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GABOR FRAMES ON FINITE NON-CYCLIC ABELIAN GROUPS

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Abstract. We attempt to analyze Gabor frames for L^2 spaces on finite non cyclic abelian groups from their natural perspectives and establish their equivalence with Gabor frames on the isomorphic product groups of finite cyclic groups by means of unitary and non unitary invertible linear transformations. Gabor frames and their canonical dual frames on finite non cyclic groups are identified as the images of the same Gabor frame on a finite product of finite cyclic groups under invertible linear transformations.

1. Introduction and preliminaries

Theory of frames, introduced and flourished through the works [3, 13], is one of the fast growing and promising research area in mathematics due to its wide applications. Time-frequency analysis of signals in $L^2(\mathbb{R})$, as suggested by D. Gabor [7], aims at representing functions (signals) as superposition of translated and modulated versions of a fixed function $g \in L^2(\mathbb{R})$. In 1980's, it became an independent topic of mathematical investigation through the contributions like [9]. The idea of combining Gabor analysis with frame theory came up in the fundamental work [2]. Analyzing the images of Gabor frames under appropriate invertible operators on $L^2(\mathbb{R})$ is an interesting topic [4, 5] in this area. More details on the classical theory can be seen in [1, 8].

On the general aspects of the Gabor theory, Gabor systems on finite abelian groups have been studied specifically in [6, 11, 12]. In the present article, we discuss unitary and non unitary approaches for the analysis of Gabor frames on a finite abelian (non cyclic) group, without bringing in the abstract representation theory. In this aspect, Gabor frames and their canonical dual frames on finite non cyclic groups are viewed as the images of the same Gabor frame on an appropriate finite product of finite cyclic groups under invertible linear transformations.

We now go through some of the fundamental terminologies and results that were required for the presentation of the key findings in this article.
A sequence $\{u_k\}_{k=1}^{\infty}$ in a Hilbert space $\mathcal{H}$ is said to be a *frame* for $\mathcal{H}$, if there are positive constants α, β such that $\alpha\|x\|^2 \leq \sum_{k=1}^{\infty} |\langle x, u_k\rangle|^2 \leq \beta\|x\|^2$ for all

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$x \in \mathcal{H}$. A frame $\{u_k\}_{k=1}^{\infty}$ is called a *Parseval frame* (or *normalized tight frame*) when $\alpha = \beta = 1$.

If $\{u_k\}_{k=1}^{\infty}$ is a frame for a Hilbert space $\mathcal{H}$, then the map S defined by $Sx = \sum_{k=1}^{\infty} \langle x, u_k \rangle u_k$ on $\mathcal{H}$ is a bounded, positive and invertible operator on $\mathcal{H}$, called the *frame operator* associated with the frame $\{u_k\}_{k=1}^{\infty}$.

If S is the frame operator of a frame $\{u_k\}_{k=1}^{\infty}$ for $\mathcal{H}$, then the frame $\{S^{-1}u_k\}_{k=1}^{\infty}$ is called the *(canonical) dual frame* of $\{u_k\}_{k=1}^{\infty}$. Every frame $\{u_k\}_{k=1}^{\infty}$ for $\mathcal{H}$ with frame operator S admits the *frame decomposition* in two ways:
$x = \sum_{k=1}^{\infty} \langle x, S^{-1}u_k \rangle u_k$ and $x = \sum_{k=1}^{\infty} \langle x, u_k \rangle S^{-1}u_k$, where both the series converge unconditionally for all $x \in \mathcal{H}$.

Our focus is on Gabor frames for L^2 spaces on finite abelian groups. Corresponding to a finite abelian group G, the space $\mathbb{C}^G$ is identified with the Hilbert space $L^2(G)$ of complex valued functions on G equipped with the inner product given by $\langle f, h \rangle = \sum\limits_{g \in G} f(g)\overline{h(g)} \quad f,\ h \in L^2(G)$.

