Gulf Journal of Mathematics
Vol 2 , Issue 1 (2014) 58-74

GROWTH RATES OF WRONSKIANS GENERATED BY COMPLEX VALUED FUNCTIONS

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Abstract. In the paper we establish some new results depending on the comparative growth properties of composite entire or meromorphic functions using generalized L^*-order of Wronskians generated by one of the factors.

1. Introduction, Definitions and Notations

Let f be an entire function defined in the open complex plane $\mathbb{C}$. The maximum modulus function $M(r,f)$ corresponding to f is defined on $|z|=r$ as follows:

$$M(r,f)=\max_{|z|=r}|f(z)| \ .$$

When f is meromorphic, $M(r,f)$ cannot be defined as f is not analytic throughout the complex plane. In this situation, one may introduce another function $T(r,f)$ known as Nevanlinna's characteristic function of f, playing the same role as $M(r,f)$.

The integrated counting function $N(r,a;f)$ $\left(\bar{N}(r,a;f)\right)$ of a-points (distinct a-points) of f is defined as

$$N(r,a;f)=\int_0^r \frac{n(t,a;f)-n(0,a;f)}{t}dt+n(0,a;f)\log r$$

$$\left(\bar{N}(r,a;f)=\int_0^r \frac{\bar{n}(t,a;f)-\bar{n}(r,a;f)}{t}dt+\bar{n}(0,a;f)\log r\right) \ ,$$

where we denote by $n(t,a;f)$ $\left(\bar{n}(t,a;f)\right)$ the number of a-points (distinct a-points) of f in $|z|\leq t$ and an ∞ -point is a pole of f .

In many occasions $N(r,\infty;f)$ and $\bar{N}(r,\infty;f)$ are denoted by $N(r,f)$ and $\bar{N}(r,f)$ respectively. The function $N(r,a;f)$ is called the enumerative function.

Date: Received: Jul 20, 2013; Accepted: Sep 16, 2013.

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2010 *Mathematics Subject Classification.* Primary 30D30; Secondary 30D35.

Key words and phrases. Transcendental entire and meromorphic function, composition, growth, generalized L^* -order, Wronskian, slowly changing function.

On the other hand, the function $m(r,f) \equiv m(r,\infty;f)$ known as the proximity function is defined as

$$m(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|f\left(re^{i\theta}\right)\right| d\theta,$$

$$\text{where } \log^+ x = \max(\log x, 0) \text{ for all } x \geqslant 0$$

and an ∞ -point is a pole of f .

Analogously, $m\left(r, \frac{1}{f-a}\right) \equiv m(r,a;f)$ is defined when a is not an ∞-point of f.

Thus the Nevanlinna's characteristic function $T(r,f)$ corresponding to f is defined as

$$T(r,f) = N(r,f) + m(r,f) \ .$$

When f is entire, $T(r,f)$ coincides with $m(r,f)$ as $N(r,f) = 0$.

For an entire function f, Sato [9] defined the generalized order $\rho_f^{[m]}$ and generalized lower order $\lambda_f^{[m]}$ as follows:

$$\rho_f^{[m]} = \limsup_{r\to\infty} \frac{\log^{[m+1]} M(r,f)}{\log r} \text{ and } \lambda_f^{[m]} = \liminf_{r\to\infty} \frac{\log^{[m+1]} M(r,f)}{\log r}$$

where $\log^{[k]} x = \log\left(\log^{[k-1]} x\right)$ for $k = 1,2,3,....$ and $\log^{[0]} x = x$.

When f is meromorphic, one can easily verify that

$$\rho_f^{[m]} = \limsup_{r\to\infty} \frac{\log^{[m]} T(r,f)}{\log r} \text{ and } \lambda_f^{[m]} = \liminf_{r\to\infty} \frac{\log^{[m]} T(r,f)}{\log r}.$$

Let $L = L(r)$ be a positive continuous function increasing slowly *i.e.,* $L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant a. *Singh and Barker* [10] defined it in the following way:

Definition 1.1. [10] A positive continuous function $L(r)$ is called a slowly changing function if for $\varepsilon\,(>0)$,

$$\frac{1}{k^\varepsilon} \leq \frac{L(kr)}{L(r)} \leq k^\varepsilon \ for\ r \geq r(\varepsilon) \text{ and}$$

uniformly for $k\,(\geq 1)$.

If further, $L(r)$ is differentiable, the above condition is equivalent to

$$\lim_{r\to\infty} \frac{rL'(r)}{L(r)} = 0 \ .$$

Somasundaram and Thamizharasi [11] introduced the notions of L-order for entire functions defined in the open complex plane $\mathbb{C}$. The more generalized concept for L-order for entire and meromorphic functions are L^*-order respectively. Their definitions are as follows:

Definition 1.2. [11] The L^*-order $\rho_f^{L^*}$ and the L^*-lower order $\lambda_f^{L^*}$ of an entire function f are defined as

$$\rho_f^{L^*} = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log\left[re^{L(r)}\right]} \text{ and } \lambda_f^{L^*} = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log\left[re^{L(r)}\right]},$$

When f is meromorphic, one can easily verify that

$$\rho_f^{L^*} = \limsup_{r\to\infty}\frac{\log T(r,f)}{\log\left[re^{L(r)}\right]} \text{ and } \lambda_f^{L^*} = \liminf_{r\to\infty}\frac{\log T(r,f)}{\log\left[re^{L(r)}\right]}.$$

Combining the concepts of Sato[9] and Somasundaram and Thamizharasi [11] we may state the following definition:

Definition 1.3. The generalized L^*-order $\rho_f^{[m]L^*}$ and the generalized L^*-lower order $\lambda_f^{[m]L^*}$ of an entire function f are defined as

$$\rho_f^{[m]L^*} = \limsup_{r\to\infty}\frac{\log^{[m+1]} M(r,f)}{\log\left[re^{L(r)}\right]} \text{ and } \lambda_f^{[m]L^*} = \liminf_{r\to\infty}\frac{\log^{[m+1]} M(r,f)}{\log\left[re^{L(r)}\right]},$$

where m is any positive integer.
When f is meromorphic, it can be easily verified that

$$\rho_f^{[m]L^*} = \limsup_{r\to\infty}\frac{\log^{[m]} T(r,f)}{\log\left[re^{L(r)}\right]} \text{ and } \lambda_f^{[m]L^*} = \liminf_{r\to\infty}\frac{\log^{[m]} T(r,f)}{\log\left[re^{L(r)}\right]}.$$

For $m=1$, Definition 1.3 reduces to Definition 1.2.

The following definitions are also well known.

Definition 1.4. A meromorphic function $a \equiv a(z)$ is called small with respect to f if $T(r,a) = S(r,f)$ where $S(r,f) = o\{T(r,f)\}$ i.e., $\frac{S(r,f)}{T(r,f)} \to 0$ as $r\to\infty$.

