GROWTH RATES OF WRONSKIANS GENERATED BY COMPLEX VALUED FUNCTIONS

SANJIB KUMAR DATTA\textsuperscript{1*}, TANMAY BISWAS\textsuperscript{2} AND SULTAN ALI\textsuperscript{3}

Abstract. In the paper we establish some new results depending on the comparative growth properties of composite entire or meromorphic functions using generalized $L^*$-order of Wronskians generated by one of the factors.

1. Introduction, Definitions and Notations

Let $f$ be an entire function defined in the open complex plane $\mathbb{C}$. The maximum modulus function $M(r, f)$ corresponding to $f$ is defined on $|z| = r$ as follows:

$$M(r, f) = \max_{|z| = r} |f(z)|.$$

When $f$ is meromorphic, $M(r, f)$ cannot be defined as $f$ is not analytic throughout the complex plane. In this situation, one may introduce another function $T(r, f)$ known as Nevanlinna’s characteristic function of $f$, playing the same role as $M(r, f)$.

The integrated counting function $N(r, a; f)\left(\tilde{N}(r, a; f)\right)$ of $a$-points (distinct $a$-points) of $f$ is defined as

$$N(r, a; f) = \int_0^r \frac{n(t, a; f) - n(0, a; f)}{t} dt + n(0, a; f) \log r$$

$$\left(\tilde{N}(r, a; f) = \int_0^r \frac{\tilde{n}(t, a; f) - \tilde{n}(r, a; f)}{t} dt + \tilde{n}(0, a; f) \log r\right),$$

where we denote by $n(t, a; f)\left(\tilde{n}(t, a; f)\right)$ the number of $a$-points (distinct $a$-points) of $f$ in $|z| \leq t$ and an $\infty$-point is a pole of $f$.

In many occasions $N(r, \infty; f)$ and $\tilde{N}(r, \infty; f)$ are denoted by $N(r, f)$ and $\tilde{N}(r, f)$ respectively. The function $N(r, a; f)$ is called the enumerative function.

\textit{Date:} Received: Jul 20, 2013; Accepted: Sep 16, 2013.
\textit{*} Corresponding author.
2010 \textit{Mathematics Subject Classification.} Primary 30D30; Secondary 30D35.
\textit{Key words and phrases.} Transcendental entire and meromorphic function, composition, growth, generalized $L^*$-order, Wronskian, slowly changing function.
On the other hand, the function \( m(r,f) \equiv m(r,\infty; f) \) known as the proximity function is defined as
\[
m(r,f) = \frac{1}{2\pi} \int_{0}^{2\pi} \log^+ |f(re^{i\theta})| \, d\theta,
\]
where \( \log^+ x = \max(\log x, 0) \) for all \( x \geq 0 \) and an \( \infty \)-point is a pole of \( f \).

Analogously, \( m\left(r, \frac{1}{f-a}\right) \equiv m(r,a; f) \) is defined when \( a \) is not an \( \infty \)-point of \( f \).

Thus the Nevanlinna’s characteristic function \( T(r,f) \) corresponding to \( f \) is defined as
\[
T(r,f) = N(r,f) + m(r,f).
\]
When \( f \) is entire, \( T(r,f) \) coincides with \( m(r,f) \) as \( N(r,f) = 0 \).

For an entire function \( f \), Sato [9] defined the generalized order \( \rho_f^{[m]} \) and generalized lower order \( \lambda_f^{[m]} \) as follows:
\[
\rho_f^{[m]} = \limsup_{r \to \infty} \frac{\log^{[m+1]} M(r,f)}{\log r} \quad \text{and} \quad \lambda_f^{[m]} = \liminf_{r \to \infty} \frac{\log^{[m+1]} M(r,f)}{\log r},
\]
where \( \log^{[k]} x = \log \left( \log^{[k-1]} x \right) \) for \( k = 1, 2, 3, \ldots \) and \( \log^{[0]} x = x \).

Let \( L = L(r) \) be a positive continuous function increasing slowly i.e., \( L(ar) \sim L(r) \) as \( r \to \infty \) for every positive constant \( a \). Singh and Barker [10] defined it in the following way:

**Definition 1.1.** [10] A positive continuous function \( L(r) \) is called a slowly changing function if for \( \varepsilon(>0) \),
\[
\frac{1}{k^\varepsilon} \leq \frac{L(kr)}{L(r)} \leq k^\varepsilon \quad \text{for} \quad r \geq r(\varepsilon) \quad \text{and}
\]
uniformly for \( k \geq 1 \).

If further, \( L(r) \) is differentiable, the above condition is equivalent to
\[
\lim_{r \to \infty} \frac{rL'(r)}{L(r)} = 0.
\]

Somasundaram and Thamizharasi [11] introduced the notions of \( L \)-order for entire functions defined in the open complex plane \( \mathbb{C} \). The more generalized concept for \( L \)-order for entire and meromorphic functions are \( L^* \)-order respectively. Their definitions are as follows:

**Definition 1.2.** [11] The \( L^* \)-order \( \rho_f^{L*} \) and the \( L^* \)-lower order \( \lambda_f^{L*} \) of an entire function \( f \) are defined as
\[
\rho_f^{L*} = \limsup_{r \to \infty} \frac{\log^{[2]} M(r,f)}{\log [rL(r)]} \quad \text{and} \quad \lambda_f^{L*} = \liminf_{r \to \infty} \frac{\log^{[2]} M(r,f)}{\log [rL(r)]},
\]
When \( f \) is meromorphic, one can easily verify that
\[
\rho_f^L = \limsup_{r \to \infty} \frac{\log T(r, f)}{\log [re^L(r)]} \quad \text{and} \quad \lambda_f^L = \liminf_{r \to \infty} \frac{\log T(r, f)}{\log [re^L(r)]},
\]

Combining the concepts of Sato\([9]\) and Somasundaram and Thamizharasi\([11]\) we may state the following definition:

**Definition 1.3.** The generalized \( L^* \)-order \( \rho_f^{[m]L^*} \) and the generalized \( L^* \)-lower order \( \lambda_f^{[m]L^*} \) of an entire function \( f \) are defined as
\[
\rho_f^{[m]L^*} = \limsup_{r \to \infty} \frac{\log^{[m]+1} M(r, f)}{\log [re^{L(r)}]} \quad \text{and} \quad \lambda_f^{[m]L^*} = \liminf_{r \to \infty} \frac{\log^{[m]+1} M(r, f)}{\log [re^{L(r)}]},
\]
where \( m \) is any positive integer.

When \( f \) is meromorphic, it can be easily verified that
\[
\rho_f^{[m]L^*} = \limsup_{r \to \infty} \frac{\log^{[m]} T(r, f)}{\log [re^{L(r)}]} \quad \text{and} \quad \lambda_f^{[m]L^*} = \liminf_{r \to \infty} \frac{\log^{[m]} T(r, f)}{\log [re^{L(r)}]},
\]

For \( m = 1 \), Definition 1.3 reduces to Definition 1.2.

