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ROBUST OPTIMIZATION FOR QUADRATIC PROGRAMS WITH DATA UNCERTAINTY USING THE CONCEPT OF STABILITY RADIUS

SATAFA SANOGO[1]*, MOHAMED ZONGO[2], MOUSSA BARRO[2] AND SADO TRAORE[2]

Abstract. In this work, we study an uncertain quadratic optimization problem by using the concept of stability radius. That allows to prove robustness of uncertain quadratic optimization models. Firstly, we determine the stability radius of this problem by applying Ascoli formula. Secondly, we show that its robust counterpart has at least an optimal solution. Finally, we deal with an application and provide some numerical methods for such an uncertain quadratic optimization problem.

1. Introduction

Let us consider the succeeding quadratic optimization problem

$$\begin{aligned} \text{Minimize} \quad & \frac{1}{2}x^TAx + a^Tx, \\ \text{s.t.} \quad & \frac{1}{2}x^TBx + b^Tx + \beta \quad \leqslant 0, \end{aligned} \tag{1.1}$$

where A, B are $n \times n$ real symmetric matrices; a, b are in $\mathbb{R}^n$ and β is a real number.

This problem (1.1) plays a key role in many theoretical and engineering problems such as in optimal design problems, optimal control theory, signal processing and telecommunication [5, 18, 19, 22]. Data of optimization problem are often subjected to errors of measurements or modelling and those affecte the quality of the obtained results by adding some noises on these computed results. Hence, it is important and most realist to study models whose data are subjected by uncertainties. That allows to handle the previous errors in the optimization problem model, then such a model can give us most realist solution. That is why some authors [4, 5, 21] are interested in the following uncertain quadratic optimization problem

$$\begin{aligned} \text{Minimize} \quad & p(x), \\ \text{s.t.} \quad & q(x,u) \quad \leqslant 0, \end{aligned} \tag{1.2}$$

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* Corresponding author.

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where $p(x) = \frac{1}{2}x^T Ax + a^T x$ and $q(x,u) = \frac{1}{2}x^T B_u x + b_u^T x + \beta_u$.
Recall that B_u denote an uncertain symmetric $n \times n$ matrix, b_u an uncertain vector and β_u is an uncertain scalar. One of the most well-known approaches to study problem (1.2) is robust optimization [4, 5, 7, 8, 9, 29]. A solution of such an optimization problem is called robust optimal solution. This kind of solution is known to be immunized against uncertainties and noises of a quadratic model. The staple research in the purview of optimization is robust strong duality [2, 3, 20] characterization of robust optimal solutions [4, 5, 21, 23, 30, 31]. M. Sniedovich has used the stability radius model to analyse and quantify strict uncertainty [25, 26, 27, 28]. A new approach has been highlighted by M. Ciligot-Travain in [11] for optimization problem under data uncertainty that was the robustness of optimization problem. He presented and made a survey about the notion of stability radius. These two approaches have been compared by Barro et al. in [1, 2]. In the present work, we apply these approaches and concepts of stability radius to handle quadratic optimization problems subject to uncertainties. Moreover in modern engineering science, taking into account of stability radius in design process is very important. Indeed, it allows to classify errors margin during the manufacturing of industrial products. Stability radius intervene in optimization, decision making, operational research, design engineering and dependability whenever models are subjected by uncertainties.

The main aim of this article is to resolve (1.2) by using optimization of robustness approach. We explicit its stability radius and show that the robust counterpart of (1.2) in the sense of robustness optimization has at least an optimal solution.

The outline of the rest of this work is as follows. In Section 2, we recall some fundamental definitions and preliminary results which will be used. Section 3 is devoted to the computation of the stability radius of the problem (1.2). Then, we make an application in Section 4 to illustrate our approaches. Moreover in this section, we present some numerical methods. The Section 5 concludes our work.

2. Preliminaries

We start this section by mentioning some necessary results which will be used throughout this work. In all of the suite, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{R}^n$ the n-dimensional euclidean space. The set of $n \times n$ real symmetric matrices is denoted by $\mathcal{S}_n(\mathbb{R})$. For all $t \in \mathbb{R}$, we note $t^+ := \max(t, 0)$ known as the positive part of t.
Let $(U, \| \,.\, \|)$ be a normed real vector space and $U^\star$ its topological dual space. $\langle ., . \rangle$ denotes the standard coupling function between U and $U^\star$. The dual norm on $U^\star$ denoted $\| \,.\, \|^\star$ is define by $\| u^\star \|^\star = \max\limits_{\|u\| \leqslant 1} | \langle u^\star, u \rangle |$.
The complement set of $A \subset U$ is denoted A^c and defined by $A^c = \{ u \in U \mid u \notin A \}$.