A *character* on G (see [12]) is a (continuous) group homomorphism from G into the multiplicative group $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. The set of all characters on G forms a group $\widehat{G}$ under pointwise multiplication and this group is called the *dual group of* G (see [11]). Modulation operators on $L^2(G)$ are the multiplication operators corresponding to the characters on G. Thus for $\xi \in \widehat{G}$, the modulation operator $M_\xi : L^2(G) \to L^2(G)$ is given by $M_\xi f(g) = \xi(g) f(g)$ for all $g \in G$. For $g \in G$, the translation operator $T_g : L^2(G) \to L^2(G)$, is defined by $T_g f(h) = f(hg^{-1})$ for all $h \in G$.

Definition 1.1. [11] A Gabor system in $L^2(G)$ is a family $\mathcal{G}(\varphi, \Lambda) = \{M_\xi T_g \varphi : (g, \xi) \in \Lambda\}$ where φ is a non zero element known as the *window function* or the *generator* of $\mathcal{G}(\varphi, \Lambda)$ in $L^2(G)$ and Λ is a subset (preferably a subgroup) of the product group $G \times \widehat{G}$. A Gabor system which spans $L^2(G)$ is a frame and is called a *Gabor frame* for $L^2(G)$.

The frame operator S on $L^2(G)$ associated with the Gabor frame $\mathcal{G}(\varphi, \Lambda)$ is given by $S(f) = \sum\limits_{(g,\xi) \in \Lambda} \langle f, M_\xi T_g \varphi \rangle M_\xi T_g \varphi$ for all $f \in L^2(G)$.

If a group G of order N is cyclic then it is isomorphic to $\mathbb{Z}_N$. In this case, Gabor frames for $L^2(G)$ are similar to those in $L^2(\mathbb{Z}_N)$. On the otherhand, if the group G of order N is non cyclic, by Fundamental theorem of finitely generated abelian groups, G will be isomorphic to a direct product of the form $\Gamma_G = \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \cdots \times \mathbb{Z}_{m_p}$, where $m_1, m_2, \ldots m_p \in \mathbb{N}$ with $m_1 m_2 \ldots m_p = N$. It is quiet interesting in this case, to view Gabor frames for $L^2(G)$ using Gabor frames for $L^2(\Gamma_G)$.

Towards this, we need the special case when the group G is the finite product $\Gamma = \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \cdots \times \mathbb{Z}_{m_p}$ where $m_1, m_2, \ldots m_p$ are positive integers. The corresponding Hilbert space $L^2(\Gamma)$ is a finite dimensional Hilbert space w.r.to the standard inner product given by

$$\langle f, g \rangle = \sum_{i_1=1}^{m_1} \sum_{i_2=1}^{m_2} \cdots \sum_{i_p=1}^{m_p} f(i_1, i_2, \ldots i_p)\overline{g(i_1, i_2, \ldots i_p)}, \qquad f,\ g \in L^2(\Gamma).$$

For each $k = (k_1, k_2, \dots k_p) \in \Gamma$, the *translation operator* $T_k : L^2(\Gamma) \to L^2(\Gamma)$ is defined by

$$T_k g(r_1, r_2, \dots r_p) = g(s_1, s_2, \dots s_p), \qquad (r_1, r_2, \dots r_p) \in \Gamma,$$

where $s = (s_1, s_2, \dots s_p) \in \Gamma$ is such that $r_j - k_j \equiv s_j (mod\ m_j)$ for $j = 1, 2, \dots p$.

Similarly for $l = (l_1, l_2, \dots l_p) \in \Gamma$, the *modulation operator* $M_l : L^2(\Gamma) \to L^2(\Gamma)$ is defined by

$$M_l g(r_1, r_2, \dots r_p) = e^{2\pi i[\frac{l_1 r_1}{m_1} + \frac{l_2 r_2}{m_2} + \dots + \frac{l_p r_p}{m_p}]} g(r_1, r_2, \dots r_p), \qquad (r_1, r_2, \dots r_p) \in \Gamma.$$

Thus for $g \in L^2(\Gamma) \setminus \{0\}$, a *Gabor System* generated by g in $L^2(\Gamma)$ is a family of elements of the form $\{M_l T_k g;\ k, l \in \Lambda \subset \Gamma\}$.