The following definitions are also well known.

**Definition 1.4.** A meromorphic function \( a \equiv a(z) \) is called small with respect to \( f \) if \( T(r, a) = S(r, f) \) where \( S(r, f) = o\{T(r, f)\} \) i.e., \( \frac{S(r, f)}{T(r, f)} \to 0 \) as \( r \to \infty \).

**Definition 1.5.** Let \( a_1, a_2, \ldots, a_k \) be linearly independent meromorphic functions and small with respect to \( f \). We denote by \( L(f) = W(a_1, a_2, \ldots, a_k; f) \), the Wronskian determinant of \( a_1, a_2, \ldots, a_k, f \) i.e.,
\[
L(f) = \begin{vmatrix}
a_1 & a_2 & \cdots & a_k & f \\
 a_1' & a_2' & \cdots & a_k' & f' \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
a_1^{(k)} & a_2^{(k)} & \cdots & a_k^{(k)} & f^{(k)}
\end{vmatrix}.
\]

**Definition 1.6.** If \( a \in \mathbb{C} \cup \{\infty\}, \) the quantity
\[
\delta(a; f) = 1 - \limsup_{r \to \infty} \frac{N(r, a; f)}{T(r, f)}
\]
\[
= \liminf_{r \to \infty} \frac{m(r, a; f)}{T(r, f)}
\]
is called the Nevanlinna deficiency of the value ‘\( a \)’.

From the second fundamental theorem it follows that the set of values of \( a \in \mathbb{C} \cup \{\infty\} \) for which \( \delta(a; f) > 0 \) is countable and \( \sum_{a \neq \infty} \delta(a; f) + \delta(\infty; f) \leq 2 \) (cf.\([3]\), p.43). If in particular, \( \sum_{a \neq \infty} \delta(a; f) + \delta(\infty; f) = 2 \), we say that \( f \) has the maximum deficiency sum.
Lakshminarasimhan [4] introduced the idea of the functions of L-bounded index. Later Lahiri and Bhattacharjee [6] worked on the entire functions of L-bounded index and of non uniform L-bounded index. Since the natural extension of a derivative is a differential polynomial, in this paper we prove our results for a special type of linear differential polynomials viz. the Wronskians. In the paper we establish some new results depending on the comparative growth properties of composite entire or meromorphic functions using $L^*$-order and Wronskians generated by one of the factors. We use the standard notations and definitions in the theory of entire and meromorphic functions which are available in [3] and [12].

2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

**Lemma 2.1.** [1] If $f$ be meromorphic and $g$ be entire then for all sufficiently large values of $r$,

$$ T(r, f \circ g) \leq \{1 + o(1)\} \frac{T(r, g)}{\log M(r, g)} T(M(r, g), f) . $$

**Lemma 2.2.** [8] Let $f$ and $g$ be two entire functions. Then for all $r > 0$,

$$ T(r, f \circ g) \geq \frac{1}{3} \log M \left\{ \frac{1}{8} M \left( \frac{r}{4}, g \right) + o(1), f \right\} . $$

**Lemma 2.3.** [5] Let $g$ be an entire function with $\lambda_g < \infty$ and $a_i (i = 1, 2, 3, \cdots, n; n \leq \infty)$ are entire functions satisfying $T(r, a_i) = o\{T(r, g)\}$. If

$$ \sum_{i=1}^{n} \delta(a_i, g) = 1 $$

then

$$ \lim_{r \to \infty} \frac{T(r, g)}{\log M(r, g)} = \frac{1}{\pi} . $$

**Lemma 2.4.** [7] Let $f$ be a transcendental meromorphic function having the maximum deficiency sum. Then

$$ \lim_{r \to \infty} \frac{T(r, L(f))}{T(r, f)} = 1 + k - k \delta(\infty; f) . $$

**Lemma 2.5.** [2] If $f$ be a transcendental meromorphic function having the maximum deficiency sum. Then the generalized $L^*$-order (generalized $L^*$-lower order) of $L(f)$ and that of $f$ are same.

3. Theorems

In this section we present the main results of the paper.

**Theorem 3.1.** Let $f$ be a transcendental meromorphic function having the maximum deficiency sum and $g$ be entire such that $0 < \rho_g^{L^*} < \lambda_f^{L^*} \leq \rho_f^{L^*} < \infty$. Then

$$ \lim_{r \to \infty} \frac{\log \{T(r, f \circ g) \log M(r, g)\}}{T(r, L(f)) \cdot K(r, g; L)} = 0 , $$

where $K(r, g; L) = \begin{cases} 1 & \text{if } L(M(r, g)) = o\left\{ r^{\alpha} e^{\alpha L(r)} \right\} \text{ as } r \to \infty \\ L(M(r, g)) & \text{otherwise} \end{cases}$ for some $\alpha < \lambda_f^{L^*}$.
Proof. In view of Lemma 2.1 we have for all sufficiently large values of $r$ that
\[ T(r, f \circ g) \log M(r, g) \leq (1 + o(1)) \left[ T(r, g) T(M(r, g), f) \right. \]
i.e., \[ \log \left\{ T(r, f \circ g) \log M(r, g) \right\} \leq \log \left\{ 1 + o(1) \right\} + \log T(r, g) \]
\[ \left. + \log T(M(r, g), f) \right\} \]
i.e., \[ \log \left\{ T(r, f \circ g) \log M(r, g) \right\} \leq o(1) + (\rho^L_g + \varepsilon) \log [r e^{L(r)}] \]
\[ + (\rho^L_f + \varepsilon) \left[ \log M(r, g) e^{L(M(r, g))} \right] \]
i.e., \[ \log \left\{ T(r, f \circ g) \log M(r, g) \right\} \leq o(1) + (\rho^L_g + \varepsilon) [\log r + L(r)] \]
\[ + (\rho^L_f + \varepsilon) \left[ \log M(r, g) + L(M(r, g)) \right] \]
i.e., \[ \log \left\{ T(r, f \circ g) \log M(r, g) \right\} \leq o(1) + (\rho^L_g + \varepsilon) [\log r + L(r)] \]
\[ + (\rho^L_f + \varepsilon) \left[ \{r e^{L(r)}\}^{\left(\rho^L_g + \varepsilon\right)} + L(M(r, g)) \right]. \quad (3.1) \]

Also in view of Lemma 2.5 we obtain for all sufficiently large values of $r$ that
\[ \log T(r, L(f)) \geq (\lambda^L_{L(f)} - \varepsilon) \log [r e^{L(r)}] \]
i.e., \[ \log T(r, L(f)) \geq (\lambda^L_f - \varepsilon) \log [r e^{L(r)}] \]
i.e., \[ T(r, L(f)) \geq [r e^{L(r)}]^{\left(\lambda^L_f - \varepsilon\right)}. \quad (3.2) \]

