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ON THE METRIC DIMENSION OF EXTENDED ZERO DIVISOR GRAPHS

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Abstract. Let R be a commutative ring with unity, the extended zero divisor graph $\mathbb{E}(R)$ is defined by considering the non-zero zero divisors $Z(R)^*$ as the vertices and two distinct vertices x and y of R are adjacent whenever there exist two positive integers m and n such that $x^m y^m = 0$ with $x^m \neq 0$ and $y^n \neq 0$. In this paper, we investigate metric dimension, fault tolerant metric dimension and local metric dimension of the extended zero divisor graph for the ring of integers modulo m and the ring of Gaussian integers modulo m.

1. Introduction

Numerous mathematicians from all over the world are interested in the interaction between ring theoretic and graph theoretic properties. Anderson and Livingston [1] introduced the zero divisor graph in 1999. This gave a large area to explore the commutative rings with the graph theory. Bennis and others introduced the extended zero divisor graph in 2016 [4]. Various other extensions from algebraic structures have also been researched [9], [10]. The concept of metric dimension was first introduced by Slater.P.J in [16] and it was also studied for graphs from algebraic structures.

If u and v are adjacent to exactly same vertices, then we define $u \sim v$. If u and v are adjacent and there isn't a vertex w of G that is adjacent to both u and v, two unique vertices u and v for any graph G are said to be orthogonal, denoted by $u \perp v$. The graph G is said to be complemented if for any vertex u of G, there is a vertex v (v is called the complement of u) such that $u \perp v$. The graph G is said to be uniquely complemented if G is complemented and whenever $u \perp v$ and $u \perp w$, then $v \sim w$. Consider a graph G with vertex set cardinality larger than 1. Any two distinct vertices x and y of G are distance similar if $d(x, a) = d(y, a)$ for all $a \in V(G) - \{x, y\}$. Any two distinct vertices are distance similar if either $xy \in E(G)$ and $N(x) = N(y)$ or $xy \in E(G)$ and $N[x] = N[y]$. The distance similar relation is an equivalence relation on $V(G)$.

Throughout this paper, we denote R as commutative ring with identity, $Nil(R)$ as the set of all nilpotent elements and $Nil(R)^* = Nil(R)\backslash\{0\}$. The index of

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nilpotency for any non-zero nilpotent element x is denoted by n_x . We denote $Z(R)$ as the set of all zero divisors of the ring R and $Z(R)^* = Z(R)\backslash\{0\}$. The graph $\mathbb{E}(R)$ is said to be the *extended zero divisor graph* of a commutative ring R with all its non-zero zero divisors $Z(R)^*$ as vertices and two distinct vertices x and y of R are adjacent whenever there exist two positive integers m and n such that $x^m y^m = 0$ with $x^m \neq 0$ and $y^n \neq 0$. For an ordered subset $W = \{w_1, w_2, ..., w_k\}$ of vertices in a connected graph G and a vertex v of G, the metric representation of v with respect to W is the k-vector ordered set, $r(v|W) = (d(v, w_1), d(v, w_2), ..., d(v, w_k))$. The set W is a resolving set for G if $r(u|W) = r(v|W)$ implies that $u = v$ for all pairs u, v of vertices of G. The *metric dimension*, $\beta(G)$ is the minimum cardinality of the resolving set for G. A resolving set W' of a graph G is a fault-tolerant resolving set if the removal of any vertex keeps it resolving. The minimal fault-tolerant resolving set is called fault-tolerant metric basis and its cardinality is called fault-tolerant metric dimension which is denoted by $\beta'(G)$. For an ordered subset $W = \{w_1, w_2, ..., w_k\}$ of vertices in a connected graph G and a vertex v of G, the local metric representation of v with respect to W is the k-vector ordered set, $r(v|W) = (d(v, w_1), d(v, w_2), ..., d(v, w_k))$. The set W is a local resolving set for G if $r(u|W) \neq r(v|W)$ for every pair u, v of adjacent vertices of G. The *local metric dimension* $lmd(G)$ is the minimum cardinality of the local resolving set for G. A local metric basis is a local resolving set of order $lmd(G)$. Refer [3], [6], [8], [17] for other basic definitions from ring theory and graph theory.

In Section 2, 3, 4, we calculate metric dimension, fault tolerant metric dimension, local metric dimension of the extended zero divisor graph of $\mathbb{Z}_m$, the ring of integers modulo m and $\mathbb{Z}_m[i]$, the ring of gaussian integers modulo m respectively.

2. Metric dimension of $\mathbb{E}(R)$

Theorem 2.1. *If R is a ring, then the following assertions hold:*

(i) R is an integral domain if and only if $\beta(\mathbb{E}(R))$ is undefined
(ii) R is finite if and only if $\beta(\mathbb{E}(R))$ is finite.

Proof. (i) Since the vertex of the integral domain is empty, the metric dimension of integral domain is undefined.
(ii) Since the diameter of $\mathbb{E}(R)$ is less than or equal to 3, by Corollary 2.2 of [13] the result follows.

