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NUMBERS WITH n DIFFERENT PRIME FACTORS IN THEIR PRIME FACTORIZATION

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Abstract. In this note we give a simple proof of the well-known result that the numbers with a fixed number of different prime factors in their prime factorization have density zero.

1. Introduction and preliminaries

Let us consider the prime factorization of a positive integer $n > 1$. Namely

$$n = q_1^{e_1} q_2^{e_2} q_3^{e_3} \cdots q_k^{e_k}, \tag{1.1}$$

where $q_1 < q_2 < q_3 < \cdots < q_k$ $(k \geq 1)$ are the different primes in the prime factorization of n and the e_i $(i = 1, \ldots, k)$ are the corresponding exponents. We define (see [1, Chapter XXII]) $\omega(n)$ as the number of different prime factors in the prime factorization of n. That is $\omega(n) = k$. Let us consider equation (1.1), we define q_1 as the first prime factor, q_2 as the second prime factor, ..., q_k as the k-th prime factor.

Let $A_k(x)$ be the number of numbers n not exceeding x such that $\omega(n) = k$, where k is an arbitrary but fixed positive integer. For example, if $k = 1$ we obtain the prime powers. In this note we prove the equation

$$A_k(x) = o(x). \tag{1.2}$$

That is, the density of these numbers is zero.

A more precise asymptotic formula due to Landau is well-known, namely

$$A_k(x) \sim \frac{x(\log\log x)^{k-1}}{(k-1)!\log x}.$$

However, our weak result is of interest, since the proof is very elementary. We only use the inclusion-exclusion principle. In addition, we obtain infinite series on prime numbers with sum 1.

We need the following theorem, it is sometimes called either the principle of inclusion-exclusion or the principle of cross-classification. We now enunciate the principle.

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Theorem 1.1. *Let S be a set of N distinct elements, and let $S_1, \ldots, S_r$ be arbitrary subsets of S containing $N_1, \ldots, N_r$ elements, respectively. For $1 \le i < j < \ldots < l \le r$, let $S_{ij\ldots l}$ be the intersection of $S_i, S_j, \ldots, S_l$ and let $N_{ij\ldots l}$ be the number of elements of $S_{ij\ldots l}$. Then the number K of elements of S not in any of $S_1, \ldots, S_r$ is*

$$K = N - \sum_{1\le i\le r} N_i + \sum_{1\le i<j\le r} N_{ij} - \sum_{1\le i<j<k\le r} N_{ijk} + \cdots + (-1)^r N_{12\ldots r}. \tag{1.3}$$

Proof. See, for example, either [1, page 233] or [2, page 84]. □

In this note p_n denotes the n-th prime. We also need the following theorem.

Theorem 1.2. *The following formula holds,*

$$\sum_{h=1}^{n} \left(\prod_{i=1}^{h-1} \left(1 - \frac{1}{p_i}\right) \right) \frac{1}{p_h} = 1 - \prod_{i=1}^{n} \left(1 - \frac{1}{p_i}\right) \qquad (n \ge 1). \tag{1.4}$$

Besides

$$\sum_{h=1}^{\infty} \left(\prod_{i=1}^{h-1} \left(1 - \frac{1}{p_i}\right) \right) \frac{1}{p_h} = \frac{1}{2} + \left(1 - \frac{1}{2}\right)\frac{1}{3} + \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\frac{1}{5}$$
$$+ \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{5}\right)\frac{1}{7} + \cdots = 1. \tag{1.5}$$

Proof. Equation (1.4) can be proved easily using mathematical induction. Equation (1.5) is an immediate consequence of (1.4) and the following well-known limit [1, Chapter XXII]

$$\lim_{n\to\infty} \prod_{i=1}^{n} \left(1 - \frac{1}{p_i}\right) = 0.$$

□

2. Main results

Let $m \ge 2$ be an arbitrary but fixed positive integer. Let $D_{p_h}(x)$ be the number of positive integers n not exceeding x such that the fixed prime p_h $(h \ge m)$ is their m-th prime factor (see (1.1)).

Theorem 2.1. *The following formula holds $(h \ge m)$*

$$D_{p_h}(x) = \frac{x}{p_h} \prod_{j=1}^{h-1} \left(1 - \frac{1}{p_j}\right) \sum_{1\le i_1<\cdots<i_{m-1}\le h-1} \frac{1}{(p_{i_1} - 1)\cdots(p_{i_{m-1}} - 1)} + O(1). \tag{2.1}$$

Therefore, these numbers have positive density $(h \ge m)$

$$D_{p_h} = \frac{1}{p_h} \prod_{j=1}^{h-1} \left(1 - \frac{1}{p_j}\right) \sum_{1\le i_1<\cdots<i_{m-1}\le h-1} \frac{1}{(p_{i_1} - 1)\cdots(p_{i_{m-1}} - 1)}. \tag{2.2}$$