Now from (3.1) and (3.2) we get for all sufficiently large values of $r$ that
\[ \frac{\log \left\{ T(r, f \circ g) \log M(r, g) \right\}}{T(r, L(f))} \leq \frac{o(1) + (\rho^L_g + \varepsilon) [\log r + L(r)]}{T(r, L(f))} \]
\[ + \frac{(\rho^L_f + \varepsilon) \left[ \{r e^{L(r)}\}^{\left(\rho^L_g + \varepsilon\right)} + L(M(r, g)) \right]}{\{r e^{L(r)}\}^{\left(\lambda^L_f - \varepsilon\right)}}. \quad (3.3) \]

Since $\rho^L_g < \lambda^L_f$, we can choose $\varepsilon (> 0)$ in such a way that
\[ \rho^L_g + \varepsilon < \lambda^L_f - \varepsilon. \quad (3.4) \]

**Case I.** Let $L(M(r, g)) = o\left\{ r^\alpha e^{\alpha L(r)} \right\}$ as $r \to \infty$ and for some $\alpha < \lambda^L_f$.

As $\alpha < \lambda^L_f$ we can choose $\varepsilon (> 0)$ in such a way that
\[ \alpha < \lambda^L_f - \varepsilon. \quad (3.5) \]

Since $L(M(r, g)) = o\left\{ r^\alpha e^{\alpha L(r)} \right\}$ as $r \to \infty$ we get on using (3.5) that
\[ \frac{L(M(r, g))}{r^\alpha e^{\alpha L(r)}} \to 0 \text{ as } r \to \infty \]
i.e., \[ \frac{L(M(r, g))}{[r e^{L(r)}]^{\left(\lambda^L_f - \varepsilon\right)}} \to 0 \text{ as } r \to \infty. \quad (3.6) \]
Now in view of (3.3), (3.4) and (3.6) we get that
\[
\lim_{r \to \infty} \frac{\log \{ T(r, f \circ g) \log M(r, g) \}}{T(r, L(f))} = 0 . \tag{3.7}
\]

Case II. If \( L(M(r, g)) \neq o \{ r^\alpha e^{\alpha L(r)} \} \) as \( r \to \infty \) and for some \( \alpha < \lambda_f^L \) then we get from (3.3) that for a sequence of values of \( r \) tending to infinity,
\[
\log \{ T(r, f \circ g) \log M(r, g) \} \leq o(1) + \left( p_g^{L^*} + \varepsilon \right) \left( \log \{ r e^{L(r)} \} \right) \frac{\log L(M(r, g))}{\log \left( \lambda_f^{L^*} \right)}
+ \left( p_f^{L^*} + \varepsilon \right) \left( \log \{ r e^{L(r)} \} \right) \frac{\log L(M(r, g))}{\log \left( \lambda_f^{L^*} \right)}
+ \frac{1}{\log \left( \lambda_f^{L^*} \right)} L(M(r, g)). \tag{3.8}
\]

Now using (3.4) it follows from (3.8) that
\[
\lim_{r \to \infty} \frac{\log \{ T(r, f \circ g) \log M(r, g) \}}{T(r, L(f)) \log M(r, g)} = 0 . \tag{3.9}
\]

Combining (3.7) and (3.9) we obtain that
\[
\lim_{r \to \infty} \frac{\log \{ T(r, f \circ g) \log M(r, g) \}}{T(r, L(f)) \cdot K(r, g; L)} = 0 ,
\]
where \( K(r, g; L) = \begin{cases} 1 & \text{if } L(M(r, g)) = o \{ r^\alpha e^{\alpha L(r)} \} \text{ as } r \to \infty \\ L(M(r, g)) & \text{otherwise.} \end{cases} \)

Thus the theorem is established. \( \Box \)

**Theorem 3.2.** Suppose \( f \) be a transcendental meromorphic function with \( \sum \delta(a; f) + \delta(\infty; f) = 2 \) and \( g \) be entire such that \( 0 < p_g^{L^*} < p_f^{L^*} < \infty \). Then
\[
\liminf_{r \to \infty} \frac{\log \{ T(r, f \circ g) \log M(r, g) \}}{T(r, L(f)) \cdot K(r, g; L)} = 0 ,
\]
where \( K(r, g; L) = \begin{cases} 1 & \text{if } L(M(r, g)) = o \{ r^\alpha e^{\alpha L(r)} \} \text{ as } r \to \infty \\ L(M(r, g)) & \text{otherwise.} \end{cases} \)

The proof of Theorem 3.2 is omitted because it can be carried out in the line of Theorem 3.1.

**Theorem 3.3.** Let \( f \) be meromorphic with \( \lambda_f^{[m]L^*} < \infty \) where \( m \geq 1 \) and \( g \) be transcendental entire with finite lower order and \( \sum \delta(a; g) + \delta(\infty; g) = 2 \). Also let there exists entire functions \( a_i \ (i = 1, 2, 3, \cdots, n; n \leq \infty) \) such that \( T(r, a_i) = \)}
$o\{T(r,g)\}$ and $\sum_{i=1}^{n}\delta(a_i,g) = 1$. If

$L(M(r,g)) = o\{r^{\alpha}e^{\alpha L(r)}\}$ as $r \to \infty$ and for some $\alpha$ with $0 < \alpha < \lambda_g^{L^*}$ then

$$\liminf_{r \to \infty} \frac{\log^{[m]} T(r, f \circ g)}{T(r, L(g))} \leq \frac{\pi \lambda_f^{[m]L^*}}{(1 + k - k\delta(\infty; g))},$$

otherwise

$$\liminf_{r \to \infty} \frac{\log^{[m]} T(r, f \circ g)}{T(r, L(g)) \cdot L(M(r,g))} = 0.$$ 

Proof. In view of the inequality $T(r,g) \leq \log^+ M(r,g)$ and by Lemma 2.1 we get for a sequence of values of $r$ tending to infinity that

$$T(r, f \circ g) \leq \{1 + o(1)\} T(M(r,g), f)$$
i.e., $\log T(r, f \circ g) \leq \log \{1 + o(1)\} + \log T(M(r,g), f)$
i.e., $\log^{[m]} T(r, f \circ g) \leq \left(\lambda_f^{[m]L^*} - \varepsilon\right) (\log M(r,g) + L(M(r,g))) + O(1)$
i.e., $\frac{\log^{[m]} T(r, f \circ g)}{T(r, L(g))} \leq \frac{\left(\lambda_f^{[m]L^*} - \varepsilon\right) (\log M(r,g) + L(M(r,g))) + O(1)}{T(r, L(g))}$
i.e.,

$$\frac{\log^{[m]} T(r, f \circ g)}{T(r, L(g))} \leq \left(\lambda_f^{[m]L^*} - \varepsilon\right) \frac{\log M(r,g) + L(M(r,g))}{T(r, L(g))} + O(1). \quad (3.10)$$