□

Theorem 2.2. *If R is a ring, then the following assertions hold:*

(i) $\mathbb{E}(R)$ is a path if and only if $\beta(\mathbb{E}(R)) = 1$
(ii) $\mathbb{E}(R)$ is a complete graph if and only if $\beta(\mathbb{E}(R)) = |Z(R)^*| - 1$
(iii) If $\mathbb{E}(R)$ is a cycle, then $\beta(\mathbb{E}(R)) = 2$
(iv) If $\mathbb{E}(R)$ is a complete bipartite graph (except $K_{1,1}$), then $\beta(\mathbb{E}(R)) = |Z(R)^*| - 2$.

Proof. These results follows from Lemma 2.1, 2.2, 2.3 and Corollary 2.1 of [13].
□

Theorem 2.3. *Let $\mathbb{Z}_m$ be a ring with $m = p_1^{\alpha}\prod_{i=2}^{k} p_i$ where p_i's are prime and $m \neq 4p$ for odd primes p, then there exist exactly one single vertex distance similar class if and only if $p_1 = 2, \alpha = 1, 2$.*

Proof. Suppose $p_1 = 2$ and $\alpha = 1, m = 2 \times \prod_{i=1}^{k-1} p_i$, then there exist exactly one distance similar class $\{p_2...p_k = \frac{m}{2}\}$, which is a single vertex set. If $\alpha = 2, m = 4 \times \prod_{i=1}^{k-1} p_i$, then there exist exactly one distance similar class of product of all p_i's, $\{p_1p_2...p_k = \frac{m}{2}\}$. If $\alpha \neq 1, 2$, then the distance similar class of all combinations of p_i's is given by $\{t \times 2p_2p_3...p_k < m, t \in \mathbb{N}\}$ has more than one element which is a contradiction. If $p_1 \neq 2$ or $\alpha \neq 1$, then the distance similar class of all combinations of p_i's is given by $\{t \times 2p_2p_3...p_k < m, t \in \mathbb{N}\}$ has more than one element which is a contradiction. □

Theorem 2.4. *Let $m = p_1^{\alpha}\prod_{i=2}^{k} p_i$ be the prime factorization of an integer m, then $\beta(\mathbb{E}(\mathbb{Z}_m)) = |Z(\mathbb{Z}_m)^*| - 2^k + 2$ or $|Z(\mathbb{Z}_m)^*| - 2^k + 1$.*

Proof. If $m = 4p$, p is an odd prime, it is clear that the vertices are partitioned into 2 distinct distance similar equivalence classes of $V_1 = \{pi \mid 1 \leq i \leq 3\}, V_2 = \{2j \mid 2j < m$ and $j \neq p, j \in \mathbb{N}\}$ which forms a complete bipartite graph. Hence $\beta(\mathbb{E}(\mathbb{Z}_m)) = |Z(R)^*| - 2$ from Theorem 2.2.

The vertices of $\mathbb{E}(R)$ can be partitioned into distinct distance similar classes as follows. Consider a distance similar class of set of all multiples of p_1 removing the multiples of all other p_i's, $1 \leq i \leq k$ i.e, $\{tp_1 < m \mid t \in \mathbb{N}\}\backslash\{\bigcup_{i=2}^{k} tp_i \mid t \in \mathbb{N}\}$. Continuing the same method for all p_i's, we have the set of all distance similar classes to be $\{tp_{i_1} < m \mid 1 \leq i_1 \leq k, t \in \mathbb{N}\}\backslash\{\bigcup_{i=1, i\neq i_1}^{k} tp_i \mid t \in \mathbb{N}\}$. The cardinality of the above distance similar classes is given by ${}^kC_1$. Now, consider a distance similar class of multiples of p_1p_2 removing the multiples of all other p_i's i.e, $\{tp_1p_2 < m \mid t \in \mathbb{N}\}\backslash\{\bigcup_{i=3}^{k} tp_i \mid t \in \mathbb{N}\}$. Continuing the same process for all other combination of product of two p_i's, we have $\{tp_{i_1}p_{i_2} < m \mid 1 \leq i_1 \leq i_2 \leq k, t \in \mathbb{N}\}\backslash\{\bigcup_{i=1, i\neq i_1, i_2}^{k} tp_i \mid t \in \mathbb{N}\}$. The cardinality of the above distance similar classes is given by ${}^kC_2$. Next, consider a distance similar class of multiples of $p_1p_2p_3$ removing the multiples of all other p_i's, the distance similar classes is given by $\{tp_{i_1}p_{i_2}p_{i_3} < m \mid 1 \leq i_1 \leq i_2 \leq i_3 \leq k, t \in \mathbb{N}\}\backslash\{\bigcup_{i=1, i\neq i_1, i_2, i_3}^{k} tp_i \mid t \in \mathbb{N}\}$. The cardinality of the above distance similar class is ${}^kC_3$. Continuing the process for all the $k-1$ combinations of p_i's , we have $\{tp_{i_1}...p_{i_m} < m \mid 1 \leq i_1 \leq ... \leq i_m \leq k, t \in \mathbb{N}\}\backslash\{tp_i \mid i \neq i_1, ..., i_m, t \in \mathbb{N}\}$. The cardinality of the distance similar classes is ${}^kC_{k-1}$. Finally, we consider the distance similar class of product of all p_i's i.e, $\{tp_1p_2...p_k < m \mid t \in N\}$. The cardinality of this distance similar class is ${}^kC_k = 1$. Therefore, the cardinality of all the above distance similar equivalence classes is ${}^kC_1 + {}^kC_2 + {}^kC_3 + ... + {}^kC_{k-1} + {}^kC_k = 2^k - 1$.