For all subset $A \subset U$, the distance to A is a functional $d_{\|.\|}(., A)$ defined from the normed space $(U, \| \cdot \|)$ to $[0, +\infty]$ and given by:

$$d_{\|.\|}(u, A) = \inf_{a \in A} \|u - a\|, \quad \forall u \in U.$$

Let us recall Ascoli formula.

Lemma 2.1 ([2, 6, 17, 32]). *For all nonzero linear continuous form $u^\star \in U^\star$ and for all $(\alpha, a) \in \mathbb{R} \times U$, one has:*

$$\inf_{\langle u,u^\star\rangle > \alpha} \| u - a \| \;=\; \frac{(\alpha - \langle a, u^\star\rangle)^+}{\|u^\star\|^\star}. \tag{2.1}$$

Definition 2.2 ([2, 10, 12]). Given $r \in [1, n]$, the r-norm introduce by D. Bertsimas et *al.* [10] denoted $\| \,.\, \|_{(r)}$ is a norm on $\mathbb{R}^n$ defined for $x = (x_1, ..., x_n)^T \in \mathbb{R}^n$ by:

$$\| x \|_{(r)} = \begin{cases} \displaystyle\sum_{i=1}^{\lfloor r \rfloor} |x_{(i)}| + (r - \lfloor r \rfloor)\big|x_{(\lfloor r \rfloor + 1)}\big| & \text{if } r \in [1, n[\ ; \\ \displaystyle\sum_{i=1}^{n} |x_i| & \text{if } r = n. \end{cases}$$

where $\mid x_{(i)} \mid$ is the greatest i-th coordinate of $\{\mid x_1 \mid, \mid x_2 \mid, ..., \mid x_n \mid\}$, and $\lfloor r \rfloor$ denote the integer part of the real r.
We remark that $\| \,.\, \|_{(1)} = \| \,.\, \|_\infty$ and $\| \,.\, \|_{(n)} = \| \,.\, \|_1$.

The **dual norm of the r-norm** is given by [10]:

$$\forall x \in \mathbb{R}^n, \qquad \| x \|^\star_{(r)} = \max\left\{ \| x \|_\infty, \frac{1}{r} \| x \|_1 \right\}. \tag{2.2}$$

More details on the r-norm and its dual can be found in [10].

3. Data uncertainty and optimization of robustness

In this section, we explicit stability radius for a quadratic optimization problem subjected to uncertainties. Moreover, we show that its robust counterpart in the sense of robustness optimization is tractable optimization problem.
In this paper, we define data uncertainties entries by an affine interpolation between nominal data as follows:

$$B_u = B_0 + u_1 B_1 \ ; \ b_u = b_0 + u_2 b_1 \text{ and } \beta_u = \beta_0 + u_3 \beta_1.$$

Noting that $A, B_0, B_1 \in \mathcal{S}_n(\mathbb{R})$; $a, b_0, b_1 \in \mathbb{R}^n$; $u = (u_1, u_2, u_3)^T \in \mathbb{R}^3$ and $\beta_0, \beta_1 \in \mathbb{R}$.
Let $f : \mathbb{R}^n \times \mathbb{R}^3 \longrightarrow \mathbb{R} \bigcup \{+\infty\}$ defined by:

$$f(x, u) = \begin{cases} p(x) & \text{if } q(x, u) \leqslant 0, \\ +\infty & \text{else.} \end{cases}$$

Therefore, (1.2) can be rewritten as:

$$\begin{array}{rl} \text{Minimize} & f(x, u), \\ \text{s.t.} & x \in \mathbb{R}^n, \end{array} \tag{3.1}$$

where $u \in \mathbb{R}^3$. Hence problem (1.2) and the problem (3.1) are equivalent.