Case I. Let $L(M(r,g)) = o\{r^{\alpha}e^{\alpha L(r)}\}$ as $r \to \infty$ and for some $\alpha$ with $0 < \alpha < \lambda_g^{L^*}$. Since $\alpha < \lambda_g^{L^*}$, we can choose $\varepsilon(>0)$ in such a way that

$$\alpha < \lambda_g^{L^*} - \varepsilon. \quad (3.11)$$

As $L(M(r,g)) = o\{r^{\alpha}e^{\alpha L(r)}\}$ as $r \to \infty$ we get in view of (3.11) that

$$\lim_{r \to \infty} \frac{L(M(r,g))}{[re^{L(r)}]^{\lambda_g^{L^*} - \varepsilon}} = 0. \quad (3.12)$$

Again in view of Lemma 2.5 we obtain for all sufficiently large values of $r$,

$$\log T(r, L(g)) \geq \left(\lambda_{E(g)}^{L^*} - \varepsilon\right) \log \{re^{L(r)}\}$$
i.e., $\log T(r, L(g)) \geq \left(\lambda_{E(g)}^{L^*} - \varepsilon\right) \log \{re^{L(r)}\}$
i.e., $T(r, L(g)) \geq \left[re^{L(r)}\right]^{\lambda_{E(g)}^{L^*} - \varepsilon}. \quad (3.13)$

Now from (3.10) and (3.13) we obtain for a sequence of values of $r$ tending to infinity that

$$\frac{\log^{[m]} T(r, f \circ g)}{T(r, L(g))} \leq \left(\lambda_f^{[m]L^*} - \varepsilon\right) \left[\frac{\log M(r,g)}{T(r, L(g))} + \frac{L(M(r,g))}{[re^{L(r)}]^{\lambda_g^{L^*} - \varepsilon}}\right] + O(1)$$
i.e.,

$$\frac{\log^{[m]} T(r, f \circ g)}{T(r, L(g))} \leq \left(\lambda_f^{[m]L^*} - \varepsilon\right) \left[\frac{\log M(r,g)}{T(r, L(g))} + \frac{L(M(r,g))}{[re^{L(r)}]^{\lambda_g^{L^*} - \varepsilon}}\right] + O(1).$$
\[
\leq \left( \lambda_j^{[m]L^*} - \varepsilon \right) \left[ \frac{\log M(r,g)}{T(r,g)} \cdot \frac{T(r,g)}{T(r,L(g))} + \frac{L(M(r,g))}{[r^{eL(r)}]^{\lambda_j^{[m]L^*} - \varepsilon}} \right] + O(1). \tag{3.14}
\]

Now combining (3.12) and (3.14) and in view of Lemma 2.3 and Lemma 2.4 it follows that
\[
\liminf_{r \to \infty} \frac{\log^{[m]} T(r,f \circ g)}{T(r,L(g)) L(M(r,g))} \leq \frac{\pi \lambda_j^{[m]L^*}}{(1 + k - k\delta(\infty;g))}. \tag{3.15}
\]

**Case II.** If \( L(M(r,g)) \neq o \{r^\alpha e^{\lambda(L(r))} \} \) as \( r \to \infty \) and for some \( \alpha \) with \( 0 < \alpha < \lambda_g^{L^*} \), then from (3.10) we get for a sequence of values of \( r \) tending to infinity that
\[
\frac{\log^{[m]} T(r,f \circ g)}{T(r,L(g)) L(M(r,g))} \leq \left( \lambda_j^{[m]L^*} - \varepsilon \right) \cdot \frac{\log M(r,g)}{T(r,L(g)) L(M(r,g))} + \{1 + O(1)\}.
\]

i.e.,
\[
\liminf_{r \to \infty} \frac{\log^{[m]} T(r,f \circ g)}{T(r,L(g)) L(M(r,g))} = 0.
\]

Thus combining Case I and Case II the theorem follows. \hfill \Box

In the line of Theorem 3.3 the following theorem can be proved:

**Theorem 3.4.** Let \( f \) be meromorphic with \( \rho_f^{[m]L^*} < \infty \) and \( g \) be transcendental entire with finite lower order and \( \sum_{a \neq \infty} \delta(a,g) + \delta(\infty;g) = 2 \). Also let there exist entire functions \( a_i \) \( (i = 1, 2, 3, \cdots, n; n \leq \infty) \) such that \( T(r,a_i) = o \{ T(r,g) \} \) and \( \sum_{i=1}^{n} \delta(a_i,g) = 1 \). If
\[
L(M(r,g)) = o \{r^\alpha e^{\lambda(L(r))} \} \text{ as } r \to \infty \text{ and for some } \alpha \text{ with } 0 < \alpha < \lambda_g^{L^*} \text{ then}
\]
\[
\limsup_{r \to \infty} \frac{\log^{[m]} T(r,f \circ g)}{T(r,L(g)) L(M(r,g))} \leq \frac{\pi \rho_f^{[m]L^*}}{(1 + k - k\delta(\infty;g))},
\]

otherwise
\[
\limsup_{r \to \infty} \frac{\log^{[m]} T(r,f \circ g)}{T(r,L(g)) L(M(r,g))} = 0.
\]

**Theorem 3.5.** Let \( f \) be meromorphic and \( g \) be transcendental entire with \( \rho_f^{[m]L^*} < \infty \), \( 0 < \lambda_g^{L^*} \leq \rho_f^{L^*} < \infty \) and \( \sum_{a \neq \infty} \delta(a,g) + \delta(\infty;g) = 2 \) where \( m \) is any positive integer. Then
(a) If \( L(M(r,g)) = o \{\log T(r,L(g))\} \) then
\[
\limsup_{r \to \infty} \frac{\log^{[m+1]} T(r,f \circ g)}{\log T(r,L(g)) + L(M(r,g))} \leq \frac{\rho_f^{L^*}}{\lambda_g^{L^*}},
\]
and (b) if \( T(r,L(g)) = o \{L(M(r,g))\} \) then
\[
\lim_{r \to \infty} \frac{\log^{[m+1]} T(r,f \circ g)}{\log T(r,L(g)) + L(M(r,g))} = 0.
\]
Proof. For all sufficiently large values of \( r \), we obtain in view of \( T (r, g) \leq \log^+ M (r, g) \), by Lemma 2.1 and also using \( \log \left\{ 1 + \frac{L (M (r, g))}{\log M (r, g)} \right\} \sim \frac{L (M (r, g))}{\log M (r, g)} \)

\[
T (r, f \circ g) \leq \{1 + o (1)\} T (M (r, g), f)
\]

i.e., \( \log T (r, f \circ g) \leq \log \{1 + o (1)\} + \log T (M (r, g), f) \)

i.e., \( \log^{[m]} T (r, f \circ g) \leq o (1) + \log^{[m]} T (M (r, g), f) \)