If $m = \prod_{i=1}^{k} p_i$, then the product of all p_i's, $p_1p_2...p_k = m \ (mod\ m) = 0$. Hence the cardinality of the distance similar classes is ${}^kC_1+{}^kC_2+{}^kC_3+...+{}^kC_{k-1} = 2^k-2$.

Therefore, by Theorem 2.1 of [13], any minimal resolving set W consists of all vertices except one from each distance similar classes. Also from Theorem 2.3, the vertex $\frac{m}{2} \notin W$ since it has a unique metric representation. If $\alpha = 1$, then the cardinality of distance similar classes is $2^k - 2$ and the metric dimension is

$|Z(\mathbb{Z}_m)^*| - 2^k + 2$. If $\alpha \neq 1$, then the number of distance similar classes is $2^k - 1$ and the metric dimension is given by $|Z(\mathbb{Z}_m)^*| - 2^k + 1$. □

Theorem 2.5. *Let $m = p^2q^2$, where p and q are primes with $p \neq q, p, q \geq 2$, then $\beta(\mathbb{E}(\mathbb{Z}_m)) = |Z(R)^*| - 5$.*

Proof. The vertices can be partitioned into 5 distance similar equivalence classes as $V_1 = \{tp < m \mid t \in \mathbb{N}\} \backslash \{tq \mid t \in \mathbb{N}\}$, $V_2 = \{tq < m \mid t \in \mathbb{N}\} \backslash \{tp \mid t \in \mathbb{N}\}$, $V_3 = \{tp^2q < m \mid t \in \mathbb{N}\}$, $V_4 = \{tpq^2 < m \mid t \in \mathbb{N}\}$, $V_5 = \{tpq < m \mid t \in \mathbb{N}\} \backslash \{tp^2q \cup tpq^2 \mid t \in \mathbb{N}\}$. Also, when $p = 2$, there exist a distance similar class $\{pq^2 = \frac{m}{2} \mid t \in \mathbb{N}\}$ which is a single vertex set. Therefore, by Theorem 2.1 of [13], any minimal resolving set W consists of all vertices except one from each distance similar classes and the vertex $\frac{m}{2} \notin W$ since it has a unique metric representation with respect to the resolving set W. Hence, the metric dimension is given by $|Z(\mathbb{Z}_m)^*| - 5$. □

Example 2.6. For the ring $R = \mathbb{Z}_{36}$, the vertices are partitioned into $V_1 = \{2, 4, 8, 10, 14, 16, 20, 22, 26, 28, 32, 34\}$, $V_2 = \{3, 9, 15, 21, 27, 33\}$, $V_3 = \{12, 24\}$, $V_4 = \{18\}$, $V_5 = \{6, 30\}$ and so the resolving set $W = \{2, 4, 8, 10, 14, 16, 20, 22, 26, 28, 32, 3, 9, 15, 21, 27, 12, 6\}$ and the metric representation are as follows:
$r(34|W) = (2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 1, 1, 1, 1, 1, 2, 2)$,
$r(33|W) = (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 1, 2)$,
$r(24|W) = (2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 1, 1, 1, 1, 1, 1, 1)$,
$r(18|W) = (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 1, 1)$,
$r(30|W) = (2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 1, 1)$.
Hence, $\beta(\mathbb{E}(\mathbb{Z}_{36})) = 18$.

Corollary 2.7. *If p and q are prime numbers, then the following assertions hold:*
(i) $\beta(\mathbb{E}(\mathbb{Z}_m)) = |Z(\mathbb{Z}_m)^*| - 2$ *if* $m = 2p$ *for* $p > 2$
(ii) $\beta(\mathbb{E}(\mathbb{Z}_m)) = |Z(\mathbb{Z}_m)^*| - 2$ *if* $m = pq$ *for* $p \neq q, p, q \geq 2$
(iii) $\beta(\mathbb{E}(\mathbb{Z}_m)) = |Z(\mathbb{Z}_m)^*| - 1$ *if* $m = p^k$ *for* $p \geq 2, k \geq 2$.

Theorem 2.8. *(i) Let $R \cong \prod_{j=1}^{t} R_j$, and R_j be integral domains with $|R_j| > 2$ for some j. Then $\beta(\mathbb{E}(R)) = |Z(R)^*| - 2^t + 2$*
(ii) For any $t \geq 2, \beta(\mathbb{E}(\prod_{j=1}^{t} \mathbb{Z}_2)) \leq t$
(iii) $\beta(\mathbb{E}(\prod_{j=1}^{5} \mathbb{Z}_2)) = 5$.