3.1. **Robust counterpart of (3.1) and stability radius.** In this subsection, we recall the robust counterpart formulation of the problem (1.2) in the sense of optimization of robustness following the work of Barro et al. [1]. Given a tolerance threshold $\alpha \in \mathbb{R}$, we define the satisfaction set of a decision $x \in \mathbb{R}^n$ with respect to α by

$$S_\alpha(x) := [f(x,.) \leqslant \alpha] = \left\{u \in \mathbb{R}^3 \;\mid\; f(x,u) \leqslant \alpha\right\}. \tag{3.2}$$

Let us denote by $\overline{u}$ the nominal data. A decision variable x in $\mathbb{R}^n$ is said to be robust admissible if $\overline{u}$ belongs to the satisfaction set of x i.e. $f(x,\overline{u}) \leqslant \alpha$. For a tolerance threshold $\alpha > \inf\{f(x,\overline{u}) | x \in \mathbb{R}^n\}$, the stability radius of a decision x in $\mathbb{R}^n$, related to α and the nominal data $\overline{u}$ is the distance of $\overline{u}$ to the complementary in $\mathbb{R}^3$ of the corresponding satisfaction set of x (given in (3.2)). Thus, this stability radius is denoted $r_\alpha(x,\overline{u})$ and is given by the formula:

$$r_\alpha(x,\overline{u}) \;:=\; d_{\|\cdot\|_{(r)}}(\overline{u}, S_\alpha(x)^c) = \inf\left\{\|\overline{u} - u\|_{(r)} \mid u \in [f(x,.) > \alpha]\right\}. \tag{3.3}$$

The robust counterpart of (3.1) in the sense of optimization of robustness is given by:

$$\begin{array}{rl} \text{Maximize} & r_\alpha(x,\overline{u}), \\ \text{s.t.} & x \in \mathbb{R}^n. \end{array} \tag{3.4}$$

3.2. **Stability radius computation.** In this subsection, we give an explicit formula of the stability radius of $S_\alpha(x)$ by using Lemma 2.1.

The following result explicit the stability radius of $S_\alpha(x)$ by using Lemma 2.1.

Proposition 3.1. *Assume that $\beta_1 \neq 0$. Then for all $x \in \mathbb{R}^n$, the stability radius of $S_\alpha(x)$ is given by*

$$r_\alpha(x,\overline{u}) = \begin{cases} \dfrac{(-q(x,\overline{u}))^+}{\|X(x)\|^\star_{(r)}} & \text{if} \quad p(x) \leqslant \alpha, \\ 0 & \text{if} \quad p(x) > \alpha, \end{cases}$$

Where: $X(x) = [\frac{1}{2}x^T B_1 x, b_1^T x, \beta_1]^T$.

Proof. If $p(x) > \alpha$ then $S_\alpha(x) = \emptyset$ and $r_\alpha(x, \overline{u}) = d_{\|.\|_{(r)}}(\overline{u}, \mathbb{R}^3) = 0$.
Else, one has:

$$\begin{aligned}
S_\alpha(x) &= \left\{ u \in \mathbb{R}^3 \;\middle|\; f(x,u) \leqslant \alpha \right\} \\
&= \left\{ u \in \mathbb{R}^3 \;\middle|\; \begin{cases} p(x) \leqslant \alpha \\ q(x,u) \leqslant 0 \end{cases} \right\} \\
&= \left\{ u \in \mathbb{R}^3 \;\middle|\; \begin{cases} p(x) \leqslant \alpha \\ \frac{1}{2} x^T B_u x + b_u^T x + \beta_u \leqslant 0 \end{cases} \right\} \\
&= \left\{ u \in \mathbb{R}^3 \;\middle|\; \begin{cases} p(x) \leqslant \alpha \\ \frac{u_1}{2} x^T B_1 x + u_2 b_1^T x + u_3 \beta_1 \leqslant -\frac{1}{2} x^T B_0 x - b_0^T x - \beta_0 \end{cases} \right\} \\
&= \left\{ u \in \mathbb{R}^3 \;\middle|\; \begin{cases} p(x) \leqslant \alpha \\ u \in [\langle ., X(x) \rangle \leqslant -q(x,0)] \end{cases} \right\}.
\end{aligned}$$

Hence: $r_\alpha(x, \overline{u}) = d_{\|.\|_{(r)}}(\overline{u}, S_\alpha(x)^c) = d_{\|.\|_{(r)}}\Big(\overline{u}, \big[\langle ., X(x) \rangle > -q(x,0)\big]\Big)$.