Definition 1.5. Let $a_1, a_2, ...a_k$ be linearly independent meromorphic functions and small with respect to f. We denote by $L(f) = W(a_1, a_2, ...a_k; f)$, the Wronskian determinant of $a_1, a_2, ..., a_k$, f i.e.,

$$L(f) = \begin{vmatrix} a_1 & a_2 & . & . & . & a_k & f \\ a_1' & a_2' & . & . & . & a_k' & f' \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ a_1^{(k)} & a_2^{(k)} & . & . & . & a_k^{(k)} & f^{(k)} \end{vmatrix}.$$

Definition 1.6. If $a \in \mathbb{C}\cup\{\infty\}$,the quantity

$$\begin{aligned}\delta(a;f) &= 1 - \limsup_{r\to\infty}\frac{N(r,a;f)}{T(r,f)} \\ &= \liminf_{r\to\infty}\frac{m(r,a;f)}{T(r,f)}\end{aligned}$$

is called the Nevanlinna deficiency of the value 'a'.

From the second fundamental theorem it follows that the set of values of $a \in \mathbb{C}\cup\{\infty\}$ for which $\delta(a;f) > 0$ is countable and $\sum_{a\neq\infty}\delta(a;f) + \delta(\infty;f) \leq 2$ (cf.[[3], p.43]). If in particular, $\sum_{a\neq\infty}\delta(a;f) + \delta(\infty;f) = 2$, we say that f has the maximum deficiency sum.

Lakshminarasimhan [4] introduced the idea of the functions of L-bounded index. Later Lahiri and Bhattacharjee [6] worked on the entire functions of L-bounded index and of non uniform L-bounded index. Since the natural extension of a derivative is a differential polynomial, in this paper we prove our results for a special type of linear differential polynomials viz. the Wronskians. In the paper we establish some new results depending on the comparative growth properties of composite entire or meromorphic functions using L^*-order and Wronskians generated by one of the factors. We use the standard notations and definitions in the theory of entire and meromorphic functions which are available in [3] and [12].

2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 2.1. [1] *If f be meromorphic and g be entire then for all sufficiently large values of r,*

$$T(r, f\circ g) \leq \{1+o(1)\}\frac{T(r,g)}{\log M(r,g)}T(M(r,g),f) \ .$$

Lemma 2.2. [8] *Let f and g be two entire functions. Then for all $r>0$,*

$$T(r,f\circ g)\geq \frac{1}{3}\log M\left\{\frac{1}{8}M\left(\frac{r}{4},g\right)+o(1),f\right\} \ .$$

Lemma 2.3. [5] *Let g be an entire function with $\lambda_g<\infty$ and $a_i (i=1,\ 2,\ 3,\ \cdots,n;\ n\leq\infty)$ are entire functions satisfying $T(r,a_i)=o\{T(r,g)\}$. If $\sum\limits_{i=1}^{n}\delta(a_i,g)=1$ then $\lim\limits_{r\to\infty}\frac{T(r,g)}{\log M(r,g)}=\frac{1}{\pi}$.*

Lemma 2.4. [7] *Let f be a transcendental meromorphic function having the maximum deficiency sum . Then*

$$\lim_{r\to\infty}\frac{T(r,L(f))}{T(r,f)}=1+k-k\delta(\infty;f) \ .$$

Lemma 2.5. [2] *If f be a transcendental meromorphic function having the maximum deficiency sum. Then the generalized L^*-order (generalized L^*-lower order) of $L(f)$ and that of f are same.*

3. Theorems

In this section we present the main results of the paper.

Theorem 3.1. *Let f be a transcendental meromorphic function having the maximum deficiency sum and g be entire such that $0<\rho_g^{L^*}<\lambda_f^{L^*}\leq\rho_f^{L^*}<\infty$. Then*

$$\lim_{r\to\infty}\frac{\log\{T(r,f\circ g)\log M(r,g)\}}{T(r,L(f))\cdot K(r,g;L)}=0 \ ,$$

where $K(r,g;L)=\begin{cases} 1 \text{ if } L(M(r,g))=o\left\{r^{\alpha}e^{\alpha L(r)}\right\} \text{ as } r\to\infty \\ \qquad\qquad\qquad \text{and for some } \alpha<\lambda_f^{L^*} \\ L(M(r,g)) \text{ otherwise.}\end{cases}$

Proof. In view of Lemma 2.1 we have for all sufficiently large values of r that

$$T(r, f\circ g)\log M(r,g) \leq \{1+o(1)\}\, T(r,g)\, T(M(r,g),f)$$

$$i.e.,\ \log\{T(r,f\circ g)\log M(r,g)\} \leq \log\{1+o(1)\} + \log T(r,g) + \log T(M(r,g),f)$$

$$i.e.,\ \log\{T(r,f\circ g)\log M(r,g)\} \leq o(1) + \left(\rho_g^{L^*}+\varepsilon\right)\log\left[re^{L(r)}\right] + \left(\rho_f^{L^*}+\varepsilon\right)\left[\log M(r,g)\, e^{L(M(r,g))}\right]$$

$$i.e.,\ \log\{T(r,f\circ g)\log M(r,g)\} \leq o(1) + \left(\rho_g^{L^*}+\varepsilon\right)[\log r + L(r)] + \left(\rho_f^{L^*}+\varepsilon\right)[\log M(r,g) + L(M(r,g))]$$

$$i.e.,\ \log\{T(r,f\circ g)\log M(r,g)\} \leq o(1) + \left(\rho_g^{L^*}+\varepsilon\right)[\log r + L(r)] + \left(\rho_f^{L^*}+\varepsilon\right)\left[\left\{re^{L(r)}\right\}^{\left(\rho_g^{L^*}+\varepsilon\right)} + L(M(r,g))\right]. \tag{3.1}$$

Also in view of Lemma 2.5 we obtain for all sufficiently large values of r that

$$\log T(r,L(f)) \geq \left(\lambda_{L(f)}^{L^*}-\varepsilon\right)\log\left[re^{L(r)}\right]$$

$$i.e.,\ \log T(r,L(f)) \geq \left(\lambda_f^{L^*}-\varepsilon\right)\log\left[re^{L(r)}\right]$$

$$i.e.,\ T(r,L(f)) \geq \left[re^{L(r)}\right]^{\left(\lambda_f^{L^*}-\varepsilon\right)}. \tag{3.2}$$

Now from (3.1) and (3.2) we get for all sufficiently large values of r that

$$\frac{\log\{T(r,f\circ g)\log M(r,g)\}}{T(r,L(f))} \leq \frac{o(1)+\left(\rho_g^{L^*}+\varepsilon\right)[\log r + L(r)]}{T(r,L(f))} + \frac{\left(\rho_f^{L^*}+\varepsilon\right)\left[\left\{re^{L(r)}\right\}^{\left(\rho_g^{L^*}+\varepsilon\right)} + L(M(r,g))\right]}{\left\{re^{L(r)}\right\}^{\left(\lambda_f^{L^*}-\varepsilon\right)}}. \tag{3.3}$$