2. Main results

Here onwards, a finite abelian group G of order N and its isomorphic direct product Γ_G are assumed as $\{g_0, g_1, \dots, g_{N-1}\}$ and $\mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \dots \times \mathbb{Z}_{m_p}$ respectively, where $m_1, m_2, \dots m_p \in \mathbb{N}$ with $m_1 m_2 \dots m_p = N$. We begin with a fundamental lemma which is very crucial in our discussions.

Lemma 2.1. *For each isomorphism $\theta : \Gamma_G \to G$, the map $B_\theta : L^2(\Gamma_G) \to L^2(G)$ defined by $B_\theta(x)(g_j) = x(\theta^{-1}(g_j))$ for all $x \in L^2(\Gamma_G)$ and $g_j \in G$, is linear, bounded and unitary with adjoint $B_\theta^* : L^2(G) \to L^2(\Gamma_G)$ given by*
$B_\theta^*(\eta)(r_1, r_2, \dots, r_p) = \eta(\theta(r_1, r_2, \dots, r_p))$, $\eta \in L^2(G)$, $(r_1, r_2, \dots, r_p) \in \Gamma_G$.

Proof. The map B_θ is linear, since for $x, y \in L^2(\Gamma_G)$ and $c \in \mathbb{C}$,

$$\begin{aligned} B_\theta(cx + y)(g_j) &= (cx + y)(\theta^{-1}(g_j)) \\ &= (cx)(\theta^{-1}(g_j)) + y(\theta^{-1}(g_j)) \\ &= c(x(\theta^{-1}(g_j))) + y(\theta^{-1}(g_j)) \\ &= cB_\theta(x)(g_j) + B_\theta(y)(g_j) \\ &= (cB_\theta x + B_\theta y)(g_j), \quad \text{for all } j = 0, 1, \dots, N-1. \end{aligned}$$

Since both the spaces $L^2(G)$ and $L^2(\Gamma_G)$ are finite dimensional, B_θ is bounded. The adjoint $B_\theta^* : L^2(G) \to L^2(\Gamma_G)$ of B_θ is defined by $B_\theta^*(\eta)(r_1, r_2, ..., r_p) = \eta(\theta(r_1, r_2, ..., r_p))$ for all $\eta \in L^2(G)$ and $(r_1, r_2, ..., r_p) \in \Gamma_G$. To see this, let

$x \in L^2(\Gamma_G)$ and $\eta \in L^2(G)$. Then

$$\begin{aligned}
\langle B_\theta x, \eta \rangle &= \sum_{j=0}^{N-1} B_\theta x(g_j)\overline{\eta(g_j)} \\
&= \sum_{j=0}^{N-1} x(\theta^{-1}(g_j))\overline{\eta(g_j)} \quad .... (1) \\
\langle x, B_\theta^* \eta \rangle &= \sum_{(r_1,r_2,...,r_p)\in\Gamma_G} x(r_1,r_2,...,r_p)\overline{(B_\theta^*\eta)(r_1,r_2,...,r_p)} \\
&= \sum_{(r_1,r_2,...,r_p)\in\Gamma_G} x(r_1,r_2,...,r_p)\overline{\eta(\theta(r_1,r_2,...,r_p))} \\
&= \sum_{j=0}^{N-1} x(\theta^{-1}(g_j))\overline{\eta(g_j)} \quad .... (2)
\end{aligned}$$

where $\theta(r_1, r_2, ..., r_p) = g_j$.
From (1)and (2), $\langle B_\theta x, \eta\rangle = \langle x, B_\theta^*\eta\rangle$ for all $x \in L^2(\Gamma_G)$ and $\eta \in L^2(G)$.

