)$, then by Lemma 4, for every real α, $R > r \ge 1$ and $q > 0$, we get

$$\int_0^{2\pi}\int_0^{2\pi} |F(\theta) + e^{i\alpha}G(\theta)|^q d\theta d\alpha \le 2\pi |(R^n - \beta r^n)z + e^{i\alpha}(1-\overline{\beta})|^q \int_0^{2\pi} |P(e^{i\theta})|^q d\theta. \tag{4.6}$$

Now for every real α, $A \ge B \ge 1$ and $q > 0$, we have

$$\int_0^{2\pi} |A + e^{i\alpha}|^q d\alpha \ge \int_0^{2\pi} |B + e^{i\alpha}|^q d\alpha. \tag{4.7}$$

If $F(\theta) \ne 0$, we take $A = \left|\frac{G(\theta)}{F(\theta)}\right|$ and $B = c^*_{\mu,k}$, and making use of (4.7), we get

$$\begin{aligned}
\int_0^{2\pi} |F(\theta) + e^{i\alpha}G(\theta)|^q d\alpha &= |F(\theta)|^q \int_0^{2\pi} \left|1 + e^{i\alpha}\left|\frac{G(\theta)}{F(\theta)}\right|\right|^q d\alpha \\
&= |F(\theta)|^q \int_0^{2\pi} \left|\left|\frac{G(\theta)}{F(\theta)}\right| + e^{i\alpha}\right|^q d\alpha \\
&= |F(\theta)|^q \int_0^{2\pi} |A + e^{i\alpha}|^q d\alpha \\
&\ge |F(\theta)|^q \int_0^{2\pi} |B + e^{i\alpha}|^q d\alpha \\
&= |F(\theta)|^q \int_0^{2\pi} |c^*_{\mu,k} + e^{i\alpha}|^q d\alpha.
\end{aligned}$$

This gives

$$\int_0^{2\pi} |F(\theta)|^q d\theta \int_0^{2\pi} |c^*_{\mu,k} + e^{i\alpha}|^q d\alpha \le \int_0^{2\pi}\int_0^{2\pi} |F(\theta) + e^{i\alpha}G(\theta)|^q d\theta d\alpha. \tag{4.8}$$

Combining (4.8) and (4.6), we get

$$\int_0^{2\pi} |P(Re^{i\theta}) - \beta P(re^{i\theta})|^q d\theta \int_0^{2\pi} |c^*_{\mu,k} + e^{i\alpha}|^q d\alpha$$
$$\le 2\pi |(R^n - \beta r^n)z + e^{i\alpha}(1-\overline{\beta})|^q \int_0^{2\pi} |P(e^{i\theta})|^q d\theta.$$

This gives for every $q > 0$,

$$\|P(Rz) - \beta P(rz)\|_q \le \frac{|(R^n - \beta r^n)z + e^{i\alpha}(1-\overline{\beta})|}{\|c^*_{\mu,k} + z\|_q}\|P\|_q.$$

This completes the proof of Theorem 2.

Proof of Theorem 3. Here $P(z) = (z - z_0)^s \phi(z)$, where $\phi(z) = a_0 + \sum_{\nu=\mu}^{n-s} a_\nu z^\nu$ $1 \le \mu \le n-s$, is a polynomial of degree $n-s$ having no zero in $|z| < k$, $k \ge 1$. Then by Lemma 5, for $0 \le r \le R \le k$ and every complex number α with $|\alpha| \ge R$, we have

$$\begin{aligned}\max_{|z|=R} |D_\alpha \phi(z)| \le \frac{(n-s)(R^{\mu-1}|\alpha| + k^\mu)}{R^\mu + k^\mu}\Bigg[\left(\frac{R^\mu + k^\mu}{r^\mu + k^\mu}\right)^{n-s/\mu} \max_{|z|=r}|\phi(z)| \\ - \left\{\left(\frac{R^\mu + k^\mu}{r^\mu + k^\mu}\right)^{n-s/\mu} - 1\right\}\min_{|z|=k}|\phi(z)|\Bigg] \\ - \frac{(n-s)(R^{\mu-1}|\alpha| - R^\mu)}{R^\mu + k^\mu}\min_{|z|=k}|\phi(z)|.\end{aligned} \tag{4.9}$$

Now by definition, the polar derivative of $P(z)$ with respect to α is given by

$$\begin{aligned} D_\alpha P(z) &= nP(z) + (\alpha - z)P^{'}(z) \\ &= n(z-z_0)^s\phi(z) + (\alpha - z)\left\{s(z-z_0)^{s-1}\phi(z) + (z-z_0)^s\phi^{'}(z)\right\} \\ &= (z-z_0)^s\left\{n\phi(z) + (\alpha - z)\phi^{'}(z)\right\} + s(\alpha - z)(z-z_0)^{s-1}\phi(z) \\ &= (z-z_0)^s D_\alpha\phi(z) + s(\alpha - z)(z-z_0)^{s-1}\phi(z). \end{aligned} \tag{4.10}$$