Since $\rho_g^{L^*} < \lambda_f^{L^*}$, we can choose $\varepsilon\,(>0)$ in such a way that

$$\rho_g^{L^*} + \varepsilon < \lambda_f^{L^*} - \varepsilon\ . \tag{3.4}$$

Case I. Let $L(M(r,g)) = o\left\{r^\alpha e^{\alpha L(r)}\right\}$ as $r\to\infty$ and for some $\alpha < \lambda_f^{L^*}$. As $\alpha < \lambda_f^{L^*}$ we can choose $\varepsilon\,(>0)$ in such a way that

$$\alpha < \lambda_f^{L^*} - \varepsilon\ . \tag{3.5}$$

Since $L(M(r,g)) = o\left\{r^\alpha e^{\alpha L(r)}\right\}$ as $r\to\infty$ we get on using (3.5) that

$$\frac{L(M(r,g))}{r^\alpha e^{\alpha L(r)}} \to 0 \text{ as } r\to\infty$$

$$i.e.,\ \frac{L(M(r,g))}{\left[re^{L(r)}\right]^{\left(\lambda_f^{L^*}-\varepsilon\right)}} \to 0 \text{ as } r\to\infty\ . \tag{3.6}$$

Now in view of (3.3), (3.4) and (3.6) we get that

$$\lim_{r\to\infty}\frac{\log\{T(r,f\circ g)\log M(r,g)\}}{T(r,L(f))}=0\ . \tag{3.7}$$

Case II. If $L(M(r,g))\neq o\left\{r^{\alpha}e^{\alpha L(r)}\right\}$ as $r\to\infty$ and for some $\alpha<\lambda_f^{L^*}$ then we get from (3.3) that for a sequence of values of r tending to infinity,

$$\begin{aligned}\frac{\log\{T(r,f\circ g)\log M(r,g)\}}{T(r,L(f))\,L(M(r,g))}&\leq\frac{o(1)+\left(\rho_g^{L^*}+\varepsilon\right)\left[\log\left\{re^{L(r)}\right\}\right]}{\left\{re^{L(r)}\right\}^{\left(\lambda_f^{L^*}-\varepsilon\right)}L(M(r,g))}\\&\quad+\frac{\left(\rho_f^{L^*}+\varepsilon\right)\left\{re^{L(r)}\right\}^{\left(\rho_g^{L^*}+\varepsilon\right)}}{\left\{re^{L(r)}\right\}^{\left(\lambda_f^{L^*}-\varepsilon\right)}L(M(r,g))}\\&\quad+\frac{1}{\left\{re^{L(r)}\right\}^{\left(\lambda_f^{L^*}-\varepsilon\right)}L(M(r,g))}\ .\end{aligned} \tag{3.8}$$

Now using (3.4) it follows from (3.8) that

$$\lim_{r\to\infty}\frac{\log\{T(r,f\circ g)\log M(r,g)\}}{T(r,L(f))\,L(M(r,g))}=0\ . \tag{3.9}$$

Combining (3.7) and (3.9) we obtain that

$$\lim_{r\to\infty}\frac{\log\{T(r,f\circ g)\log M(r,g)\}}{T(r,L(f))\cdot K(r,g;L)}=0\ ,$$

where $K(r,g;L)=\begin{cases}1 \text{ if } L(M(r,g))=o\left\{r^{\alpha}e^{\alpha L(r)}\right\} \text{ as } r\to\infty\\ \qquad\qquad\qquad\text{and for some } \alpha<\lambda_f^{L^*}\\ L(M(r,g)) \text{ otherwise.}\end{cases}$

Thus the theorem is established. □

Theorem 3.2. *Suppose f be a transcendental meromorphic function with $\sum_{a\neq\infty}\delta(a;f)+\delta(\infty;f)=2$ and g be entire such that $0<\rho_g^{L^*}<\rho_f^{L^*}<\infty$. Then*

$$\liminf_{r\to\infty}\frac{\log\{T(r,f\circ g)\log M(r,g)\}}{T(r,L(f))\cdot K(r,g;L)}=0\ ,$$

where $K(r,g;L)=\begin{cases}1 \textit{ if } L(M(r,g))=o\left\{r^{\alpha}e^{\alpha L(r)}\right\} \textit{ as } r\to\infty\\ \qquad\qquad\qquad\textit{and for some } \alpha<\rho_f^{L^*}\\ L(M(r,g)) \textit{ otherwise.}\end{cases}$

The proof of Theorem 3.2 is omitted because it can be carried out in the line of Theorem 3.1.

Theorem 3.3. *Let f be meromorphic with $\lambda_f^{[m]L^*}<\infty$ where $m\geq 1$ and g be transcendental entire with finite lower order and $\sum_{a\neq\infty}\delta(a;g)+\delta(\infty;g)=2$. Also let there exists entire functions $a_i\,(i=1,2,3,\cdots,n;n\leq\infty)$ such that $T(r,a_i)=$*

$o\{T(r,g)\}$ *and* $\sum_{i=1}^{n}\delta(a_i,g)=1$. *If* $L(M(r,g))=o\left\{r^{\alpha}e^{\alpha L(r)}\right\}$ *as* $r\to\infty$ *and for some* α *with* $0<\alpha<\lambda_g^{L^*}$ *then*

$$\liminf_{r\to\infty}\frac{\log^{[m]}T(r,f\circ g)}{T(r,L(g))}\leq\frac{\pi\lambda_f^{[m]L^*}}{(1+k-k\delta(\infty;g))}\ ,$$

otherwise

$$\liminf_{r\to\infty}\frac{\log^{[m]}T(r,f\circ g)}{T(r,L(g))\cdot L(M(r,g))}=0\ .$$

Proof. In view of the inequality $T(r,g)\leq\log^{+}M(r,g)$ and by Lemma 2.1 we get for a sequence of values of r tending to infinity that

$$\begin{aligned}T(r,f\circ g)&\leq\{1+o(1)\}T(M(r,g),f)\\ i.e.,\ \log T(r,f\circ g)&\leq\log\{1+o(1)\}+\log T(M(r,g),f)\\ i.e.,\ \log^{[m]}T(r,f\circ g)&\leq\left(\lambda_f^{[m]L^*}-\varepsilon\right)(\log M(r,g)+L(M(r,g)))+O(1)\\ i.e.,\ \frac{\log^{[m]}T(r,f\circ g)}{T(r,L(g))}&\leq\frac{\left(\lambda_f^{[m]L^*}-\varepsilon\right)(\log M(r,g)+L(M(r,g)))+O(1)}{T(r,L(g))}\end{aligned}$$

$$i.e.,\ \frac{\log^{[m]}T(r,f\circ g)}{T(r,L(g))}\leq\left(\lambda_f^{[m]L^*}-\varepsilon\right)\cdot\frac{\log M(r,g)+L(M(r,g))}{T(r,L(g))}+O(1)\ .\quad(3.10)$$

Case I. Let $L(M(r,g))=o\left\{r^{\alpha}e^{\alpha L(r)}\right\}$ as $r\to\infty$ and for some α with $0<\alpha<\lambda_g^{L^*}$.