Now

$$\begin{aligned}
(B_\theta^* B_\theta)(x)(r_1, r_2, \ldots, r_p) &= B_\theta x(\theta(r_1, r_2, \ldots, r_p)) \\
&= x(\theta^{-1}(\theta(r_1, r_2, \ldots, r_p))) \\
&= x(r_1, r_2, \ldots, r_p)
\end{aligned}$$

for all $(r_1, r_2, \ldots, r_p) \in \Gamma_G$ and $x \in L^2(\Gamma_G)$. Hence $B_\theta^* B_\theta$ is the identity operator on $L^2(\Gamma_G)$. Similarly,

$$\begin{aligned}
(B_\theta B_\theta^*)(\eta)(g_j) &= B_\theta^*\eta(\theta^{-1}(g_j)) \\
&= \eta(\theta(\theta^{-1}(g_j))) \\
&= \eta(g_j)
\end{aligned}$$

for all $g_j \in G$ and $\eta \in L^2(G)$. Thus $B_\theta B_\theta^*$ is the identity operator on $L^2(G)$ and hence the map $B_\theta : L^2(\Gamma_G) \to L^2(G)$ is unitary. □

Now we provide an explicit description of the dual group $\widehat{G}$ of a finite abelian group $G = \{g_0, g_1, \ldots, g_{N-1}\}$ in terms of its isomorphic product group Γ_G.

For $l = (l_1, l_2, \ldots, l_p) \in \Gamma_G$ define $\xi_l : G \to S^1$ by

$$\xi_l(g) = e^{2\pi i[\frac{l_1 r_1}{m_1} + \frac{l_2 r_2}{m_2} + \cdots + \frac{l_p r_p}{m_p}]}$$

where $(r_1, \ldots, r_p) = \theta^{-1}(g)$ and $g \in G$. The map $\xi_l : G \to S^1$ is a homomorphism. To see this, let $g_j, g_k \in G$, $(r_1, r_2, ..., r_p) = \theta^{-1}(g_j)$ and $(r_1', r_2', ..., r_p') = \theta^{-1}(g_k)$. Then,

$$\theta(r_1 +_{m_1} r_1', r_2 +_{m_2} r_2', ..., r_p +_{m_p} r_p') = \theta(r_1, r_2, ..., r_p) + \theta(r_1', r_2', ..., r_p') = g_j * g_k.$$

Hence

$$\begin{aligned}\xi_l(g_j)\xi_l(g_k) &= e^{2\pi i[\frac{l_1r_1}{m_1}+\frac{l_2r_2}{m_2}+.....+\frac{l_pr_p}{m_p}]}e^{2\pi i[\frac{l_1r_1'}{m_1}+\frac{l_2r_2'}{m_2}+.....+\frac{l_pr_p'}{m_p}]}\\ &= e^{2\pi i[\frac{l_1(r_1+m_1r_1')}{m_1}+\frac{l_2(r_2+m_2r_2')}{m_2}+.....+\frac{l_p(r_p+m_pr_p')}{m_p}]}\\ &= \xi_l(g_j * g_k)\end{aligned}$$

Thus ξ_l is a homomorphism from G into S^1.
Also if $\xi_l = \xi_k$ for $l, k \in \Gamma_G$, then for all $(r_1, r_2, \ldots, r_p) \in \Gamma_G$

$$e^{2\pi i[\frac{l_1r_1}{m_1}+\frac{l_2r_2}{m_2}+\cdots+\frac{l_pr_p}{m_p}]} = e^{2\pi i[\frac{k_1r_1}{m_1}+\frac{k_2r_2}{m_2}+\cdots+\frac{k_pr_p}{m_p}]} \quad \cdots\cdots\cdots (*)$$

By choosing $r_1 = 1$ and $r_2 = r_3 = \cdots = r_p = 0$, we obtain from $(*)$ that $l_1 = k_1$. In general, for each $j \in \{1, 2, \ldots, p\}$, by choosing $r_n = 1$ if $n = j$ and $r_n = 0$ for $n \in \{1, 2, \ldots, p\}$ but $n \neq j$, we obtain from $(*)$ that $l_j = k_j$ for all $j = 1, 2, \ldots, p$ and hence $l = k$. On the otherhand, if $l = k$, then evidently $\xi_l = \xi_k$. Hence $\{\xi_l : l \in \Gamma_G\}$ is precisely the collection of distinct characters of G. Thus $\widehat{G} = \{\xi_l : l \in \Gamma_G\}$

The next Proposition brings out the actions of B_θ and B_θ^* on modulation and translation operators.