i.e., \( \log^{[m]} T (r, f \circ g) \leq o (1) + \left( \rho_f^{[m]L^*} + \varepsilon \right) \{ \log M (r, g) + L (M (r, g)) \} \) \( (3.16) \)

i.e., \( \log^{[m]} T (r, f \circ g) \leq o (1) + \left( \rho_f^{[m]L^*} + \varepsilon \right) \log M (r, g) \)

\[
\log \left\{ 1 + \frac{L (M (r, g))}{\log M (r, g)} \right\}
\]

i.e., \( \log^{[m+1]} T (r, f \circ g) \leq o (1) + \log \left( \rho_f^{[m]L^*} + \varepsilon \right) + \log^{[2]} M (r, g) \)

\[
+ \log \left\{ 1 + \frac{L (M (r, g))}{\log M (r, g)} \right\}
\]

i.e., \( \log^{[m+1]} T (r, f \circ g) \leq o (1) + \log \left( \rho_f^{[m]L^*} + \varepsilon \right) + \log \log \left( \rho_f^{[m]L^*} + \varepsilon \right) \log \left\{ \log T (r, L (g)) \right\} \)

\[
+ \log \left\{ 1 + \frac{L (M (r, g))}{\log M (r, g)} \right\}
\]

i.e., \( \log^{[m+1]} T (r, f \circ g) \leq o (1) + \left( \rho_f^{[m]L^*} + \varepsilon \right) \{ \log r + L (r) \} + \frac{L (M (r, g))}{\log M (r, g)} \) \( (3.17) \)

Again in view of Lemma 2.5 we get from the definition of \( L^* \)-lower order for all sufficiently large values of \( r \) that

\[
\log T (r, L (g)) \geq \left( \lambda_{L (g)}^{L^*} - \varepsilon \right) \log \rho_f^{L^*} \]

i.e., \( \log T (r, L (g)) \geq \left( \lambda_{L (g)}^{L^*} - \varepsilon \right) \log \rho_f^{L^*} \]

i.e., \( \log T (r, L (g)) \geq \left( \lambda_{L (g)}^{L^*} - \varepsilon \right) \left[ \log r + L (r) \right] \)

i.e., \( \log r + L (r) \leq \frac{\log T (r, L (g))}{\lambda_{L (g)}^{L^*} - \varepsilon} \) \( (3.18) \)

Hence from (3.17) and (3.18) it follows for all sufficiently large values of \( r \) that

\[
\log^{[m+1]} T (r, f \circ g) \leq o (1) + \left( \frac{\rho_f^{L^*} + \varepsilon}{\lambda_{L (g)}^{L^*} - \varepsilon} \right) \log T (r, L (g)) + \frac{L (M (r, g))}{\log M (r, g)} \]

i.e.,

\[
\frac{\log^{[m+1]} T (r, f \circ g)}{\log T (r, L (g)) + L (M (r, g))} \leq o (1) + \left( \frac{\rho_f^{L^*} + \varepsilon}{\lambda_{L (g)}^{L^*} - \varepsilon} \right) \cdot \frac{\log T (r, L (g))}{\log T (r, L (g)) + L (M (r, g))} \]

\[
+ \frac{L (M (r, g))}{\log T (r, L (g)) + L (M (r, g))} \log M (r, g)
\]
GROWTH RATES OF WRONSKIANS ....... 67

\[ i.e, \quad \frac{\log^{[m+1]} T(r, f \circ g)}{\log T(r, L(g)) + L(M(r, g))} \leq o(1) + \frac{\left( \frac{\rho_g^{L^*} + \varepsilon}{\lambda_g^{L^*} - \varepsilon} \right)}{1 + \frac{L(M(r, g))}{\log T(r, L(g))}} \]

\[ + \frac{1}{1 + \frac{\log T(r, L(g))}{L(M(r, g))}} \log M(r, g). \]  \hspace{1cm} (3.19)

Since \( L(M(r, g)) = o\{\log T(r, L(g))\} \) as \( r \to \infty \) and \( \varepsilon (> 0) \) is arbitrary we obtain from (3.19) that

\[ \limsup_{r \to \infty} \frac{\log^{[m+1]} T(r, f \circ g)}{\log T(r, L(g)) + L(M(r, g))} \leq \frac{\rho_g^{L^*}}{\lambda_g^{L^*}}. \]  \hspace{1cm} (3.20)

Again if \( \log T(r, g) = o\{L(M(r, g))\} \) then from (3.19) we get that

\[ \lim_{r \to \infty} \frac{\log^{[m+1]} T(r, f \circ g)}{\log T(r, L(g)) + L(M(r, g))} = 0. \]  \hspace{1cm} (3.21)

Thus from (3.20) and (3.21) the theorem is established. \( \Box \)

**Corollary 3.6.** Let \( f \) be meromorphic and \( g \) be transcendental entire with \( \rho_f^{[m]L^*} < \infty, 0 < \rho_g^{L^*} < \infty \) and \( \sum_{a \neq \infty} \delta(a; g) + \delta(\infty; g) = 2 \) where \( m \geq 1 \). Then

(a) If \( L(M(r, g)) = o\{\log T(r, L(g))\} \) then

\[ \liminf_{r \to \infty} \frac{\log^{[m+1]} T(r, f \circ g)}{T(r, L(g)) + L(M(r, g))} \leq 1 \]

and (b) if \( T(r, L(g)) = o\{L(M(r, g))\} \) then

\[ \liminf_{r \to \infty} \frac{\log^{[m+1]} T(r, f \circ g)}{T(r, L(g)) + L(M(r, g))} = 0. \]

We omit the proof of Corollary 3.6 because it can be carried out in the line of Theorem 3.5.

**Remark 3.7.** The equality sign in Theorem 3.5 and Corollary 3.6 cannot be removed as we see in the following example:

**Example 3.8.** Let \( f = g = \exp z, m = 1 \) and \( L(r) = \frac{1}{p} \exp \left( \frac{1}{r} \right) \) where \( p \) is any positive real number. Then

\[ \rho_f^{L^*} = \lambda_g^{L^*} = \rho_g^{L^*} = 1 \]  \hspace{1cm} and \( \sum_{a \neq \infty} \delta(a; g) + \delta(\infty; g) = 2. \]

Now taking \( a_1 = 1, a_2 = \ldots a_k = 0 \) we get that

\[ L(g) = \begin{vmatrix} a_1 & g \\ a'_1 & g' \end{vmatrix} = \begin{vmatrix} 1 & \exp z \\ 0 & \exp z \end{vmatrix} = \exp z. \]

Also

\[ T(r, f \circ g) \sim \frac{\exp r}{(2\pi^3r)^3} (r \to \infty), \quad T(r, g) = \frac{r}{\pi} \quad \text{and} \quad M(r, g) = \exp r. \]
So
\[ L(M(r, g)) = L(\exp r) = \frac{1}{p} \exp \left( \frac{1}{\exp r} \right). \]