Since $\alpha<\lambda_g^{L^*}$, we can choose $\varepsilon\,(>0)$ in such a way that

$$\alpha<\lambda_g^{L^*}-\varepsilon\ .\quad(3.11)$$

As $L(M(r,g))=o\left\{r^{\alpha}e^{\alpha L(r)}\right\}$ as $r\to\infty$ we get in view of (3.11) that

$$\lim_{r\to\infty}\frac{L(M(r,g))}{\left[re^{L(r)}\right]^{\lambda_g^{L^*}-\varepsilon}}=0\ .\quad(3.12)$$

Again in view of Lemma 2.5 we obtain for all sufficiently large values of r,

$$\begin{aligned}\log T(r,L(g))&\geq\left(\lambda_{L(g)}^{L^*}-\varepsilon\right)\log\left\{re^{L(r)}\right\}\\ i.e.,\ \log T(r,L(g))&\geq\left(\lambda_{Lg}^{L^*}-\varepsilon\right)\log\left\{re^{L(r)}\right\}\\ i.e.,\ T(r,L(g))&\geq\left[re^{L(r)}\right]^{\lambda_g^{L^*}-\varepsilon}\ .\end{aligned}\quad(3.13)$$

Now from (3.10) and (3.13) we obtain for a sequence of values of r tending to infinity that

$$\frac{\log^{[m]}T(r,f\circ g)}{T(r,L(g))}\leq\left(\lambda_f^{[m]L^*}-\varepsilon\right)\left[\frac{\log M(r,g)}{T(r,L(g))}+\frac{L(M(r,g))}{\left[re^{L(r)}\right]^{\lambda_g^{L^*}-\varepsilon}}\right]+O(1)$$

$$i.e.,\ \frac{\log^{[m]}T(r,f\circ g)}{T(r,L(g))}$$

$$\leq \left(\lambda_f^{[m]L^*} - \varepsilon\right)\left[\frac{\log M(r,g)}{T(r,g)} \cdot \frac{T(r,g)}{T(r,L(g))} + \frac{L(M(r,g))}{\left[re^{L(r)}\right]^{\lambda_g^{L^*}-\varepsilon}}\right] + O(1). \tag{3.14}$$

Now combining (3.12) and (3.14) and in view of Lemma 2.3 and Lemma 2.4 it follows that

$$\liminf_{r\to\infty} \frac{\log^{[m]} T(r, f\circ g)}{T(r,L(g))} \leq \frac{\pi\lambda_f^{[m]L^*}}{(1+k-k\delta(\infty;g))}\ . \tag{3.15}$$

Case II. If $L(M(r,g)) \neq o\left\{r^\alpha e^{\alpha L(r)}\right\}$ as $r\to\infty$ and for some α with $0<\alpha<\lambda_g^{L^*}$, then from (3.10) we get for a sequence of values of r tending to infinity that

$$\frac{\log^{[m]} T(r,f\circ g)}{T(r,L(g))\,L(M(r,g))} \leq \left(\lambda_f^{[m]L^*} - \varepsilon\right)\cdot \frac{\log M(r,g)}{T(r,L(g))\,L(M(r,g))} + \frac{\{1+O(1)\}}{T(r,L(g))}\ .$$

$$i.e.,\ \liminf_{r\to\infty} \frac{\log^{[m]} T(r,f\circ g)}{T(r,L(g))\,L(M(r,g))} = 0\ .$$

Thus combining Case I and Case II the theorem follows. □

In the line of Theorem 3.3 the following theorem can be proved:

Theorem 3.4. *Let f be meromorphic with $\rho_f^{[m]L^*} < \infty$ and g be transcendental entire with finite lower order and $\sum_{a\neq\infty}\delta(a;g) + \delta(\infty;g) = 2$. Also let there exists entire functions $a_i\ (i=1,2,3,\cdots,n; n\leq\infty)$ such that $T(r,a_i) = o\{T(r,g)\}$ and $\sum_{i=1}^{n}\delta(a_i,g) = 1$. If $L(M(r,g)) = o\left\{r^\alpha e^{\alpha L(r)}\right\}$ as $r\to\infty$ and for some α with $0<\alpha<\lambda_g^{L^*}$ then*

$$\limsup_{r\to\infty} \frac{\log^{[m]} T(r,f\circ g)}{T(r,L(g))} \leq \frac{\pi\rho_f^{[m]L^*}}{(1+k-k\delta(\infty;g))}\ ,$$

otherwise

$$\limsup_{r\to\infty} \frac{\log^{[m]} T(r,f\circ g)}{T(r,L(g))\,L(M(r,g))} = 0\ .$$

Theorem 3.5. *Let f be meromorphic and g be transcendental entire with $\rho_f^{[m]L^*} < \infty$, $0<\lambda_g^{L^*}\leq\rho_g^{L^*}<\infty$ and $\sum_{a\neq\infty}\delta(a;g) + \delta(\infty;g) = 2$ where m is any positive integer. Then*
(a) If $L(M(r,g)) = o\{\log T(r,L(g))\}$ then

$$\limsup_{r\to\infty} \frac{\log^{[m+1]} T(r,f\circ g)}{\log T(r,L(g)) + L(M(r,g))} \leq \frac{\rho_g^{L^*}}{\lambda_g^{L^*}}$$

and (b) if $T(r,L(g)) = o\{L(M(r,g))\}$ then

$$\lim_{r\to\infty} \frac{\log^{[m+1]} T(r,f\circ g)}{\log T(r,L(g)) + L(M(r,g))} = 0\ .$$

Proof. For all sufficiently large values of r, we obtain in view of $T(r,g) \leq \log^{+} M(r,g)$, by Lemma 2.1 and also using $\log\left\{1+\frac{L(M(r,g))}{\log M(r,g)}\right\} \sim \frac{L(M(r,g))}{\log M(r,g)}$

$$T(r, f\circ g) \leq \{1+o(1)\} T(M(r,g), f)$$
$$i.e.,\ \log T(r, f\circ g) \leq \log\{1+o(1)\} + \log T(M(r,g), f)$$
$$i.e.,\ \log^{[m]} T(r, f\circ g) \leq o(1) + \log^{[m]} T(M(r,g), f)$$
$$i.e.,\ \log^{[m]} T(r, f\circ g) \leq o(1) + \left(\rho_f^{[m]L^*} + \varepsilon\right)\{\log M(r,g) + L(M(r,g))\} \quad (3.16)$$
$$i.e.,\ \log^{[m]} T(r, f\circ g) \leq o(1) + \left(\rho_f^{[m]L^*} + \varepsilon\right)\log M(r,g)\left\{1+\frac{L(M(r,g))}{\log M(r,g)}\right\}$$
$$i.e.,\ \log^{[m+1]} T(r, f\circ g) \leq o(1) + \log\left(\rho_f^{[m]L^*} + \varepsilon\right) + \log^{[2]} M(r,g) + \log\left\{1+\frac{L(M(r,g))}{\log M(r,g)}\right\}$$
$$i.e.,\ \log^{[m+1]} T(r, f\circ g) \leq o(1) + \log\left(\rho_f^{[m]L^*} + \varepsilon\right) + \left(\rho_g^{L^*} + \varepsilon\right)\log\left\{re^{L(r)}\right\} + \log\left\{1+\frac{L(M(r,g))}{\log M(r,g)}\right\}$$
$$i.e.,\ \log^{[m+1]} T(r, f\circ g) \leq o(1) + \left(\rho_g^{L^*} + \varepsilon\right)\{\log r + L(r)\} + \frac{L(M(r,g))}{\log M(r,g)}\ . \quad (3.17)$$