Proposition 2.2. *Let G be a finite abelian group and $L^2(\Gamma_G)$, θ and B_θ be as in Lemma 2.1. Also let (M_l, T_k) and (M_{ξ_l}, T_{g_k}) denote the modulation-translation pairs on $L^2(\Gamma_G)$ and $L^2(G)$ respectively. Then for $k, l \in \Gamma_G$, $g_k \in G$ and $\xi_l \in \widehat{G}$,*

(i) *$B_\theta M_l = M_{\xi_l} B_\theta$ and $B_\theta T_k = T_{g_k} B_\theta$.*
(ii) *$B_\theta^* M_{\xi_l} = M_l B_\theta^*$ and $B_\theta^* T_{g_k} = T_k B_\theta^*$.*

Proof. Let $l \in \Gamma_G$ and $g_j \in G$ with $\theta^{-1}(g_j) = (r_1, r_2, ..., r_p)$.
Then for all $x \in L^2(\Gamma_G)$,

$$\begin{aligned}(M_{\xi_l}B_\theta)(x)(g_j) &= M_{\xi_l}(B_\theta(x))(g_j)\\ &= e^{2\pi i[\frac{l_1r_1}{m_1}+\frac{l_2r_2}{m_2}+.....+\frac{l_pr_p}{m_p}]}(x(\theta^{-1}(g_j))) \quad \text{and}\\ (B_\theta M_l)(x)(g_j) &= B_\theta(M_l(x))(g_j) = M_l(x)(\theta^{-1}(g_j))\\ &= M_l(x)(r_1, r_2, ..., r_p)\\ &= e^{2\pi i[\frac{l_1r_1}{m_1}+\frac{l_2r_2}{m_2}+.....+\frac{l_pr_p}{m_p}]}(x(r_1, r_2, ..., r_p))\\ &= e^{2\pi i[\frac{l_1r_1}{m_1}+\frac{l_2r_2}{m_2}+\cdots+\frac{l_pr_p}{m_p}]}(x(\theta^{-1}(g_j)))\end{aligned}$$

Thus $B_\theta M_l = M_{\xi_l} B_\theta$.

Similarly if $k \in \Gamma_G$, g_j, $g_k \in G$ with $\theta^{-1}(g_j) = (r_1, r_2, \ldots, r_p)$, $\theta^{-1}(g_k^{-1}) = (q_1, \ldots, q_p)$, $r_j - q_j \equiv s_j (mod\ m_j)$ for $j = 1, 2, \ldots, p$ and $x \in L^2(\Gamma_G)$,

$$\begin{aligned}
\text{then} \quad (T_{g_k} B_\theta)(x)(g_j) &= T_{g_k}(B_\theta(x))(g_j) = (B_\theta(x))(g_j g_k^{-1}) \\
&= x(\theta^{-1}(g_j g_k^{-1})) = x(\theta^{-1}(g_j) - \theta^{-1}(g_k^{-1})) \\
&= x(s_1, s_2, ..., s_p) \\
\text{and} \quad (B_\theta T_k)(x)(g_j) &= B_\theta(T_k(x))(g_j) = T_k(x)\theta^{-1}(g_j) \\
&= T_k(x)(r_1, r_2, ..., r_p) \\
&= x(s_1, s_2, ..., s_p).
\end{aligned}$$