Proof. Since integral domains are reduced rings, these statements follow from Proposition 2.8 of [4] and Theorem 6.1, 6.2, 6.3 of [14]. □

Now, we calculate the metric dimension when $\mathbb{E}(R)$ is uniquely complemented. This following theorem is applicable for both reduced as well as non-reduced cases.

Theorem 2.9. *Let R be a ring. If $\mathbb{E}(R)$ is uniquely complemented, then $\beta(\mathbb{E}(R)) = |Z(R)^*| - 2$.*

Proof. Since $\mathbb{E}(R)$ is uniquely complemented, $\mathbb{E}(R)$ is a triangle free graph which implies $gr(\mathbb{E}(R)) \in \{4, \infty\}$. If $gr(\mathbb{E}(R)) = 4$, then by Theorem 2.2 of [2] and Theorem 4.5 of [4], $\mathbb{E}(R)$ is a $K_{m,n}$ complete bipartite graph. Moreover, for any three vertices x, y, z in $\mathbb{E}(R)$, $x \perp y$ and $x \perp z$ such that $y \nsim z$. Hence $gr(\mathbb{E}(R)) = \infty$ which implies $\beta(\mathbb{E}(R)) = |Z(R)^*| - 2$ according to Theorem 2.2, indicating $\mathbb{E}(R)$ is a star graph. □

We now discuss the metric dimension of extended zero divisor graph of gaussian integers modulo n. The gaussian integers modulo n is defined as $\mathbb{Z}_m[i] = \{a + ib \mid a, b \in \mathbb{Z}\}$. The units of gaussian integers are $\pm 1, \pm i$. Refer [5] for the number of units in Gaussian integers.

Theorem 2.10. *If p and q are prime integers where $p \equiv 1 (mod\ 4)$ and $q \equiv 3 (mod\ 4)$, then the following assertions hold:*

(i) If $m = q$, then $\beta(\mathbb{E}(\mathbb{Z}_m[i]))$ is undefined
(ii) If $m = 2^k, k \geq 1$, then $\beta(\mathbb{E}(\mathbb{Z}_m[i])) = 2^{m-1} - 2$
(iii) If $m = q^k, k \geq 2$, then $\beta(\mathbb{E}(\mathbb{Z}_m[i])) = 2^{m-2} - 2$
(iv) If $m = p$ or $m = q_1 q_2$, then $\beta(\mathbb{E}(\mathbb{Z}_m[i])) = 2p - 4$.

Proof. (i) For $m = q$, $\mathbb{E}(\mathbb{Z}_m[i])$ is an integral domain with no non-zero zero divisors. Hence $\beta(\mathbb{E}(\mathbb{Z}_m[i]))$ is undefined.
(ii) For $m = 2$, then $\mathbb{E}(\mathbb{Z}_m[i])$ is a single vertex graph. For $k > 1$, $\mathbb{E}(\mathbb{Z}_m[i])$ is a complete graph with $2^{m-1} - 1$ vertices. Hence $\beta(\mathbb{E}(\mathbb{Z}_m[i])) = 2^{m-1} - 2$.
(iii) For $m = q^k, k \geq 2$, $\mathbb{E}(\mathbb{Z}_m[i])$ is a complete graph with $2^{m-2} - 1$ vertices. Thus $\beta(\mathbb{E}(\mathbb{Z}_m[i])) = 2^{m-2} - 2$.
(iv) For $m = p$ or $m = q_1 q_2$, where q_1 and q_2 are gaussian primes, then $\mathbb{E}(\mathbb{Z}_m[i])$ is a $K_{p-1,p-1}$ complete bipartite graph. Hence $\beta(\mathbb{E}(\mathbb{Z}_m[i])) = 2p - 4$.

□

Theorem 2.11. *Let $m = 2 \times \prod_{i=1}^{s} p_i \times \prod_{j=1}^{t} q_j$ such that $p_i \equiv 1 (mod\ 4)$ and $q_j \equiv 3 (mod\ 4)$, then $\beta(\mathbb{E}(\mathbb{Z}_m[i])) = m^2 - [2 \times \prod_{i=1}^{s}(p_i - 1)^2 \times \prod_{j=1}^{t}(q_j - 1)] - 2^k + 1$ or $m^2 - [2 \times \prod_{i=1}^{s}(p_i - 1)^2 \times \prod_{j=1}^{t}(q_j - 1)] - 2^k$.*