Again in view of Lemma 2.5 we get from the definition of L^*-lower order for all sufficiently large values of r that

$$\log T(r, L(g)) \geq \left(\lambda_{L(g)}^{L^*} - \varepsilon\right)\log\left[re^{L(r)}\right]$$
$$i.e.,\ \log T(r, L(g)) \geq \left(\lambda_g^{L^*} - \varepsilon\right)\log\left[re^{L(r)}\right]$$
$$i.e.,\ \log T(r, L(g)) \geq \left(\lambda_g^{L^*} - \varepsilon\right)[\log r + L(r)]$$
$$i.e.,\ \log r + L(r) \leq \frac{\log T(r, L(g))}{\left(\lambda_g^{L^*} - \varepsilon\right)}\ . \quad (3.18)$$

Hence from (3.17) and (3.18) it follows for all sufficiently large values of r that

$$\log^{[m+1]} T(r, f\circ g) \leq o(1) + \left(\frac{\rho_g^{L^*} + \varepsilon}{\lambda_g^{L^*} - \varepsilon}\right)\cdot \log T(r, L(g)) + \frac{L(M(r,g))}{\log M(r,g)}$$

$$i.e,\ \frac{\log^{[m+1]} T(r, f\circ g)}{\log T(r, L(g)) + L(M(r,g))} \leq o(1) + \left(\frac{\rho_g^{L^*} + \varepsilon}{\lambda_g^{L^*} - \varepsilon}\right)\cdot \frac{\log T(r, L(g))}{\log T(r, L(g)) + L(M(r,g))} + \frac{L(M(r,g))}{[\log T(r, L(g)) + L(M(r,g))]\log M(r,g)}$$

$$i.e,\ \frac{\log^{[m+1]} T(r, f\circ g)}{\log T(r, L(g)) + L(M(r,g))} \leq o(1) + \frac{\left(\frac{\rho_g^{L^*}+\varepsilon}{\lambda_g^{L^*}-\varepsilon}\right)}{1+\frac{L(M(r,g))}{\log T(r,L(g))}} + \frac{1}{\left[1+\frac{\log T(r,L(g))}{L(M(r,g))}\right]\log M(r,g)}. \tag{3.19}$$

Since $L(M(r,g)) = o\{\log T(r, L(g))\}$ as $r \to \infty$ and $\varepsilon\,(>0)$ is arbitrary we obtain from (3.19) that

$$\limsup_{r\to\infty} \frac{\log^{[m+1]} T(r, f\circ g)}{\log T(r, L(g)) + L(M(r,g))} \leq \frac{\rho_g^{L^*}}{\lambda_g^{L^*}}\ . \tag{3.20}$$

Again if $\log T(r,g) = o\{L(M(r,g))\}$ then from (3.19) we get that

$$\lim_{r\to\infty} \frac{\log^{[m+1]} T(r, f\circ g)}{\log T(r, L(g)) + L(M(r,g))} = 0\ . \tag{3.21}$$

Thus from (3.20) and (3.21) the theorem is established. □

Corollary 3.6. *Let f be meromorphic and g be transcendental entire with $\rho_f^{[m]L^*} < \infty$, $0 < \rho_g^{L^*} < \infty$ and $\sum_{a\neq\infty} \delta(a;g) + \delta(\infty;g) = 2$ where $m \geq 1$. Then*

(a) If $L(M(r,g)) = o\{\log T(r, L(g))\}$ then

$$\liminf_{r\to\infty} \frac{\log^{[m+1]} T(r, f\circ g)}{T(r, L(g)) + L(M(r,g))} \leq 1$$

and (b) if $T(r, L(g)) = o\{L(M(r,g))\}$ then

$$\liminf_{r\to\infty} \frac{\log^{[m+1]} T(r, f\circ g)}{T(r, L(g)) + L(M(r,g))} = 0\ .$$

We omit the proof of Corollary 3.6 because it can be carried out in the line of Theorem 3.5.

Remark 3.7. The equality sign in Theorem 3.5 and Corollary 3.6 cannot be removed as we see in the following example:

Example 3.8. Let $f = g = \exp z$, $m = 1$ and $L(r) = \frac{1}{p}\exp\left(\frac{1}{r}\right)$ where p is any positive real number.
Then

$$\rho_f^{L^*} = \lambda_g^{L^*} = \rho_g^{L^*} = 1 \text{ and } \sum_{a\neq\infty} \delta(a;g) + \delta(\infty;g) = 2.$$

Now taking $a_1 = 1$, $a_2 = ... = a_k = 0$ we get that

$$L(g) = \begin{vmatrix} a_1 & g \\ a_1' & g' \end{vmatrix} = \begin{vmatrix} 1 & \exp z \\ 0 & \exp z \end{vmatrix} = \exp z.$$

Also

$$T(r, f\circ g) \sim \frac{\exp r}{(2\pi^3 r)^{\frac{1}{2}}}\,(r\to\infty)\,,\ T(r,g) = \frac{r}{\pi} \text{ and } M(r,g) = \exp r\ .$$

So

$$L\left(M\left(r,g\right)\right)=L\left(\exp r\right)=\frac{1}{p}\exp\left(\frac{1}{\exp r}\right)\ .$$

Hence

$$\begin{aligned}\liminf_{r\to\infty}\frac{\log^{[2]}T\left(r,f\circ g\right)}{\log T\left(r,L(g)\right)+L\left(M\left(r,g\right)\right)} &= \limsup_{r\to\infty}\frac{\log^{[2]}T\left(r,f\circ g\right)}{\log T\left(r,L(g)\right)+L\left(M\left(r,g\right)\right)}\\ &= \limsup_{r\to\infty}\frac{\log\left[r-\frac{1}{2}\log r+O\left(1\right)\right]}{\log r+O\left(1\right)+\frac{1}{p}\exp\left(\frac{1}{\exp r}\right)}=1\ .\end{aligned}$$