Proof. Let $B_{i_1,\ldots,i_{m-1}}(x)$ be the number of positive integers n not exceeding x such that p_{i_1} is their first prime factor, ..., $p_{i_{m-1}}$ is their $(m-1)$-th prime factor and p_h is their m-th prime factor (see (1.1)). Therefore (see (1.3)) we have

$$\begin{aligned} B_{i_1,\ldots,i_{m-1}}(x) &= \left[\frac{x}{p_{i_1}\cdots p_{i_{m-1}}p_h}\right] - \sum_{\substack{1\le j\le h-1\\ j\neq i_1,\ldots i_{m-1}}} \left[\frac{x}{p_{i_1}\cdots p_{i_{m-1}}p_hp_j}\right] + \cdots \\ &= \frac{x}{p_{i_1}\cdots p_{i_{m-1}}p_h} - \sum_{\substack{1\le j\le h-1\\ j\neq i_1,\ldots i_{m-1}}} \frac{x}{p_{i_1}\cdots p_{i_{m-1}}p_hp_j} + \cdots + O(1) \\ &= \frac{x}{p_{i_1}\cdots p_{i_{m-1}}p_h} \prod_{\substack{1\le j\le h-1\\ j\neq i_1,\ldots i_{m-1}}} \left(1-\frac{1}{p_j}\right) + O(1) \\ &= \frac{x}{p_h}\prod_{j=1}^{h-1}\left(1-\frac{1}{p_j}\right)\frac{1}{(p_{i_1}-1)\cdots(p_{i_{m-1}}-1)} + O(1). \end{aligned} \tag{2.3}$$

On the other hand, we have

$$D_{p_h}(x) = \sum_{1\le i_1<\cdots<i_{m-1}\le h-1} B_{i_1,\ldots,i_{m-1}}(x). \tag{2.4}$$

Equation (2.1) is an immediate consequence of (2.3) and (2.4). □

Theorem 2.2. *The series* $(m\ge 2)$ *(see (2.2))*

$$\sum_{h=m}^{\infty} D_{p_h} = \sum_{h=m}^{\infty}\frac{1}{p_h}\prod_{j=1}^{h-1}\left(1-\frac{1}{p_j}\right)\sum_{1\le i_1<\cdots<i_{m-1}\le h-1}\frac{1}{(p_{i_1}-1)\cdots(p_{i_{m-1}}-1)}$$

converges. Besides

$$\begin{aligned} \sum_{h=m}^{\infty} D_{p_h} &= \sum_{h=m}^{\infty}\frac{1}{p_h}\prod_{j=1}^{h-1}\left(1-\frac{1}{p_j}\right)\sum_{1\le i_1<\cdots<i_{m-1}\le h-1}\frac{1}{(p_{i_1}-1)\cdots(p_{i_{m-1}}-1)} \\ &= 1 \qquad (m\ge 2). \end{aligned} \tag{2.5}$$

Proof. Let n be an arbitrary but fixed positive integer. We have

$$\sum_{h=m}^{n} D_{p_h}(x) \le x.$$

That is

$$\sum_{h=m}^{n}\frac{D_{p_h}(x)}{x} \le 1.$$

Therefore (see (2.1) and (2.2))

$$\lim_{x\to\infty}\left(\sum_{h=m}^{n}\frac{D_{p_h}(x)}{x}\right) = \sum_{h=m}^{n}\lim_{x\to\infty}\frac{D_{p_h}(x)}{x} = \sum_{h=m}^{n} D_{p_h} \le 1.$$

Hence

$$\sum_{h=m}^{\infty} D_{p_h} \leq 1.$$

That is, the series $\sum_{h=m}^{\infty} D_{p_h}$ converges. Now, we shall prove that $\sum_{h=m}^{\infty} D_{p_h} = 1$. We shall apply mathematical induction. First, we shall prove (2.5) if $m = 2$. We have (see (2.5) with $m = 2$, (1.4) and (1.5))

$$\begin{aligned}
\sum_{h=2}^{\infty} D_{p_h} &= \sum_{h=2}^{\infty} \frac{1}{p_h} \prod_{j=1}^{h-1} \left(1 - \frac{1}{p_j}\right) \sum_{1 \leq i_1 \leq h-1} \frac{1}{(p_{i_1} - 1)} \\
&= \sum_{h=2}^{\infty} \left(\sum_{j=h}^{\infty} \frac{1}{p_j} \prod_{i=1}^{j-1} \left(1 - \frac{1}{p_i}\right) \frac{1}{p_{h-1} - 1} \right) \\
&= \sum_{h=2}^{\infty} \left(\left(1 - \sum_{j=1}^{h-1} \frac{1}{p_j} \prod_{i=1}^{j-1} \left(1 - \frac{1}{p_i}\right)\right) \frac{1}{p_{h-1} - 1} \right) \\
&= \sum_{h=2}^{\infty} \prod_{i=1}^{h-1} \left(1 - \frac{1}{p_i}\right) \frac{1}{p_{h-1} - 1} = \sum_{h=1}^{\infty} \prod_{i=1}^{h} \left(1 - \frac{1}{p_i}\right) \frac{1}{p_h - 1} \\
&= \sum_{h=1}^{\infty} \frac{1}{p_h} \prod_{i=1}^{h-1} \left(1 - \frac{1}{p_i}\right) = 1,
\end{aligned}$$