Proof. Let $\pi_\alpha, 1 \leq \alpha \leq r$ be the gausaian primes. Now, the vertices of $\mathbb{E}(R)$ can be partitioned into distinct distance similar classes as follows. Consider a distance similar class of set of all multiples of π_1 removing the multiples of other π_α's, $1 \leq \alpha \leq r$ i.e, $\{a + ib \times \pi_1 (mod\ m) \mid a + ib \in \mathbb{Z}_m[i]\} \backslash \{\bigcup_{i=2}^{r} a + ib \times \pi_i \mid a + ib \in \mathbb{Z}_m[i]\}$. Continuing the same method for all π_i's, we have the set of all distance similar classes to be $\{a + ib \times \pi_{i_1} (mod\ m) \mid 1 \leq i_1 \leq r, a + ib \in \mathbb{Z}_m[i]\} \backslash \{\bigcup_{i=1, i \neq i_1}^{r} a + ib \times \pi_i \mid a + ib \in \mathbb{Z}_m[i]\}$. The cardinality of the above distance similar classes is ${}^rC_1$. Now consider a distance similar class of multiples of $\pi_1 \pi_2$ removing the multiples of other π_α's i.e, $\{a + ib \times \pi_1 \pi_2 (mod\ m) \mid a + ib \in \mathbb{Z}_m[i]\} \backslash \{\bigcup_{i=3}^{r} a + ib \times \pi_i \mid a + ib \in \mathbb{Z}_m[i]\}$. Continuing the same process for all other combination of product of two π_i's, we have $\{a + ib \times \pi_{i_1} \pi_{i_2} (mod\ m) \mid 1 \leq i_1 \leq i_2 \leq r, a + ib \in \mathbb{Z}_m[i]\} \backslash \{\bigcup_{i=1, i \neq i_1, i_2}^{r} a + ib \times \pi_i \mid a + ib \in \mathbb{Z}_m[i]\}$. The cardinality of the above distance similar classes is given by ${}^rC_2$. Now considering a distance similar class of multiples of $\pi_1 \pi_2 \pi_3$ removing the multiples of all other π_i's, the distance similar classes is given by $\{a + ib \times \pi_{i_1} \pi_{i_2} \pi_{i_3} (mod\ m) \mid 1 \leq i_1 \leq i_2 \leq i_3 \leq r, a + ib \in \mathbb{Z}_m[i]\} \backslash \{\bigcup_{i=1, i \neq i_1, i_2, i_3}^{r} a + ib \times \pi_i \mid a + ib \in \mathbb{Z}_m[i]\}$. The cardinality of the above distance similar class is ${}^rC_3$. Continuing the process for all the $r - 1$ combinations of π_i's , we have $\{a + ib \times \pi_{i_1} ... \pi_{i_n} (mod\ m) \mid 1 \leq i_1 \leq ... \leq i_n \leq$

$r, a+ib \in \mathbb{Z}_m[i]\}\backslash\{a+ib \times \pi_i \mid i \neq i_1, ..., i_n, a+ib \in \mathbb{Z}_m[i]\}$. The cardinality of the distance similar classes is ${}^rC_{r-1}$. Finally, we consider the distance similar class of product of all π_i's i.e, $\{a+ib \times \pi_1\pi_2...\pi_r\}$. The cardinality of this distance similar class is ${}^rC_r = 1$. Therefore, the cardinality of the distance similar classes is given by ${}^rC_1 + {}^rC_2 + {}^rC_3 + ... + {}^rC_{r-1} + {}^rC_r = 2^r - 1$.

If $m = \prod_{i=1}^{s} p_i \times \prod_{j=1}^{t} q_j$, then the product of all π_i's, $\pi_1\pi_2...\pi_k = n \ (mod\ n) = 0$. Hence the cardinality of the distance similar class is given by ${}^rC_1 + {}^rC_2 + {}^rC_3 + ... + {}^rC_{r-1} = 2^r - 2$.

Therefore, by Theorem 2.1 of [13], any minimal resolving set W consists of all vertices except one from each distance similar classes. Also, when $m = 2 \times \prod_{i=1}^{s} p_i \times \prod_{j=1}^{t} q_j$, $m \neq \prod_{\alpha=1}^{r} \pi_\alpha$ since $2 = (1+i)(1-i)$, there exist a distance similar class of product of all gaussian primes i.e., $\{(1+i) \times \pi_2\pi_3...\pi_r\}$ and this vertex $(1+i) \times \pi_2\pi_3...\pi_r \notin W$ since it has a unique metric representation. If $m = \prod_{i=1}^{s} p_i \times \prod_{j=1}^{t} q_j$, then the number of distance similar classes is $2^r - 2$. By Lemma 12 of [12], $\beta(\mathbb{E}(\mathbb{Z}_m[i])) = m^2 - [2 \times \prod_{i=1}^{s}(p_i-1)^2 \times \prod_{j=1}^{t}(q_j-1)] - 2^k + 1$. For all other cases the number of distance similar classes is $2^r - 1$ and so $\beta(\mathbb{E}(\mathbb{Z}_m[i])) = m^2 - [2 \times \prod_{i=1}^{s}(p_i-1)^2 \times \prod_{j=1}^{t}(q_j-1)] - 2^k$. □

3. Fault tolerant metric dimension of $\mathbb{E}(R)$

Theorem 3.1. *If G is a star graph with m vertices, then $\beta'(\mathbb{E}(R)) = m-1, m \neq 2$.*

Proof. Let $G \cong K_{1,m-1}$ and $V(G) = v_1, v_2, ...v_m$ be the set of vertices and let v_m be the vertex connected to all other vertices in $V(G)$. Then the fault tolerant resolving set of G is given by $W' = \{v_1, v_2, ..., v_{m-1}\}$ since $r'(v_i|W') \neq r'(v_j|W'), 1 \leq i, j \leq m-1$ for $v_i \neq v_j$. Hence W' forms the fault tolerant metric basis for G. □

Theorem 3.2. *If R is a ring, then the following assertions hold:*

(i) R is an integral domain or $R \cong \mathbb{Z}_4$ or $\dfrac{\mathbb{Z}_2[x]}{(X^2)}$ if and only if $\beta'(\mathbb{E}(R))$ is undefined

(ii) R is finite if and only if $\beta'(\mathbb{E}(R))$ is finite.