Theorem 3.9. *Let f be transcendental entire and g be an entire function with $0<\lambda_f^{[m]L^*}\leq\rho_f^{[m]L^*}<\infty$ where m is any positive integer , $0<\lambda_g^{L^*}\leq\rho_g^{L^*}<\infty$ and $\sum\limits_{a\neq\infty}\delta(a;f)+\delta(\infty;f)=2$. Then*

$$\limsup_{r\to\infty}\frac{\log^{[m+1]}T\left(r,f\circ g\right)}{\log^{[m]}T\left(r,L(f)\right)+L\left(\frac{1}{8}M\left(\frac{r}{4},g\right)\right)}\geq\frac{\rho_g^{L^*}}{\rho_f^{[m]L^*}}\ .$$

Proof. In view of Lemma 2.2, we have for all sufficiently large values of r,

$$T\left(r,f\circ g\right)\geq\frac{1}{3}\log M\left\{\frac{1}{8}M\left(\frac{r}{4},g\right)+o\left(1\right),f\right\}$$

$$i.e.,\ \log^{[m]}T\left(r,f\circ g\right)\geq o\left(1\right)+\log^{[m+1]}M\left\{\frac{1}{8}M\left(\frac{r}{4},g\right)+o\left(1\right),f\right\}\ . \tag{3.22}$$

$$i.e.,\ \log^{[m]}T\left(r,f\circ g\right)\geq o\left(1\right)+\left(\lambda_f^{[m]L^*}-\varepsilon\right)\left[\log\left\{\frac{1}{8}M\left(\frac{r}{4},g\right)+o\left(1\right)\right\}+L\left(\frac{1}{8}M\left(\frac{r}{4},g\right)\right)\right]$$

$$i.e.,\ \log^{[m]}T\left(r,f\circ g\right)\geq o\left(1\right)+\left(\lambda_f^{[m]L^*}-\varepsilon\right)\left[\log\left\{\frac{1}{8}M\left(\frac{r}{4},g\right)\left(1+\frac{o\left(1\right)}{\frac{1}{8}M\left(\frac{r}{4},g\right)}\right)\right\}+L\left(\frac{1}{8}M\left(\frac{r}{4},g\right)\right)\right]$$

$$i.e.,\ \log^{[m]}T\left(r,f\circ g\right)\geq\left(\lambda_f^{[m]L^*}-\varepsilon\right)\log M\left(\frac{r}{4},g\right).\left\{\frac{\log M\left(\frac{r}{4},g\right)+\log\left(1+\frac{o(1)}{\frac{1}{8}M\left(\frac{r}{4},g\right)}\right)+L\left(\frac{1}{8}M\left(\frac{r}{4},g\right)\right)}{\log M\left(\frac{r}{4},g\right)}\right\}$$

$$i.e.,\ \log^{[m+1]} T(r, f\circ g) \geq \log^{[2]} M\left(\frac{r}{4}, g\right) + \left(\frac{\lambda_g^{L^*}-\varepsilon}{\rho_f^{[m]L^*}+\varepsilon}\right) L\left(\frac{1}{8}M\left(\frac{r}{4}, g\right)\right) - \log\left[\exp\left\{\left(\frac{\lambda_g^{L^*}-\varepsilon}{\rho_f^{[m]L^*}+\varepsilon}\right) L\left(\frac{1}{8}M\left(\frac{r}{4}, g\right)\right)\right\}\right] + \log\left\{\frac{\left[\log M\left(\frac{r}{4}, g\right) + L\left(\frac{1}{8}M\left(\frac{r}{4}, g\right)\right)\right] + o(1)}{\log M\left(\frac{r}{4}, g\right)}\right\}$$

$$i.e.,\ \log^{[m+1]} T(r, f\circ g) \geq \log^{[2]} M\left(\frac{r}{4}, g\right) + \left(\frac{\lambda_g^{L^*}-\varepsilon}{\rho_f^{[m]L^*}+\varepsilon}\right) L\left(\frac{1}{8}M\left(\frac{r}{4}, g\right)\right) + \log\left\{\frac{\left[\log M\left(\frac{r}{4}, g\right) + L\left(\frac{1}{8}M\left(\frac{r}{4}, g\right)\right)\right] + o(1)}{\exp\left\{\left(\frac{\lambda_g^{L^*}-\varepsilon}{\rho_f^{[m]L^*}+\varepsilon}\right) L\left(\frac{1}{8}M\left(\frac{r}{4}, g\right)\right)\right\} \log M\left(\frac{r}{4}, g\right)}\right\}$$

$$\begin{aligned} & i.e.,\ \log^{[m+1]} T(r, f\circ g) \\ \geq\ & \log^{[2]} M\left(\frac{r}{4}, g\right) + \left(\frac{\lambda_g^{L^*}-\varepsilon}{\rho_f^{[m]L^*}+\varepsilon}\right) L\left(\frac{1}{8}M\left(\frac{r}{4}, g\right)\right) . \end{aligned} \tag{3.23}$$

Now from (3.23) it follows for a sequence of values of r tending to infinity that

$$\log^{[m+1]} T(r, f\circ g) \geq \left(\rho_g^{L^*}-\varepsilon\right) \log\left\{\frac{r}{4} e^{L\left(\frac{r}{4}\right)}\right\} + \left(\frac{\rho_g^{L^*}-\varepsilon}{\rho_f^{[m]L^*}+\varepsilon}\right) L\left(\frac{1}{8}M\left(\frac{r}{4}, g\right)\right). \tag{3.24}$$

In view of Lemma 2.5, we get for all sufficiently large values of r that

$$\begin{aligned} \log^{[m]} T(r, L(f)) &\leq \left(\rho_{L(f)}^{[m]L^*}+\varepsilon\right) \log\left\{re^{L(r)}\right\} \\ i.e.,\ \log^{[m]} T(r, L(f)) &\leq \left(\rho_f^{[m]L^*}+\varepsilon\right) \log\left\{re^{L(r)}\right\} \\ i.e.,\ \log^{[m]} T(r, L(f)) &\leq \left(\rho_f^{[m]L^*}+\varepsilon\right) \log\left\{\frac{r}{4} e^{L\left(\frac{r}{4}\right)}\right\} + \log 4. \end{aligned} \tag{3.25}$$