Hence
\[
\liminf_{r \to \infty} \frac{\log^2 T(r, f \circ g)}{\log T(r, L(g)) + L(M(r, g))} = \frac{\log^2 T(r, f \circ g)}{\log T(r, L(g)) + L(M(r, g))} = \limsup_{r \to \infty} \frac{\log [r - \frac{1}{2} \log r + O(1)]}{\log r + O(1) + \frac{1}{p} \exp \left( \frac{1}{\exp r} \right)} = 1.
\]

**Theorem 3.9.** Let \( f \) be transcendental entire and \( g \) be an entire function with \( 0 < \lambda^*[m]^{L^*} \leq \rho^*[m]^{L^*} < \infty \) where \( m \) is any positive integer, \( 0 < \lambda^*_g \leq \rho^*_g < \infty \) and \( \sum_{a \neq \infty} \delta(a; f) + \delta(\infty; f) = 2 \). Then
\[
\limsup_{r \to \infty} \frac{\log^{[m+1]} T(r, f \circ g)}{\log^{[m]} T(r, L(f)) + L \left( \frac{1}{8} M \left( \frac{r}{4}, g \right) \right)} \geq \frac{\rho^*_g}{\rho^*[m]^{L^*}}.
\]

**Proof.** In view of Lemma 2.2, we have for all sufficiently large values of \( r \),
\[
T(r, f \circ g) \geq \frac{1}{3} \log M \left\{ \frac{1}{8} M \left( \frac{r}{4}, g \right) + o(1), f \right\}
\]
\[
\text{i.e., } \log^{[m]} T(r, f \circ g) \geq o(1) + \log^{[m+1]} M \left\{ \frac{1}{8} M \left( \frac{r}{4}, g \right) + o(1), f \right\} \quad (3.22)
\]
\[
\text{i.e., } \log^{[m]} T(r, f \circ g) \geq o(1) + \left( \lambda^*[m]^{L^*} - \varepsilon \right) \left[ \log \left\{ \frac{1}{8} M \left( \frac{r}{4}, g \right) + o(1) \right\} + L \left( \frac{1}{8} M \left( \frac{r}{4}, g \right) \right) \right]
\]
\[
\text{i.e., } \log^{[m]} T(r, f \circ g) \geq o(1) + \left( \lambda^*[m]^{L^*} - \varepsilon \right) \left[ \log \left\{ \frac{1}{8} M \left( \frac{r}{4}, g \right) \left( 1 + \frac{o(1)}{\frac{1}{8} M \left( \frac{r}{4}, g \right)} \right) \right\} + L \left( \frac{1}{8} M \left( \frac{r}{4}, g \right) \right) \right]
\]
\[
\text{i.e., } \log^{[m]} T(r, f \circ g) \geq \left( \lambda^*[m]^{L^*} - \varepsilon \right) \log M \left( \frac{r}{4}, g \right) \cdot \frac{\log M \left( \frac{r}{4}, g \right) + \log \left( 1 + \frac{o(1)}{\frac{1}{8} M \left( \frac{r}{4}, g \right)} \right) + L \left( \frac{1}{8} M \left( \frac{r}{4}, g \right) \right)}{\log M \left( \frac{r}{4}, g \right)}
\]
i.e., \( \log^{[m+1]} T(r, f \circ g) \geq \log^{[2]} M \left( \frac{T}{4}, g \right) \) \( + \left( \frac{\lambda^*_g}{\rho^*_f} + \varepsilon \right) L \left( \frac{1}{8} M \left( \frac{T}{4}, g \right) \right) \) 

\[- \log \left[ \exp \left\{ \left( \frac{\lambda^*_g}{\rho^*_f} + \varepsilon \right) L \left( \frac{1}{8} M \left( \frac{T}{4}, g \right) \right) \right\} \right] \]

\[+ \log \left\{ \frac{\left[ \log M \left( \frac{T}{4}, g \right) + L \left( \frac{1}{8} M \left( \frac{T}{4}, g \right) \right) \right]}{\log M \left( \frac{T}{4}, g \right)} \right\} \]

\( i.e., \log^{[m+1]} T(r, f \circ g) \geq \log^{[2]} M \left( \frac{T}{4}, g \right) \) \( + \left( \frac{\lambda^*_g}{\rho^*_f} + \varepsilon \right) L \left( \frac{1}{8} M \left( \frac{T}{4}, g \right) \right) \) 

\[+ \log \left\{ \frac{\left[ \log M \left( \frac{T}{4}, g \right) + L \left( \frac{1}{8} M \left( \frac{T}{4}, g \right) \right) \right]}{\log M \left( \frac{T}{4}, g \right)} \right\} \]

\( i.e., \log^{[m+1]} T(r, f \circ g) \geq \log^{[2]} M \left( \frac{T}{4}, g \right) \) \( + \left( \frac{\lambda^*_g}{\rho^*_f} + \varepsilon \right) L \left( \frac{1}{8} M \left( \frac{T}{4}, g \right) \right) \) \( + \log \left\{ \frac{\left[ \log M \left( \frac{T}{4}, g \right) + L \left( \frac{1}{8} M \left( \frac{T}{4}, g \right) \right) \right]}{\log M \left( \frac{T}{4}, g \right)} \right\} \]

(3.24) \( \frac{T}{4} e^{L(\xi)} \)

(3.25) \( \log^{[m]} T(r, L(f)) \) \( \leq \left( \frac{\rho^{[m]} e^{L(r)}}{\rho^*_f + \varepsilon} \right) \log \left\{ \frac{T}{4} e^{L(\xi)} \right\} \)

(3.26) \( \log^{[m]} T(r, L(f)) \) \( \leq \left( \frac{\rho^{[m]} e^{L(r)}}{\rho^*_f + \varepsilon} \right) \log \left\{ \frac{T}{4} e^{L(\xi)} \right\} \)

In view of Lemma 2.5, we get for all sufficiently large values of \( r \) that

\( \log^{[m]} T(r, L(f)) \) \( \leq \left( \frac{\rho^{[m]} e^{L(r)}}{\rho^*_f + \varepsilon} \right) \log \left\{ \frac{T}{4} e^{L(\xi)} \right\} \)

(3.27) \( \log^{[m]} T(r, L(f)) \) \( \leq \left( \frac{\rho^{[m]} e^{L(r)}}{\rho^*_f + \varepsilon} \right) \log \left\{ \frac{T}{4} e^{L(\xi)} \right\} \)

Hence from (3.24) and (3.25) it follows for all sufficiently large values of \( r \) that