Hence, $B_\theta T_k = T_{g_k} B_\theta$. Now,

$$\begin{aligned}
(B_\theta^* M_{\xi_l})(\eta)(r_1, r_2, ..., r_p) &= M_{\xi_l}(\eta(\theta(r_1, r_2, ..., r_p))) \\
&= e^{2\pi i[\frac{l_1 r_1}{m_1} + \frac{l_2 r_2}{m_2} + ... + \frac{l_p r_p}{m_p}]}(\eta(\theta(r_1, r_2, ..., r_p))) \quad \text{and} \\
(M_l B_\theta^*)(\eta)(r_1, r_2, ..., r_p) &= M_l(B_\theta^* \eta)(r_1, r_2, ..., r_p) = M_l(\eta(\theta(r_1, r_2, ..., r_p))) \\
&= e^{2\pi i[\frac{l_1 r_1}{m_1} + \frac{l_2 r_2}{m_2} + ... + \frac{l_p r_p}{m_p}]}(\eta(\theta(r_1, r_2, ..., r_p)))
\end{aligned}$$

Thus, $B_\theta^* M_{\xi_l} = M_l B_\theta^*$ Again,

$$\begin{aligned}
(B_\theta^* T_{g_k})(\eta)(r_1, r_2, ..., r_p) &= T_{g_k}(\eta(\theta(r_1, r_2, ..., r_p))) = T_{g_k}(\eta(g_r)) \\
&= \eta(g_r g_k^{-1}) \\
&= \eta(g_q) \quad \text{and}
\end{aligned}$$

$$\begin{aligned}
(T_k B_\theta^*)(\eta)(r_1, r_2, ..., r_p) &= T_k(B_\theta^* \eta)(r_1, r_2, ..., r_p) = B_\theta^*(\eta(q)) \\
&= \eta(\theta(q_1, q_2, ..., q_p)) \\
&= \eta(g_q).
\end{aligned}$$

Hence, $B_\theta^* T_{g_k} = T_k B_\theta^*$ $\square$

Now we come to the main goals of our discussion.

Theorem 2.3. *If G is a finite abelian group, then any Gabor frame for $L^2(G)$ can be identified as the image of a Gabor frame for $L^2(\Gamma_G)$ under a unitary map.*

Proof. Let $\mathcal{G} = \{M_{\xi_l} T_{g_k} \varphi : (g_k, \xi_l) \in \Lambda \subset G \times \widehat{G}\}$ be a Gabor frame for $L^2(G)$. Fix an isomorphism θ from Γ_G onto G and define the map $B : L^2(\Gamma_G) \to L^2(G)$ by $B(x)(g_j) = x(\theta^{-1}(g_j))$ for all $g_j \in G$. By Lemma 2.1, B is unitary sothat $B^{-1}(\mathcal{G})$ is a frame in $L^2(\Gamma_G)$ since $span\{B^{-1}(\mathcal{G})\} = L^2(\Gamma_G)$.
Now $\mathcal{G} \subset \{M_{\xi_l} T_{g_k} \varphi : (g_k, \xi_l) \in G \times \widehat{G}\}$. Hence

$$\begin{aligned}
B^{-1}(\mathcal{G}) &\subset B^{-1}(\{M_{\xi_l} T_{g_k} \varphi : (g_k, \xi_l) \in G \times \widehat{G}\}) \\
&= \{B^{-1}(M_{\xi_l} T_{g_k} \varphi) : (g_k, \xi_l) \in G \times \widehat{G}\} \\
&= \{(B^{-1} M_{\xi_l} B)(B^{-1} T_{g_k} B) B^{-1} \varphi) : (g_k, \xi_l) \in G \times \widehat{G}\} \\
&= \{M_l T_k B^{-1} \varphi : k, l \in \Gamma_G\} \quad \text{by Proposition 2.2.}
\end{aligned}$$

Thus $B^{-1}(\mathcal{G})$ is a Gabor system in $L^2(\Gamma_G)$ and $\mathcal{G}_\Gamma = B^{-1}(\mathcal{G})$ is a Gabor frame for $L^2(\Gamma_G)$ with $B(\mathcal{G}_\Gamma) = \mathcal{G}$. $\square$

Unitary and non unitary approaches are equally important in quantum field theory (see [10]). In this regard, images of Gabor frames under unitary and non unitary maps are equally significant, as is observed below.