By applying Ascoli formula (2.1), we obtain:

$$\begin{aligned}
r_\alpha(x, \overline{u}) &= \frac{\big(-q(x,0) - \langle \overline{u}, X(x) \rangle\big)^+}{\|X(x)\|_{(r)}^\star} \\
&= \frac{\big(-q(x,\overline{u})\big)^+}{\|X(x)\|_{(r)}^\star}.
\end{aligned}$$

□

3.3. **Solvability of robust counterpart.** It follows from Proposition 3.1 that (3.4) can be rewritten such as:

$$\begin{array}{rc}
\text{Maximize} & \dfrac{\big(-q(x,\overline{u})\big)^+}{\|X(x)\|_{(r)}^\star}, \\
\text{s.t.} & x \in X_\alpha
\end{array} \tag{3.5}$$

- where: $X_\alpha = \left\{ x \in \mathbb{R}^n \;\middle|\; p(x) \leqslant \alpha \right\}$.

The next proposition proves the solvability of problem (3.4).

Proposition 3.2. *Assume that A is definite positive and $\beta_1 \neq 0$. If $\alpha + \frac{1}{2} a^T A^{-1} a \geqslant 0$, then problem (3.4) has at least an optimal solution. Else, (3.4) is ill-posed.*

Proof. Assume that $\alpha + \frac{1}{2}a^T A^{-1} a \geqslant 0$ and set $\widehat{x} = -A^{-1}a$.
Remark that

$$\begin{aligned} p(x) &= \frac{1}{2}x^T Ax + a^T x \\ &= \frac{1}{2}x^T Ax + a^T x + \frac{1}{2}a^T A^{-1}a - \frac{1}{2}a^T A^{-1}a \\ &= \frac{1}{2}x^T Ax + a^T A^{-1}Ax + \frac{1}{2}a^T A^{-1}AA^{-1}a - \frac{1}{2}a^T A^{-1}a \\ &= \frac{1}{2}x^T Ax - \widehat{x}^T Ax + \frac{1}{2}\widehat{x}^T A\widehat{x} - \frac{1}{2}a^T A^{-1}a \\ &= \frac{1}{2}(x-\widehat{x})^T A(x-\widehat{x}) - \frac{1}{2}a^T A^{-1}a. \end{aligned}$$

So,

$$X_\alpha = \left\{ x \in \mathbb{R}^n \;\middle|\; \frac{1}{2}(x-\widehat{x})^T A(x-\widehat{x}) - \frac{1}{2}a^T A^{-1}a - \alpha \leqslant 0 \right\}.$$

Since A is definite positive, there exists a definite positive matrix denoted by $A^{\frac{1}{2}}$ such that $(A^{\frac{1}{2}})^2 = A$. This implies that

$$X_\alpha = \left\{ x \in \mathbb{R}^n \;\middle|\; \frac{1}{2}\|A^{\frac{1}{2}}(x-\widehat{x})\|_2^2 \leqslant \alpha + \frac{1}{2}a^T A^{-1}a \right\}.$$

Hence, X_α is bounded in $\mathbb{R}^n$, then it is a compact set. Otherwise $r_\alpha(.,\overline{u})$ is defined through a distance function which is continuous. Therefore, (3.4) has an optimal solution. □

The following result establish an equivalence between optimization of an objective function and its positive part ($\max(f, 0)$).

Proposition 3.3. *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a function, $\mathcal{X}$ a no empty subset of $\mathbb{R}^n$. Assume that there exists $x_0 \in \mathcal{X}$ such that $f(x_0) > 0$. Then the maximization of f over $\mathcal{X}$ is equivalent to the maximization of f^+ over $\mathcal{X}$.*

Proof. Let $\overline{x} \in \mathcal{X}$ be a maximizer of f over $\mathcal{X}$. So, for all $x \in \mathcal{X}$, one has:

$$\begin{aligned} f(\overline{x}) \geqslant f(x) &\implies f^+(\overline{x}) \geqslant f^+(x) \\ &\implies \overline{x} \text{ maximizes } f^+ \text{ over } \mathcal{X}. \end{aligned}$$

Conversely, if $\overline{x} \in \mathcal{X}$ maximizes f^+ over $\mathcal{X}$ then for each $x \in \mathcal{X}$, we get:

$$\begin{aligned} f^+(\overline{x}) \geqslant f^+(x) &\implies f^+(\overline{x}) \geqslant f^+(x_0) = f(x_0) > 0 \\ &\implies f(\overline{x}) = f^+(\overline{x}) \geqslant f^+(x) \\ &\implies f(\overline{x}) \geqslant f(x). \end{aligned}$$

This means that $\overline{x}$ is a maximizer of f over $\mathcal{X}$. □

Proposition 3.4. *Let us consider the following optimization problem*

$$\begin{aligned} \textit{Maximize} \quad & h(x), \\ \textit{s.t.} \quad & x \in X_\alpha, \end{aligned} \tag{3.6}$$

where: $h(x) = \dfrac{-q(x,\overline{u})}{\|X(x)\|^{\star}_{(r)}}$.