Proof.
(i) If R is an integral domain, then the vertex set is empty. $\mathbb{Z}_4$ and $\frac{\mathbb{Z}_2[x]}{(X^2)}$ is a single vertex set and so the fault tolerant metric dimension is undefined.
(ii) Let $\beta'(\mathbb{E}(R))$ is finite, then the cardinality of the fault tolerant metric basis W' of $\mathbb{E}(R)$ is finite. Let $W' = \{x_1, x_2, ..., x_n\}$, then the fault tolerant metric representation, $r'(x_i|W')$ is a n-coordinate vector whose coordinate lies in the set $\{0, 1, 2, 3\}$ since the $diam(\mathbb{E}(R)) \leq 3$. Therefore, there are maximum of 4^n possibilities for $r'(x_i|W')$ and so the $|Z(R)^*| \leq 4^n$ by Theorem 1 of [7]. Hence R is finite. The converse is obvious.

□

Theorem 3.3. *If R be a ring, then the following assertions hold:*

(i) $\mathbb{E}(R)$ is a path if and only if $\beta'(\mathbb{E}(R)) = 2$

(ii) If $\mathbb{E}(R)$ is a cycle (except C_4), then $\beta'(\mathbb{E}(R)) = 3$

(iii) If $\mathbb{E}(R)$ is a complete graph or a complete bipartite graph then $\beta'(\mathbb{E}(R)) = |Z(R)^*|$
(iv) If $\mathbb{E}(R)$ is a star graph, then $\beta'(\mathbb{E}(R)) = |Z(R)^*| - 1$.

Proof. This follows from Preposition 1 of [15] and Theorem 3.1. $\square$

Theorem 3.4. *Let $m = p_1^\alpha \prod_{i=2}^k p_i$ be the prime factorization of an integer m. Then $\beta'(\mathbb{E}(\mathbb{Z}_m)) = |Z(\mathbb{Z}_m)^*|$ or $|Z(\mathbb{Z}_m)^*| - 1$.*

Proof. If $p_1 = 2, \alpha = 1, 2$ and $m \neq 4p$ for odd primes p, then there exist a single vertex distance similar class and the fault tolerant metric representation is distinct even after the removal of one vertex from any other distance similar classes. Hence, $\beta'(\mathbb{E}(\mathbb{Z}_m)) = |Z(\mathbb{Z}_m)^*| - 1$. For other cases $\beta'(\mathbb{E}(\mathbb{Z}_m)) = |Z(\mathbb{Z}_m)^*|$. $\square$

Theorem 3.5. *Let R be a ring. If $\mathbb{E}(R)$ is uniquely complemented, then $\beta'(\mathbb{E}(R)) = |Z(R)^*| - 1$.*

Proof. By the same argument as in the proof of 2.7, $\beta'(\mathbb{E}(R)) = |Z(R)^*| - 1$ since $\mathbb{E}(R)$ is a star graph. $\square$

Theorem 3.6. *If p and q are prime integers where $p \equiv 1(mod\ 4)$ and $q \equiv 3(mod\ 4)$, then the following assertions hold:*
(i) If $m = 2$ or $m = q$, then $\beta'(\mathbb{E}(\mathbb{Z}_m[i]))$ is undefined
(ii) If $m = 2^k, k \geq 2$, then $\beta'(\mathbb{E}(\mathbb{Z}_m[i])) = 2^{m-1} - 1$
(iii) If $m = q^k, k \geq 2$, then $\beta'(\mathbb{E}(\mathbb{Z}_m[i])) = 2^{m-2} - 1$
(iv) If $m = p$ or $m = q_1q_2$, then $\beta'(\mathbb{E}(\mathbb{Z}_m[i])) = 2p - 2$.

Proof. (i) For $m = q$, $\mathbb{E}(\mathbb{Z}_m[i])$ is an integral domain and for $m = 2$, $\mathbb{E}(\mathbb{Z}_m[i])$ is an single vertex graph. Hence $\beta'(\mathbb{E}(\mathbb{Z}_m[i]))$ is undefined.
(ii) If $m = 2^k, k \geq 2$, then $\mathbb{E}(\mathbb{Z}_m[i])$ is a $K_{2^{m-1}-1}$ complete graph. Hence $\beta'(\mathbb{E}(\mathbb{Z}_m[i])) = 2^{m-1} - 2$.
(iii) For $m = q^k, k \geq 2$, $\mathbb{E}(\mathbb{Z}_m[i])$ is a $K_{2^{m-2}-1}$ complete graph. Thus $\beta'(\mathbb{E}(\mathbb{Z}_m[i])) = 2^{m-2} - 1$.
(iv) For $m = p$ or $m = q_1q_2$, then $\mathbb{E}(\mathbb{Z}_m[i])$ is a $K_{p-1,p-1}$ complete bipartite graph. Hence $\beta'(\mathbb{E}(\mathbb{Z}_m[i])) = 2p - 2$.