Theorem 2.4. *For a finite abelian group G, the canonical dual frame of a non-Parseval Gabor frame for $L^2(G)$ is the image of a Gabor frame for $L^2(\Gamma_G)$ under a non unitary map $C : L^2(\Gamma_G) \longrightarrow L^2(G)$.*

Proof. Let G be a finite abelian group and $\mathcal{G} = \{M_{\xi_l} T_{g_k} \varphi : (g_k, \xi_l) \in \Lambda\}$, $\Lambda \subset G \times \widehat{G}$ be a Gabor frame for $L^2(G)$. By Theorem 2.3, $\mathcal{G}$ is the image of a Gabor frame $\mathcal{G}_{\Gamma_G}$ for $L^2(\Gamma_G)$ under a unitary map $B : L^2(\Gamma_G) \longrightarrow L^2(G)$.

Let S and S' be the frame operators of $\mathcal{G}_{\Gamma_G}$ and $\mathcal{G}$ respectively. Since $\mathcal{G}_{\Gamma_G} = B^{-1}(\mathcal{G})$, we have

$$\begin{aligned} S(x) &= \sum_{(g_k,\xi_l)\in\Lambda} \langle x, B^{-1}(M_{\xi_l} T_{g_k}\varphi)\rangle B^{-1}(M_{\xi_l} T_{g_k}\varphi) \\ &= B^{-1}\Big(\sum_{(g_k,\xi_l)\in\Lambda} \langle B(x), M_{\xi_l} T_{g_k}\varphi\rangle M_{\xi_l} T_{g_k}\varphi \Big) \\ &= B^{-1} S' B(x) \text{ for all } x \in L^2(\Gamma_G). \end{aligned}$$

Thus $S = B^{-1}S'B$ and hence $S' = BSB^{-1}$.

If $\mathcal{G}$ is a non-Parseval frame, then $\mathcal{G}_{\Gamma_G}$ is necessarily a non-Parseval frame, for otherwise $S = I$ so that $S' = BSB^{-1} = I$ and hence $\mathcal{G} = B(\mathcal{G}_{\Gamma_G})$ also will become a Parseval frame.

Now, the canonical dual frame of $\mathcal{G}$ is $(S')^{-1}(\mathcal{G}) = (BSB^{-1})^{-1}(\mathcal{G}) = BS^{-1}B^{-1}(\mathcal{G}) = B(S^{-1}(\mathcal{G}_{\Gamma_G}))$. Hence $(S')^{-1}(\mathcal{G}) = C(\mathcal{G}_{\Gamma_G})$ where $C = BS^{-1}$ is non unitary, as desired. $\square$

It is worthful to observe that in the proof of Theorem 2.4, the same frame $\mathcal{G}_{\Gamma_G}$ for $L^2(\Gamma_G)$ is being mapped to $\mathcal{G}$ as well as its canonical dual frame through the unitary map B and the non unitary map $C = BS^{-1}$ respectively.

Remark 2.5. If $\mathcal{G}$ is a Parseval frame, then so is $\mathcal{G}_{\Gamma_G}$ appearing in the proof of Theorem 2.4. In this case, the canonical dual frame of $\mathcal{G}$ is $\mathcal{G}$ itself. Let A be a non unitary invertible operator on $L^2(\Gamma_G)$ that commutes with the operators M_l and T_k involved in $\mathcal{G}_{\Gamma_G}$. Then $\mathcal{G}'_{\Gamma_G} = A(\mathcal{G}_{\Gamma_G})$ spans $L^2(\Gamma_G)$ and becomes a Gabor frame in $L^2(\Gamma_G)$. Also, the non unitary map $C = BA^{-1}$ maps the frame $\mathcal{G}'_{\Gamma_G}$ into $\mathcal{G}$ as well.

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