Hence from (3.24) and (3.25) it follows for all sufficiently large values of r that

$$i.e.,\ \log^{[m+1]} T(r, f\circ g) \geq \left(\frac{\rho_g^{L^*}-\varepsilon}{\rho_f^{[m]L^*}+\varepsilon}\right) \left(\log^{[m]} T(r, L(f)) - \log 4\right) + \left(\frac{\rho_g^{L^*}-\varepsilon}{\rho_f^{[m]L^*}+\varepsilon}\right) L\left(\frac{1}{8}M\left(\frac{r}{4}, g\right)\right)$$

$$i.e.,\ \log^{[m+1]} T(r, f\circ g) \geq \left(\frac{\rho_g^{L^*}-\varepsilon}{\rho_f^{[m]L^*}+\varepsilon}\right)\left[\log^{[m]} T(r, L(f)) + L\left(\frac{1}{8}M\left(\frac{r}{4}, g\right)\right)\right] - \left(\frac{\rho_g^{L^*}-\varepsilon}{\rho_f^{[m]L^*}+\varepsilon}\right)\log 4$$

$$i.e.,\ \frac{\log^{[m+1]} T(r, f\circ g)}{\log^{[m]} T(r, L(f)) + L\left(\frac{1}{8}M\left(\frac{r}{4}, g\right)\right)} \geq \left(\frac{\rho_g^{L^*}-\varepsilon}{\rho_f^{[m]L^*}+\varepsilon}\right) - \frac{\left(\frac{\rho_g^{L^*}-\varepsilon}{\rho_f^{[m]L^*}+\varepsilon}\right)\log 4}{\log^{[m]} T(r, L(f)) + L\left(\frac{1}{8}M\left(\frac{r}{4}, g\right)\right)}. \quad (3.26)$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows from (3.26) that

$$\limsup_{r\to\infty}\frac{\log^{[m+1]} T(r, f\circ g)}{\log^{[m]} T(r, L(f)) + L\left(\frac{1}{8}M\left(\frac{r}{4}, g\right)\right)} \geq \frac{\rho_g^{L^*}}{\rho_f^{[m]L^*}}\ .$$

This proves the theorem. □

In the line of Theorem 3.9 the following theorem may be proved:

Theorem 3.10. *Let f a transcendental entire function and g be an entire function such that $0<\lambda_f^{[m]L^*}\leq\rho_f^{[m]L^*}<\infty$ where $m\geq 1$, $0<\lambda_g^{L^*}\leq\rho_g^{L^*}<\infty$ and $\sum\limits_{a\neq\infty}\delta(a;f)+\delta(\infty;f)=2$. Then*

$$\liminf_{r\to\infty}\frac{\log^{[m+1]} T(r, f\circ g)}{\log^{[m]} T(r, L(f)) + L\left(\frac{1}{8}M\left(\frac{r}{4}, g\right)\right)} \geq \frac{\lambda_g^{L^*}}{\rho_f^{[m]L^*}}\ .$$

Theorem 3.11. *Let f a transcendental entire function and g be an entire function such that $0<\lambda_f^{[m]L^*}\leq\rho_f^{[m]L^*}<\infty$ where $m\geq 1$, $0<\lambda_g^{L^*}\leq\rho_g^{L^*}<\infty$ and $\sum\limits_{a\neq\infty}\delta(a;f)+\delta(\infty;f)=2$. Then for every constant A and any real number x,*

$$\lim_{r\to\infty}\frac{\log^{[m]} T(r, f\circ g)}{\left\{\log^{[m]} T(r^A, L(f))\right\}^{1+x}} = \infty\ .$$

Proof. If x is such that $1+x\leq 0$, then the theorem is obvious. So we suppose that $1+x>0$.
Now from (3.22) we have for all sufficiently large values of r that

$$\log^{[m]} T(r, f\circ g) \geq o(1) + \left(\lambda_f^{[m]L^*}-\varepsilon\right)\left[\log\left\{\frac{1}{8}M\left(\frac{r}{4}, g\right)+o(1)\right\} + L\left(\frac{1}{8}M\left(\frac{r}{4}, g\right)\right)\right]$$

$$i.e.,\ \log^{[m]} T(r, f\circ g) \geq o(1) + \left(\lambda_f^{[m]L^*} - \varepsilon\right)\left[\log M\left(\frac{r}{4}, g\right) + o(1) + L\left(\frac{1}{8}M\left(\frac{r}{4}, g\right)\right)\right]$$

$$i.e.,\ \log^{[m]} T(r, f\circ g) \geq o(1) + \left(\lambda_f^{[m]L^*} - \varepsilon\right)\left[\left\{\left(\frac{r}{4}\right)e^{L(r)}\right\}^{\lambda_g^{L^*}-\varepsilon} + o(1) + L\left(\frac{1}{8}M\left(\frac{r}{4}, g\right)\right)\right] \quad (3.27)$$

where we choose $0 < \varepsilon < \min\left\{\lambda_f^{[m]L^*}, \lambda_g^{L^*}\right\}$.
Also for all sufficiently large values of r we get from Lemma 2.5 that

$$\begin{aligned} \log^{[m]} T\left(r^A, L(f)\right) &\leq \left(\rho_{L(f)}^{[m]L^*} + \varepsilon\right)\log\left\{r^A e^{L\left(r^A\right)}\right\} \\ i.e.,\ \log^{[m]} T\left(r^A, L(f)\right) &\leq \left(\rho_f^{[m]L^*} + \varepsilon\right)\log\left\{r^A e^{L\left(r^A\right)}\right\} \\ i.e.,\ \left\{\log^{[m]} T\left(r^A, L(f)\right)\right\}^{1+x} &\leq \left(\rho_f^{[m]L^*} + \varepsilon\right)^{1+x}\left(\log\left\{r^A e^{L\left(r^A\right)}\right\}\right)^{1+x} \quad (3.28) \end{aligned}$$

Therefore from (3.27) and (3.28) it follows for all sufficiently large values of r that

$$\frac{\log^{[m]} T(r, f\circ g)}{\left\{\log^{[m]} T\left(r^A, L(f)\right)\right\}^{1+x}} \geq \frac{o(1) + \left(\lambda_f^{[m]L^*} - \varepsilon\right)\left\{\left(\frac{r}{4}\right)e^{L(r)}\right\}^{\lambda_g^{L^*}-\varepsilon} + o(1) + L\left(\frac{1}{8}M\left(\frac{r}{4}, g\right)\right)}{\left(\rho_f^{[m]L^*} + \varepsilon\right)^{1+x}\left(\log\left\{r^A e^{L(r^A)}\right\}\right)^{1+x}} \quad (3.29)$$

Thus from (3.29) the theorem follows. □

Theorem 3.12. *Let f an entire function and g be a transcendental entire function such that $0 < \lambda_f^{[m]L^*} \leq \rho_f^{[m]L^*} < \infty$ where $m \geq 1$, $0 < \lambda_g^{L^*} \leq \rho_g^{L^*} < \infty$ and $\sum_{a\neq\infty}\delta(a; f) + \delta(\infty; f) = 2$. Then for every constant A and any real number x,*

$$\lim_{r\to\infty}\frac{\log^{[m]} T(r, f\circ g)}{\left\{\log T\left(r^A, L(g)\right)\right\}^{1+x}} = \infty\ .$$

The proof of Theorem 3.12 is omitted as it can be carried out in the line of Theorem 3.11.

Theorem 3.13. *Let f be transcendental meromorphic and g be entire satisfying the following conditions (i) $\rho_f^{[m]L^*}$ and $\rho_g^{L^*}$ are both finite, (ii) $\rho_f^{[m]L^*}$ is positive and (iii) $\sum_{a\neq\infty}\delta(a; f) + \delta(\infty; f) = 2$. Then for each $\alpha \in (-\infty, \infty)$,*

$$\liminf_{r\to\infty}\frac{\left\{\log^{[m]} T(r, f\circ g)\right\}^{1+\alpha}}{\log^{[m]} T\left(\exp\left(r^A\right), L(f)\right)} = 0$$

for $A > (1+\alpha)\,\rho_g^{L^*}$ *and* $m \geq 1$.