$\square$

Theorem 3.7. *Let $m = 2 \times \prod_{i=1}^s p_i \times \prod_{j=1}^t q_j$ such that $p_i \equiv 1(mod\ 4)$ and $q_j \equiv 3(mod\ 4)$, then $\beta'(\mathbb{E}(\mathbb{Z}_m[i])) = m^2 - [2 \times \prod_{i=1}^s (p_i - 1)^2 \times \prod_{j=1}^t (q_j - 1)] - 1$ or $m^2 - [2 \times \prod_{i=1}^s (p_i - 1)^2 \times \prod_{j=1}^t (q_j - 1)] - 2$.*

Proof. If $m = 2 \times \prod_{i=1}^s p_i \times \prod_{j=1}^t q_j$, then there exist a single vertex set and the fault tolerant metric representation is distinct even after the removal of one vertex from any other distance similar classes. Hence, $\beta'(\mathbb{E}(\mathbb{Z}_m)) = m^2 - [2 \times \prod_{i=1}^s (p_i - 1)^2 \times \prod_{j=1}^t (q_j - 1)] - 2$. If $m = \prod_{i=1}^s p_i \times \prod_{j=1}^t q_j$, then $\beta'(\mathbb{E}(\mathbb{Z}_m)) = m^2 - [2 \times \prod_{i=1}^s (p_i - 1)^2 \times \prod_{j=1}^t (q_j - 1)] - 1$. $\square$

4. Local metric dimension of $\mathbb{E}(R)$

Theorem 4.1. *If G is a cycle with m vertices, then* $lmd(G) = \begin{cases} 1, & \text{if } m \text{ is even} \\ 2, & \text{if } m \text{ is odd.} \end{cases}$

Proof. Let G be a cycle with $C_m : v_1v_2...v_mv_1$ with m vertices. Consider $W = \{v_1\}$ be the minimum local resolving set. Suppose m is even, then $lmd(G) = 1$ since C_m is bipartite. Suppose m is odd, then any two adjacent vertices forms the local metric basis. If $W = \{v_1\}$, then $r(v_{\frac{m+1}{2}}|W) = r(v_{\frac{m+1}{2}+1}|W)$ where $v_{\frac{m+1}{2}}$ and $v_{\frac{m+1}{2}+1}$ are adjacent vertices. Similarly, all other singleton vertices do not form the local resolving set. Hence, $lmd(G) = 2$ when m is odd. □

Theorem 4.2. *If R is a ring, then the following assertions hold:*
(i) R is an integral domain if and only if $lmd(\mathbb{E}(R))$ is undefined
(ii) R is finite if and only if $lmd(\mathbb{E}(R))$ is finite.

Proof. (i) $lmd(\mathbb{E}(R))$ is undefined since the vertex set of integral domain is empty.
(ii) Since $lmd(G) \leq \beta(G)$ by [11], $lmd(\mathbb{E}(R))$ is finite and the converse is obvious.

□

Theorem 4.3. *If R is a ring, then the following assertions hold:*
(i) $\mathbb{E}(R)$ is a bipartite graph if and only if $lmd(\mathbb{E}(R)) = 1$
(ii) If $\mathbb{E}(R)$ is a cycle, then $lmd(\mathbb{E}(R)) = \begin{cases} 1, & \text{if } |Z(R)^*| \text{ is even} \\ 2, & \text{if } |Z(R)^*| \text{ is odd} \end{cases}$
(iii) $\mathbb{E}(R)$ is a complete graph if and only if $lmd(\mathbb{E}(R)) = |Z(R)^*| - 1$.

Proof. The result follows from Theorem 2.4 of [11] and Theorem 4.1. □

Corollary 4.4. *If p is a prime number, then the following assertions hold:*
(i) $lmd(\mathbb{E}(\mathbb{Z}_m)) = 1$ if $m = 2p$ or $4p$ for $p > 2$
(ii) $lmd(\mathbb{E}(\mathbb{Z}_m)) = |Z(\mathbb{Z}_m)^*| - 1$ if $m = p^k$ for $p \geq 2, k \geq 2$.

Theorem 4.5. *Let R be a ring. If $\mathbb{E}(R)$ is uniquely complemented, then $lmd(\mathbb{E}(R)) = 1$.*

Proof. By the same argument as in the proof of 2.7, $lmd(\mathbb{E}(R)) = 1$ since $\mathbb{E}(R)$ is a star graph. □

Theorem 4.6. *If p and q are prime integers where $p \equiv 1(mod\ 4)$ and $q \equiv 3(mod\ 4)$, then the following assertions hold:*
(i) If $m = q$, then $lmd(\mathbb{E}(\mathbb{Z}_m[i]))$ is undefined
(ii) If $m = 2^k, k \geq 1$, then $lmd(\mathbb{E}(\mathbb{Z}_m[i])) = 2^{m-1} - 2$
(iii) If $m = q^k, k \geq 2$, then $lmd(\mathbb{E}(\mathbb{Z}_m[i])) = 2^{m-2} - 2$
(iv) If $m = p$ or $m = q_1q_2$, then $lmd(\mathbb{E}(\mathbb{Z}_m[i])) = 1$.