Suppose that there exists $x_0 \in X_\alpha$ such that $h(x_0) > 0$.

Then, the stability radius optimization problem (3.4) corresponds to the problem (3.6).

Proof. That is an application of Proposition 3.3. □

Remark 3.5. With the relation (2.2), the cost function h of the problem (3.6) in the Proposition 3.4 above can be rewritten under the form

$$h(x) = \min\left(\frac{-q(x,\overline{u})}{\|X(x)\|_\infty}, \frac{-rq(x,\overline{u})}{\|X(x)\|_1}\right). \tag{3.7}$$

4. Example and numerical Methods

4.1. **Example.** Setting

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B_0 = \begin{pmatrix} 1 & 0.2 \\ 0.2 & 1 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 3 & -4 \\ -4 & 7 \end{pmatrix}, \quad a = \begin{pmatrix} -2 \\ 0 \end{pmatrix},$$

$$b_0 = b_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \beta_0 = 0 \quad \beta_1 = -20, \quad \text{and } r = 1.$$

In this case, we set $x = (x_1, x_2)^T \in \mathbb{R}^2$ and $p(x) = \frac{1}{2}x^T Ax + a^T x = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 - 2x_1$.
Moreover, we have $B_{u_1} = \begin{pmatrix} 1+3u_1 & 0.2-4u_1 \\ 0.2-4u_1 & 1+7u_1 \end{pmatrix}$.
So, for $u = (u_1, u_2, u_3)^T \in \mathbb{R}^3$, one has:
$q(x,u) = \frac{1}{2}\left[(1+3u_1)x_1^2 + 2(0.2-4u_1)x_1x_2 + (1+7u_1)x_2^2 - 20u_3\right]$.
Let us choose the nominal data $\overline{u} \in \mathbb{R}^3$.
We can take $\overline{u}_1 = 5.10^{-2}$, $\overline{u}_2 = 0$, $\overline{u}_3 = 1.2.10^{-1}$ and $\overline{u} = (\overline{u}_1, \overline{u}_2, \overline{u}_3)^T$.
This implies that $q(x,\overline{u}) = \frac{1}{2}\left[1.15x_1^2 + 1.35x_2^2\right] - 2.4$ and

$$f(x,\overline{u}) = \begin{cases} \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 - 2x_1 & \text{if } \frac{1}{2}\left[1.15x_1^2 + 1.35x_2^2\right] - 2.4 \leqslant 0 \\ +\infty & \text{else.} \end{cases}$$

For all $x \in \mathbb{R}^2$ such that $q(x,\overline{u}) \leqslant 0$, we get $f(x,\overline{u}) = \frac{1}{2}(x_1-2)^2 + \frac{1}{2}x_2^2 - 2 \geqslant -2$.
By setting $\overline{x} = (2,0)^T$, we get $q(\overline{x},\overline{u}) = -0.1 < 0$.
Let us remark that $\inf\limits_{\mathbb{R}^2} f = f(\overline{x},\overline{u}) = -2$. We can take α such that $\alpha > -2$.
Now, we search α such that $\alpha_p = \alpha + \frac{1}{2}a^T A^{-1} a = \alpha + 2 \geqslant 0$. To find $\alpha_p \geqslant 0$ we can take $\alpha \in \,]-2; +\infty[$.
The vector $X(x)$ is defined as $X(x) = \left(\frac{1}{2}\left[3x_1^2 - 8x_1x_2 + 7x_2^2\right], 0, -2\right)^T$.
Let us determine X_α. This set is define such as

$$\begin{aligned} X_\alpha &:= \left\{x \in \mathbb{R}^2 \mid \frac{1}{2} \parallel A^{\frac{1}{2}}x - \widehat{X} \parallel_2^2 \leqslant \alpha_p\right\} \\ &= \left\{x \in \mathbb{R}^2 \mid \frac{1}{2}\left[(x_1-2)^2 + x_2^2\right] \leqslant \alpha_p\right\}, \end{aligned}$$

- where: $\widehat{X} = (2,0)^T$.