Therefore, for $|z| = R$

$$\begin{aligned} |D_\alpha P(z)| &\le (R + |z_0|)^s |D_\alpha\phi(z)| + s(|\alpha| + R)(R + |z_0|)^{s-1}|\phi(z)| \\ &\le (R + |z_0|)^s \max_{|z|=R}|D_\alpha\phi(z)| + s(|\alpha| + R)(R + |z_0|)^{s-1}\max_{|z|=R}|\phi(z)|. \end{aligned} \tag{4.11}$$

Also

$$|\phi(z)| = \frac{|P(z)|}{|z-z_0|^s} \leq \frac{|P(z)|}{(R-z_0)^s}.$$

and

$$|\phi(z)| = \frac{|P(z)|}{|z-z_0|^s} \geq \frac{|P(z)|}{(R+|z_0|)^s}$$

Therefore, using Lemma 6, we get

$$\max_{|z|=R}|\phi(z)| \leq \frac{1}{(R-|z_0|)^s}\max_{|z|=R}|P(z)| \leq \frac{R^n}{(R-|z_0|)^s}\max_{|z|=1}|P(z)|. \tag{4.12}$$

Using (4.9) and (4.12) in (4.11), we get

$$\begin{aligned}
|D_\alpha P(z)| \leq{}& (R+|z_0|)^s\Bigg[\frac{(n-s)(R^{\mu-1}|\alpha|+k^\mu)}{R^\mu+k^\mu}\Bigg\{\left(\frac{R^\mu+k^\mu}{r^\mu+k^\mu}\right)^{n-s/\mu}\max_{|z|=r}|\phi(z)| \\
&-\left\{\left(\frac{R^\mu+k^\mu}{r^\mu+k^\mu}\right)^{n-s/\mu}-1\right\}\min_{|z|=k}|\phi(z)|\Bigg\} - \frac{(n-s)(R^{\mu-1}|\alpha|-R^\mu)}{R^\mu+k^\mu}\min_{|z|=k}|\phi(z)| \\
&+\frac{s(|\alpha|+R)R^n}{(R+|z_0|)(R-|z_0|)^s}\max_{|z|=1}|P(z)|\Bigg] \\
={}& (R+|z_0|)^s\Bigg[\frac{(n-s)(R^{\mu-1}|\alpha|+k^\mu)}{R^\mu+k^\mu}\Bigg\{\left(\frac{R^\mu+k^\mu}{r^\mu+k^\mu}\right)^{n-s/\mu}\frac{1}{(r-|z_0|)^s}\max_{|z|=r}|P(z)| \\
&-\left\{\left(\frac{R^\mu+k^\mu}{r^\mu+k^\mu}\right)^{n-s/\mu}-1\right\}\frac{1}{(k+|z_0|)^s}\min_{|z|=k}|P(z)|\Bigg\} \\
&-\frac{(n-s)(R^{\mu-1}|\alpha|-R^\mu)}{R^\mu+k^\mu}\frac{1}{(k+|z_0|)^s}\min_{|z|=k}|P(z)| \\
&+\frac{s(|\alpha|+R)R^n}{(R+|z_0|)(R-|z_0|)^s}\max_{|z|=1}|P(z)|\Bigg]
\end{aligned} \tag{4.13}$$

This gives

$$|D_\alpha P(z)| \leq (R+|z_0|)^s \Bigg[\delta_{\mu,k}\Bigg\{\frac{\Delta_1}{(r-|z_0|)^s}\max_{|z|=r}|P(z)|$$

$$-\frac{(\Delta_1-1)}{(k+|z_0|)^s}\min_{|z|=k}|P(z)|\Bigg\}-\frac{\delta_{\mu,R}}{(k+|z_0|)^s}\min_{|z|=k}|P(z)|\Bigg]$$

$$=(R+|z_0|)^s\Bigg\{\frac{\delta_{\mu,k}\Delta_1}{(r-|z_0|)^s}\max_{|z|=1}|P(z)|-\left(\frac{\delta_{\mu,k}(\Delta_1-1)+\delta_{\mu,r}}{(k+|z_0|)^s}\right)\min_{|z|=k}|P(z)|\Bigg\}$$

$$+\frac{s(|\alpha|+R)R^n}{(R+|z_0|)(R-|z_0|)^s}\max_{|z|=1}|P(z)|$$

$$=\left(\frac{R+|z_0|}{r-|z_0|}\right)^s\Delta_1\delta_{\mu,k}\max_{|z|=r}|P(z)|-\left(\frac{R+|z_0|}{k+|z_0|}\right)^s\{\delta_{\mu,k}(\Delta_1-1)+\delta_{\mu,R}\}\min_{|z|=k}|P(z)|$$

$$+\frac{s(|\alpha|+R)R^n}{(R+|z_0|)(R-|z_0|)^s}\max_{|z|=1}|P(z)|.$$

Where $\delta_{\mu,k}$ and Δ_1 are given by (2.11) and (2.12) respectively. This completes proof of Theorem 3.

Acknowledgement. The authors are grateful to the referee for his/her valuable suggestions.

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