Proof. (i) For $m = q$, $\mathbb{E}(\mathbb{Z}_m[i])$ is an integral domain with no non-zero zero divisors. Hence $lmd(\mathbb{E}(\mathbb{Z}_m[i]))$ is undefined.
(ii) For $m = 2$, then $\mathbb{E}(\mathbb{Z}_m[i])$ is a single vertex graph. For $k > 1$, $\mathbb{E}(\mathbb{Z}_m[i])$ is a complete graph with $2^{m-1} - 1$ vertices. Hence $lmd(\mathbb{E}(\mathbb{Z}_m[i])) = 2^{m-1} - 2$.
(iii) For $m = q^k, k \geq 2$, $\mathbb{E}(\mathbb{Z}_m[i])$ is a complete graph with $2^{m-2} - 1$ vertices. Thus $lmd(\mathbb{E}(\mathbb{Z}_m[i])) = 2^{m-2} - 2$.

(iv) For $m = p$ or $m = q_1q_2$, where q_1 and q_2 are gaussian primes, then $\mathbb{E}(\mathbb{Z}_m[i])$ is a $K_{p-1,p-1}$ complete bipartite graph. Hence $lmd(\mathbb{E}(\mathbb{Z}_m[i])) = 1$.

□

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References

[1] Anderson, D. F., & Livingston, P. S. (1999). The Zero-Divisor Graph of a Commutative Ring. Journal of Algebra, 217(2),434-447. https://doi.org/10.1006/jabr.1998.7840.

[2] Anderson, D. F., & Mulay, S. B. (2007). On the diameter and girth of a zero-divisor graph. Journal of pure and applied algebra, 210(2), 543-550. https://doi.org/10.1016/j.jpaa.2006.10.007.

[3] Atiyah, M. F., & Macdonald, I. G. (1969). Introduction to commutative algebra, Addison.

[4] Bennis, D., Mikram, J., & Taraza, F. (2016). On the extended zero divisor graph of commutative rings. Turkish Journal of Mathematics, 40(2), 376-388. https://doi.org/10.3906/mat-1504-61.

[5] Cross, J. T. (1983). The Euler -Function in the Gaussian Integers. The American Mathematical Monthly, 90(8), 518-528. https://doi.org/10.1080/00029890.1983.11971275.

[6] Diestel, R. (1997). Graph theory, Springer-Verlag, New York.

[7] Ganesan, N. (1964). Properties of rings with a finite number of zero divisors. Mathematische Annalen, 157(3), 215-218. https://doi.org/10.1007/BF01362435.

[8] Kaplansky, I.(1974). Commutative Rings, University of Chicago Press, Chicago.

[9] Nithya, S., & Elavarasi, G. Extended annihilating-ideal graph of a commutative ring. Discussiones Mathematicae-General Algebra and Applications, 42(2), 279-291.

[10] Nithya, S., & Elavarasi, G. (2022). Partition and local metric dimension of an extended annihilating-ideal graph. Palestine Journal of Mathematics, 11.

[11] Okamoto, F., Phinezy, B., & Zhang, P. (2010). The local metric dimension of a graph. Mathematica Bohemica, 135(3), 239-255. https://doi.org/10.21136/MB.2010.140702.

[12] Osba, E. A., Al-Addasi, S., & Jaradeh, N. A. (2008). Zero divisor graph for the ring of Gaussian integers modulo n. Communications in Algebra, 36(10), 3865-3877. https://doi.org/10.1080/00927870802160859.

[13] Pirzada, S., Raja, R., & Redmond, S. (2014). Locating sets and numbers of graphs associated to commutative rings. Journal of Algebra and Its Applications, 13(07), 1450047. https://doi.org/10.1142/S0219498814500479.

[14] Raja, R., Pirzada, S., & Redmond, S. (2016). On locating numbers and codes of zero divisor graphs associated with commutative rings. Journal of Algebra and Its Applications, 15(01), 1650014. https://doi.org/10.1142/S0219498816500146.

[15] Sharma, S., & Bhat, V. K. (2022). Fault-tolerant metric dimension of zero-divisor graphs of commutative rings. AKCE International Journal of Graphs and Combinatorics, 19(1), 24-30. https://doi.org/10.1080/09728600.2021.2009746.

[16] Slater, P. J.(1975). Leaves of trees. Proceeding of the 6th Southeastern Conference on Combinatorics, Graph Theory, and Computing, Congressus Numerantium, 14, 549-559.

[17] West, D. B. (2001). Introduction to graph theory (Vol. 2). Upper Saddle River: Prentice hall.

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