\( i.e., \log^{[m+1]} T(r, f \circ g) \geq \left( \frac{\rho^{[m]} e^{L(r)}}{\rho^*_f + \varepsilon} \right) \left( \log^{[m]} T(r, L(f)) - \log 4 \right) \)

\[+ \left( \frac{\rho^{[m]} e^{L(r)}}{\rho^*_f + \varepsilon} \right) L \left( \frac{1}{8} M \left( \frac{T}{4}, g \right) \right) \]
i.e., \( \log^{[m+1]} T(r, f \circ g) \geq \left( \frac{\rho_g^{L^*} - \epsilon}{\rho_f^{[m]L^*} + \epsilon} \right) \left[ \log^{[m]} T(r, L(f)) + L \left( \frac{1}{8} M \left( \frac{r}{4}, g \right) \right) \right] - \left( \frac{\rho_g^{L^*} - \epsilon}{\rho_f^{[m]L^*} + \epsilon} \right) \log 4 \)

\[
= \frac{\log^{[m+1]} T(r, f \circ g)}{\log^{[m]} T(r, L(f)) + L \left( \frac{1}{8} M \left( \frac{r}{4}, g \right) \right)} \geq \left( \frac{\rho_g^{L^*} - \epsilon}{\rho_f^{[m]L^*} + \epsilon} \right) - \left( \frac{\rho_g^{L^*} - \epsilon}{\rho_f^{[m]L^*} + \epsilon} \right) \log 4.
\]

(3.26)

Since \( \epsilon (> 0) \) is arbitrary, it follows from (3.26) that

\[
\limsup_{r \to \infty} \frac{\log^{[m+1]} T(r, f \circ g)}{\log^{[m]} T(r, L(f)) + L \left( \frac{1}{8} M \left( \frac{r}{4}, g \right) \right)} \geq \frac{\rho_g^{L^*}}{\rho_f^{[m]L^*}}.
\]

This proves the theorem. \( \square \)

In the line of Theorem 3.9 the following theorem may be proved:

**Theorem 3.10.** Let \( f \) a transcendental entire function and \( g \) be an entire function such that \( 0 < \lambda_f^{[m]L^*} \leq \rho_f^{[m]L^*} < \infty \) where \( m \geq 1, 0 < \lambda_g^{L^*} \leq \rho_g^{L^*} < \infty \) and \( \sum_{a \neq \infty} \delta (a; f) + \delta (\infty; f) = 2 \). Then

\[
\liminf_{r \to \infty} \frac{\log^{[m+1]} T(r, f \circ g)}{\log^{[m]} T(r, L(f)) + L \left( \frac{1}{8} M \left( \frac{r}{4}, g \right) \right)} \geq \frac{\lambda_g^{L^*}}{\rho_f^{[m]L^*}}.
\]

**Theorem 3.11.** Let \( f \) a transcendental entire function and \( g \) be an entire function such that \( 0 < \lambda_f^{[m]L^*} \leq \rho_f^{[m]L^*} < \infty \) where \( m \geq 1, 0 < \lambda_g^{L^*} \leq \rho_g^{L^*} < \infty \) and \( \sum_{a \neq \infty} \delta (a; f) + \delta (\infty; f) = 2 \). Then for every constant \( A \) and any real number \( x \),

\[
\lim_{r \to \infty} \frac{\log^{[m]} T(r, f \circ g)}{\log^{[m]} T(r^A, L(f))} = \infty.
\]

*Proof.* If \( x \) is such that \( 1 + x \leq 0 \), then the theorem is obvious. So we suppose that \( 1 + x > 0 \).

Now from (3.22) we have for all sufficiently large values of \( r \) that

\[
\log^{[m]} T(r, f \circ g) \geq o(1) + \left( \lambda_f^{[m]L^*} - \epsilon \right) \left[ \log \left( \frac{1}{8} M \left( \frac{r}{4}, g \right) \right) + o(1) \right] + L \left( \frac{1}{8} M \left( \frac{r}{4}, g \right) \right)
\]

\[
+ L \left( \frac{1}{8} M \left( \frac{r}{4}, g \right) \right)
\]
i.e., \( \log^{[m]} T(r, f \circ g) \geq o(1) + \left( \lambda_f^{[m]L^*} - \varepsilon \right) \left[ \log M \left( \frac{r}{4}, g \right) + o(1) \right. \\
+ L \left( \frac{1}{8} M \left( \frac{r}{4}, g \right) \right) \]

where we choose \( 0 < \varepsilon < \min \left\{ \lambda_f^{[m]L^*}, \lambda_g^{L^*} \right\} \).

Also for all sufficiently large values of \( r \) we get from Lemma 2.5 that

\[
\log^{[m]} T\left( r^A, L(f) \right) \leq \left( \rho_f^{[m]L^*} + \varepsilon \right) \log \left\{ r^A e^{L(r^A)} \right\} 
\]
i.e., \( \log^{[m]} T\left( r^A, L(f) \right) \leq \left( \rho_f^{[m]L^*} + \varepsilon \right) \log \left\{ r^A e^{L(r^A)} \right\} \)
i.e., \( \left\{ \log^{[m]} T\left( r^A, L(f) \right) \right\}^{1+x} \leq \left( \rho_f^{[m]L^*} + \varepsilon \right)^{1+x} \left( \log \left\{ r^A e^{L(r^A)} \right\} \right)^{1+x} \quad (3.28) \)

Therefore from (3.27) and (3.28) it follows for all sufficiently large values of \( r \) that

\[
\frac{\log^{[m]} T(r, f \circ g)}{\left\{ \log^{[m]} T\left( r^A, L(f) \right) \right\}^{1+x}} \geq \frac{o(1) + \left( \lambda_f^{[m]L^*} - \varepsilon \right) \left\{ \left( \frac{r}{4} \right) e^{L(r)} \right\} \lambda_g^{L^* - \varepsilon} + o(1) + L \left( \frac{1}{8} M \left( \frac{r}{4}, g \right) \right)}{\left( \rho_f^{[m]L^*} + \varepsilon \right)^{1+x} \left( \log \left\{ r^A e^{L(r^A)} \right\} \right)^{1+x}} \quad (3.29)
\]

Thus from (3.29) the theorem follows. \( \square \)

**Theorem 3.12.** Let \( f \) an entire function and \( g \) be a transcendental entire function such that \( 0 < \lambda_f^{[m]L^*} \leq \rho_f^{[m]L^*} < \infty \) where \( m \geq 1 \), \( 0 < \lambda_g^{L^*} \leq \rho_g^{L^*} < \infty \) and \( \sum \delta(a; f) + \delta(\infty; f) = 2 \). Then for every constant \( A \) and any real number \( x \),

\[
\lim_{r \to \infty} \frac{\log^{[m]} T(r, f \circ g)}{\left\{ \log T\left( r^A, L(g) \right) \right\}^{1+x}} = \infty.
\]

The proof of Theorem 3.12 is omitted as it can be carried out in the line of Theorem 3.11.