since the last series is (1.5). Therefore (2.5) is true if $m = 2$. Suppose that (2.5) is true for $m \geq 2$, we shall prove that (2.5) is true for $m + 1$. We have (see (2.5), (1.4) and (1.5))

$$\begin{aligned}
&\sum_{h=m+1}^{\infty} \frac{1}{p_h} \prod_{j=1}^{h-1} \left(1 - \frac{1}{p_j}\right) \sum_{1 \leq i_1 < \cdots < i_m \leq h-1} \frac{1}{(p_{i_1} - 1) \cdots (p_{i_m} - 1)} \\
= &\sum_{i_1 < \cdots < i_m} \sum_{h=i_m+1}^{\infty} \frac{1}{p_h} \prod_{j=1}^{h-1} \left(1 - \frac{1}{p_j}\right) \frac{1}{(p_{i_1} - 1) \cdots (p_{i_m} - 1)} \\
= &\sum_{i_1 < \cdots < i_{m-1}} \sum_{k=i_{m-1}+1}^{\infty} \sum_{j=k+1}^{\infty} \frac{1}{p_j} \prod_{i=1}^{j-1} \left(1 - \frac{1}{p_i}\right) \frac{1}{(p_{i_1} - 1) \cdots (p_{i_{m-1}} - 1)(p_k - 1)} \\
= &\sum_{i_1 < \cdots < i_{m-1}} \sum_{k=i_{m-1}+1}^{\infty} \prod_{j=1}^{k} \left(1 - \frac{1}{p_j}\right) \frac{1}{(p_{i_1} - 1) \cdots (p_{i_{m-1}} - 1)(p_k - 1)} \\
= &\sum_{i_1 < \cdots < i_{m-1}} \sum_{k=i_{m-1}+1}^{\infty} \frac{1}{p_k} \prod_{j=1}^{k-1} \left(1 - \frac{1}{p_j}\right) \frac{1}{(p_{i_1} - 1) \cdots (p_{i_{m-1}} - 1)} \\
= &\sum_{h=m}^{\infty} \frac{1}{p_h} \prod_{j=1}^{h-1} \left(1 - \frac{1}{p_j}\right) \sum_{1 \leq i_1 < \cdots < i_{m-1} \leq h-1} \frac{1}{(p_{i_1} - 1) \cdots (p_{i_{m-1}} - 1)} = 1,
\end{aligned}$$

since the last sum is the inductive hypothesis (2.5). Note that the symbol

$$\sum_{i_1<\cdots<i_m}$$

denotes that the sum run on all possible sets of m different positive integers $i_1, \cdots, i_m$ such that $i_1 < \cdots < i_m$. Analogously the symbol

$$\sum_{i_1<\cdots<i_{m-1}} .$$

□

Theorem 2.3. *Let $C_m(x)$ be the number of positive integers n not exceeding x such that $\omega(n) \geq m$. We have the following asymptotic formula*

$$C_m(x) \sim x. \tag{2.6}$$

Proof. Clearly we have

$$C_m(x) \leq x. \tag{2.7}$$

We can choose s such that (see (2.5))

$$\begin{aligned} \sum_{h=m}^{s} D_{p_h} = \sum_{h=m}^{s} \frac{1}{p_h} \prod_{j=1}^{h-1} \left(1 - \frac{1}{p_j}\right) \sum_{1\leq i_1<\cdots<i_{m-1}\leq h-1} \frac{1}{(p_{i_1}-1)\cdots(p_{i_{m-1}}-1)} \\ > \quad 1-\epsilon. \end{aligned} \tag{2.8}$$

Therefore, we have (see (2.1), (2.2) and (2.8))

$$\begin{aligned} C_m(x) = \sum_{p_m\leq p_h\leq x} D_{p_h}(x) \geq \sum_{p_m\leq p_h\leq p_s} D_{p_h}(x) = \sum_{h=m}^{s} D_{p_h}(x) = x \sum_{h=m}^{s} D_{p_h} \\ + \quad O(1) \geq (1-\epsilon)x + O(1) \geq (1-2\epsilon)x. \end{aligned} \tag{2.9}$$

Equations (2.7) and (2.9) give

$$1 - 2\epsilon \leq \frac{C_m(x)}{x} \leq 1. \tag{2.10}$$

Equation (2.10) gives (2.6), since ϵ is arbitrarily small. □

Corollary 2.4. *Let $D_m(x)$ be the number of positive integers n not exceeding x such that $\omega(n) \leq (m-1)$. We have the following asymptotic formula $D_m(x) = o(x)$. Therefore (see the introduction) $A_k(x) = o(x)$, since we can choose $m = k+1$.*

Proof. It is an immediate consequence of equation (2.6). □

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