Proof. If $1+\alpha < 0$, then the theorem is trivial. So we take $1+\alpha > 0$. Now from (3.16) we obtain for all sufficiently large values of r that

$$\log^{[m]} T(r, f\circ g) \leq o(1) + \left(\rho_f^{[m]L^*} + \varepsilon\right)\left[re^{L(r)}\right]^{\left(\rho_g^{L^*}+\varepsilon\right)} + \left(\rho_f^{[m]L^*} + \varepsilon\right) L\left(M(r,g)\right)$$

$$i.e.,\ \log^{[m]} T(r, f\circ g)$$
$$\leq \left[re^{L(r)}\right]^{\left(\rho_g^{L^*}+\varepsilon\right)} \left\{\left(\rho_f^{[m]L^*} + \varepsilon\right) + o(1)\right\} + \left(\rho_f^{[m]L^*} + \varepsilon\right) L\left(M(r,g)\right)$$

$$i.e.,\ \left\{\log^{[m]} T(r, f\circ g)\right\}^{1+\alpha}$$
$$\leq \left[\left[re^{L(r)}\right]^{\left(\rho_g^{L^*}+\varepsilon\right)} \left\{\left(\rho_f^{[m]L^*} + \varepsilon\right) + o(1)\right\} + \left(\rho_f^{[m]L^*} + \varepsilon\right) L\left(M(r,g)\right)\right]^{1+\alpha} \quad (3.30)$$

Again in view of Lemma 2.5 we have for a sequence of r tending to infinity and for $\varepsilon(>0)$,

$$\log^{[m]} T\left(\exp\left(r^A\right), L(f)\right) \geq \left(\rho_{L(f)}^{[m]L^*} - \varepsilon\right) \log\left[\exp\left(r^A\right) \exp\left\{L\left(\exp\left(r^A\right)\right)\right\}\right].$$

$$i.e.,\ \log^{[m]} T\left(\exp\left(r^A\right), L(f)\right) \geq \left(\rho_f^{[m]L^*} - \varepsilon\right)\left[r^A + L\left(\exp\left(r^A\right)\right)\right]. \quad (3.31)$$

Now let

$$\left[e^{L(r)}\right]^{\left(\rho_g^{L^*}+\varepsilon\right)} \left\{\left(\rho_f^{[m]L^*} + \varepsilon\right) + o(1)\right\} = k_1, \left(\rho_f^{[m]L^*} + \varepsilon\right) L\left(M(r,g)\right) = k_2,$$
$$\left(\rho_f^{[m]L^*} - \varepsilon\right) = k_3 \text{ and } \left(\rho_f^{[m]L^*} - \varepsilon\right) L\left(\exp\left(r^A\right)\right) = k_4.$$

Then from (3.30), (3.31) and above we get for a sequence of values of r tending to infinity that

$$\frac{\left\{\log^{[m]} T(r, f\circ g)\right\}^{1+\alpha}}{\log^{[m]} T\left(\exp\left(r^A\right), L(f)\right)} \leq \frac{\left[r^{\left(\rho_g^{L^*}+\varepsilon\right)} k_1 + k_2\right]^{1+\alpha}}{k_3 r^A + k_4}$$

$$i.e.,\ \frac{\left\{\log^{[m]} T(r, f\circ g)\right\}^{1+\alpha}}{\log^{[m]} T\left(\exp\left(r^A\right), L(f)\right)} \leq \frac{r^{\left(\rho_g^{L^*}+\varepsilon\right)(1+\alpha)}\left[k_1 + \frac{k_2}{r^{\left(\rho_g^{L^*}+\varepsilon\right)}}\right]^{1+\alpha}}{k_3 r^A + k_4}$$

where k_1, k_2, k_3 and k_4 are all finite.
Since$\left(\rho_g^{L^*} + \varepsilon\right)(1+\alpha) < A$, we obtain from above

$$\liminf_{r\to\infty} \frac{\left\{\log^{[m]} T(r, f\circ g)\right\}^{1+\alpha}}{\log^{[m]} T\left(\exp\left(r^A\right), L(f)\right)} = 0$$

where we choose $\varepsilon(>0)$ in such a way that

$$0 < \varepsilon < \min\left\{\rho_f^{[m]L^*}, \frac{A}{1+\alpha} - \rho_g^{L^*}\right\}.$$

This proves the theorem. □

In the line of Theorem 3.13 the following theorem may be proved:

Theorem 3.14. *Let f be transcendental meromorphic and g be entire satisfying the following conditions (i) $0 < \lambda_f^{[m]L^*} \leq \rho_f^{[m]L^*} < \infty$, (ii) $\rho_g^{L^*}$ is finite and (iii) $\sum_{a\neq\infty} \delta(a;f) + \delta(\infty;f) = 2$. Then for each $\alpha \in (-\infty,\infty)$,*

$$\lim_{r\to\infty} \frac{\left\{\log^{[m]} T\left(r, f\circ g\right)\right\}^{1+\alpha}}{\log^{[m]} T\left(\exp\left(r^A\right), L(f)\right)} = 0$$

where $A > (1+\alpha)\,\rho_g^{L^*}$ and $m \geq 1$.

Theorem 3.15. *Let f be meromorphic and g be transcendental entire such that $0 < \lambda_g^{L^*} \leq \rho_g^{L^*} < \infty$, $\rho_f^{[m]L^*} < \infty$ where $m \geq 1$ and $\sum_{a\neq\infty} \delta(a;g) + \delta(\infty;g) = 2$. Then for each $\alpha \in (-\infty,\infty)$,*

$$\lim_{r\to\infty} \frac{\left\{\log^{[m]} T\left(r, f\circ g\right)\right\}^{1+\alpha}}{\log T\left(\exp\left(r^A\right), L(g)\right)} = 0 \text{ if } A > (1+\alpha)\,\rho_g^{L^*}.$$

Theorem 3.16. *Let f be meromorphic and g be transcendental entire with $\rho_f^{[m]L^*} < \infty$ for $m \geq 1$, $0 < \rho_g^{L^*} < \infty$ and $\sum_{a\neq\infty} \delta(a;g) + \delta(\infty;g) = 2$. Then for each $\alpha \in (-\infty,\infty)$,*

$$\liminf_{r\to\infty} \frac{\left\{\log^{[m]} T\left(r, f\circ g\right)\right\}^{1+\alpha}}{\log T\left(\exp\left(r^A\right), L(g)\right)} = 0 \text{ where } A > (1+\alpha)\,\rho_g^{L^*}.$$

The proof of Theorem 3.15 and Theorem 3.16 are omitted because those can be carried out in the line of Theorem 3.14 and Theorem 3.13 respectively.

Acknowledgement. The authors are thankful to the referee for his/her valuable suggestions towards the improvement of the paper.

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