**Theorem 3.13.** Let \( f \) be transcendental meromorphic and \( g \) be entire satisfying the following conditions (i) \( \rho_f^{[m]L^*} \) and \( \rho_g^{L^*} \) are both finite, (ii) \( \rho_f^{[m]L^*} \) is positive and (iii) \( \sum \delta(a; f) + \delta(\infty; f) = 2 \). Then for each \( \alpha \in (-\infty, \infty) \),

\[
\liminf_{r \to \infty} \frac{\left\{ \log^{[m]} T(r, f \circ g) \right\}^{1+\alpha}}{\log^{[m]} T\left( \exp \left( r^A \right), L(f) \right)} = 0
\]
for $A > (1 + \alpha) \rho_g^{L^*}$ and $m \geq 1$.

**Proof.** If $1 + \alpha < 0$, then the theorem is trivial. So we take $1 + \alpha > 0$. Now from (3.16) we obtain for all sufficiently large values of $r$ that

$$
\log^m T(r, f \circ g) \leq o(1) + \left( r e^{L(r)} \right) \left( \rho_g^{L^*} + \varepsilon \right) \left( \rho_f \right) L(M(r, g))
$$

i.e.,

$$
\log^m T(r, f \circ g) \leq \left[ r e^{L(r)} \right] \left( \rho_g^{L^*} + \varepsilon \right) \left\{ \left( \rho_f^{L^*} + \varepsilon \right) + o(1) \right\} L \left( \rho_f \right)
$$

i.e.,

$$
\left\{ \log^m T(r, f \circ g) \right\}^{1+\alpha} \leq \left[ r e^{L(r)} \right] \left( \rho_g^{L^*} + \varepsilon \right) \left\{ \left( \rho_f^{L^*} + \varepsilon \right) + o(1) \right\} L \left( \rho_f \right)
$$

Again in view of Lemma 2.5 we have for a sequence of $r$ tending to infinity and for $\varepsilon(>0)$,

$$
\log^m T \left( \exp \left( r \right)^\beta, L(f) \right) \geq \left( \rho_f \right) \log \left[ \exp \left( r \right) \right] \log \left\{ \exp \left( r \right)^\beta \right\}.
$$

i.e.,

$$
\log^m T \left( \exp \left( r \right)^\beta, L(f) \right) \geq \left( \rho_f \right) L\left( \exp \left( r \right)^\beta \right) \left\{ \rho_f^{L^*} + \varepsilon \right\}
$$

Now let

$$
\left[ r e^{L(r)} \right] \left\{ \left( \rho_f^{L^*} + \varepsilon \right) \varepsilon + o(1) \right\} = k_1, \left( \rho_f^{L^*} + \varepsilon \right) L(M(r, g)) = k_2,
$$

Then from (3.30), (3.31) and above we get for a sequence of values of $r$ tending to infinity that

$$
\left\{ \log^m T \left( r, f \circ g \right) \right\}^{1+\alpha} \leq \frac{\left( r \rho_g^{L^*} + \varepsilon \right) k_1 + k_2}{k_3 r^A + k_4}
$$

i.e.,

$$
\left\{ \log^m T \left( r, f \circ g \right) \right\}^{1+\alpha} \leq \frac{\left( r \rho_g^{L^*} + \varepsilon \right) (1+\alpha) k_1 + k_2}{k_3 r^A + k_4}
$$

where $k_1, k_2, k_3$ and $k_4$ are all finite.

Since $(\rho_g^{L^*} + \varepsilon) (1 + \alpha) < A$, we obtain from above

$$
\liminf_{r \to \infty} \frac{\log^m T \left( r, f \circ g \right)}{\log^m T \left( \exp \left( r \right)^\beta, L(f) \right)} = 0
$$

where we choose $\varepsilon(>0)$ in such a way that

$$
0 < \varepsilon < \min \left\{ \rho_f^{L^*}, \frac{A}{1 + \alpha} - \rho_g^{L^*} \right\}.
$$

This proves the theorem. \(\square\)
In the line of Theorem 3.13 the following theorem may be proved:

**Theorem 3.14.** Let \( f \) be transcendental meromorphic and \( g \) be entire satisfying the following conditions (i) \( 0 < \lambda_{f}^{[m]L^{*}} \leq \rho_{f}^{[m]L^{*}} < \infty \), (ii) \( \rho_{g}^{L^{*}} \) is finite and (iii) \( \sum_{a \neq \infty} \delta(a; f) + \delta(\infty; f) = 2 \). Then for each \( \alpha \in (-\infty, \infty) \),

\[
\lim_{r \to \infty} \frac{\left\{ \log^{[m]} T(r, f \circ g) \right\}^{1+\alpha}}{\log^{[m]} T(\exp(rA), L(f))} = 0
\]

where \( A > (1 + \alpha) \rho_{g}^{L^{*}} \) and \( m \geq 1 \).

**Theorem 3.15.** Let \( f \) be meromorphic and \( g \) be transcendental entire such that \( 0 < \lambda_{g}^{L^{*}} \leq \rho_{g}^{L^{*}} < \infty \), \( \rho_{f}^{[m]L^{*}} < \infty \) where \( m \geq 1 \) and \( \sum_{a \neq \infty} \delta(a; g) + \delta(\infty; g) = 2 \). Then for each \( \alpha \in (-\infty, \infty) \),

\[
\lim_{r \to \infty} \frac{\left\{ \log^{[m]} T(r, f \circ g) \right\}^{1+\alpha}}{\log T(\exp(rA), L(g))} = 0 \text{ if } A > (1 + \alpha) \rho_{g}^{L^{*}}.
\]

**Theorem 3.16.** Let \( f \) be meromorphic and \( g \) be transcendental entire with \( \rho_{f}^{[m]L^{*}} < \infty \) for \( m \geq 1 \), \( 0 < \rho_{g}^{L^{*}} < \infty \) and \( \sum_{a \neq \infty} \delta(a; g) + \delta(\infty; g) = 2 \). Then for each \( \alpha \in (-\infty, \infty) \),

\[
\liminf_{r \to \infty} \frac{\left\{ \log^{[m]} T(r, f \circ g) \right\}^{1+\alpha}}{\log T(\exp(rA), L(g))} = 0 \text{ where } A > (1 + \alpha) \rho_{g}^{L^{*}}.
\]

The proof of Theorem 3.15 and Theorem 3.16 are omitted because those can be carried out in the line of Theorem 3.14 and Theorem 3.13 respectively.

**Acknowledgement.** The authors are thankful to the referee for his/her valuable suggestions towards the improvement of the paper.

**References**


1Department of Mathematics, University of Kalyani, Kalyani, Dist.- Nadia, PIN-741235, West Bengal, India.

E-mail address: sanjib.kr.datta@yahoo.co.in

2Rajbari, Rabindrapalli, R. N. Tagore Road, P.O. Krishnagar, P.S. Kotwali, Dist.- Nadia, PIN-741101, West Bengal, India,

E-mail address: tanmaybiswas_math@yahoo.com

E-mail address: tanmaybiswas_math@rediffmail.com

3Department of Mathematics, Kalna College, P.O. Kalna, Dist.- Burdwan, PIN-713409, West Bengal, India.

E-mail address: mailtosultanali@gmail.com