For $r = 1$, from Definition 2.2, equality (2.2) and Proposition 3.4, we get:

$$\forall x \in \mathbb{R}^2, \quad h(x) = \frac{-q(x,\overline{u})}{\|X(x)\|^{\star}_{(r)}} = \frac{-q(x,\overline{u})}{\|X(x)\|_1}.$$

By taking $x^0 = (x^0_1, x^0_2) = (1,-1)^T$, we check that :

- $x^0 \in X_\alpha$. Indeed : $\dfrac{1}{2}\left[(x^0_1-2)^2 + (x^0_2)^2\right] = 1 \leq \alpha_p$.
- $h(x^0) > 0$. Indeed : $q(x^0,\overline{u}) = \frac{1}{2}\left[1.15(x^0_1)^2 + 1.35(x^0_2)^2\right] - 2.4 = -1.15 < 0$.
- Hence $h(x) = \frac{-q(x^0,\overline{u})}{\|X(x^0)\|_1} > 0$.

Then, the stability radius optimization problem of this example corresponds to the following problem (4.1) :

$$\begin{array}{rc} \text{Maximize} & h(x) = \frac{-q(x,\overline{u})}{\|X(x)\|_1}, \\ \text{s.t.} & x \in X_\alpha. \end{array} \tag{4.1}$$

For all $x = (x_1,x_2)^T \in \mathbb{R}^2$: $\|X(x)\|_1 \geqslant 2$; and $-q(x,\overline{u}) \leqslant 2.4$.

$$\begin{aligned} -q(x,\overline{u}) \leqslant 2.4 \quad &\Rightarrow \quad \frac{-q(x,\overline{u})}{\| X(x) \|_1} \leq \frac{2.4}{\| X(x) \|_1} \\ &\Rightarrow \quad \frac{-q(x,\overline{u})}{\| X(x) \|_1} \leqslant 1.2 \\ &\Rightarrow \quad \max_{x\in\mathbb{R}^2} h(x) \leqslant 1.2. \end{aligned}$$

Let us note that $\tilde{x} = (0,0)^T \in X_\alpha$ and $h(\tilde{x}) = \dfrac{-q(\tilde{x},\overline{u})}{\| X(\tilde{x}) \|_1} = 1.2$.

So $\tilde{x}$ is a solution of (4.1).

In summary:

- A robust solution of (4.1) is $\tilde{x}$ and the corresponding (maximum) value of the criterion is $h(\tilde{x}) = 1.2$ for some threshold $\alpha > -2$.
- The satisfaction set of $\tilde{x}$ at α is :

$$S_\alpha(\tilde{x}) = \left\{ u \in \mathbb{R}^3 \;\middle|\; \begin{cases} p(\tilde{x}) \leqslant \alpha \\ q(\tilde{x},u) \leqslant 0 \end{cases} \right\},$$

  - where: $p(\tilde{x}) = 0$ and $\forall u = (u_1,u_2,u_3)^T$, $\;q(\tilde{x},u) = -20u_3$.
- Its follows that:
  - If $\alpha \in]-2;0[$ then $S_\alpha(\tilde{x}) = \emptyset$.
  - If $\alpha \in \mathbb{R}_+$ then $S_\alpha(\tilde{x}) = \mathbb{R}^2 \times \mathbb{R}_+$. In this case the stability radius is

$$r_\alpha(\tilde{x},\overline{u}) = h(\tilde{x}) = 1.2.$$

Remark 4.1. The problem (3.6) is tractable but for some other values of r with the r-norm, it could be painful to compute the solution by "hand". In these situations, we can employ some scientific computation softwares to solve (3.6) given in Proposition 3.4. Otherwise, if we are interested in theoretical algorithmic resolution procedures, we propose the following approaches which could be efficient on this type of problem.

4.2. **Algorithmic approach of optimization of robustness.** In this subsection, we present algorithm derived from those proposed by Crouzeix et al. in [13, 14, 15] in order to tract numerical solutions of (3.6). This last method is a generalization of Dinkellbach [16] iterative procedure.
Let us consider the problem

$$\begin{aligned} &\text{Maximize} \quad \min\Big(-q(x,\overline{u}) - \theta \parallel X(x) \parallel_\infty ,\ -rq(x,\overline{u}) - \theta \parallel X(x) \parallel_1 \Big) \\ &\text{s.t.} \qquad\qquad\qquad x \in X_\alpha . \end{aligned} \tag{4.2}$$

Setting

$$\overline{\theta} \ := \ \max(3.6) = \max_{x\in X_\alpha} \min\left(\frac{-q(x,\overline{u})}{\parallel X(x) \parallel_\infty} , \frac{-rq(x,\overline{u})}{\parallel X(x) \parallel_1} \right)$$

and

$$\begin{aligned} H:\ & \mathbb{R} \longrightarrow \mathbb{R} \\ & \theta \longmapsto \max_{x\in X_\alpha} \min\Big(-q(x,\overline{u}) - \theta \parallel X(x) \parallel_\infty ,\ -rq(x,\overline{u}) - \theta \parallel X(x) \parallel_1 \Big). \end{aligned}$$

- $\mathcal{S}$ is the set of optimal solutions of (3.6).
- $\mathcal{S}(\theta)$ is the set of optimal solutions of (4.2).

This following result gives a relationship between the robust counterpart and auxiliary problem (4.2).

Proposition 4.2. *The following assumptions are satisfied:*

(i) *H is finite, continuous and decreasing function on $\mathbb{R}$.*
(ii) *$H(\theta) > 0$ if and only if $\theta < \overline{\theta}$.*
(iii) *The sets $\mathcal{S}$ and $\mathcal{S}(\theta)$ are no empty and $\mathcal{S} = \mathcal{S}(\theta)$. $\overline{\theta}$ is finite and $H(\overline{\theta}) = 0$. Moreover, if $H(\theta) = 0$ then $\theta = \overline{\theta}$.*

4.3. **Application of Crouzeix-Ferland-Schaible algorithm.** This procedure consist to solve the equation $H(\theta) = 0$. Let us introduce the following iterative method which is subdivided into four steps to solve (3.6) by using (4.2).

Step 1: Start $x^0 \in X_\alpha$. Setting $\theta_1 = \min\left(\dfrac{-q(x^0,\overline{u})}{\|X(x^0)\|_\infty} , \dfrac{-rq(x^0,\overline{u})}{\|X(x^0)\|_1} \right)$.
$k = 1$.
Step 2: Solve ($\mathcal{P}_{\theta_k}$). Let $x^k \in X_\alpha$.
Step 3: If $H(\theta_k) = 0$, then stop. So, x^k and x^{k-1} are optimal solutions of (3.6) and $\theta_k = \overline{\theta}$.
Step 4: If $H(\theta_k) \neq 0$, setting $\theta_{k+1} = \min\left(\dfrac{-q(x^k,\overline{u})}{\|X(x^k)\|_\infty} , \dfrac{-rq(x^k,\overline{u})}{\|X(x^k)\|_1} \right)$.
Substitute k by $k+1$ and return in step 2.

Proposition 4.3. *Assume that the sequence (θ_k) is note finite. Therefore, (θ_k) converges to $\overline{\theta}$ and all convergent subsequence of (x^k) converges to an element of $\mathcal{S}$.*

Remark 4.4. The entire proof of Proposition 4.2 and the Proposition 4.3 can be found in the works Crouzeix et al., see for instance [13].

Proposition 4.5. *Under the assumption of the Proposition 3.4, the stability radius optimization problem (3.4) is equivalent to the minimization problem given by:*

$$\begin{array}{rl} \text{Minimize} & h(x), \\ s.t. & x \in X_\alpha, \end{array} \tag{4.3}$$

where:

$$h(x) = \frac{1}{2}\left[\frac{q(x,\overline{u})}{\|X(x)\|_\infty} + \frac{rq(x,\overline{u})}{\|X(x)\|_1} + \left|\frac{q(x,\overline{u})}{\|X(x)\|_\infty} - \frac{rq(x,\overline{u})}{\|X(x)\|_1}\right|\right]. \tag{4.4}$$

Remark 4.6. This equivalence is in term of their optimal solution but their cost function values are opposite.

Proof. We get this equivalence between (3.4) and (4.3) by using the following relations (4.5) and (4.6):

$$\min_K(f) = -\max_K(-f) \quad \text{and} \quad \max_K(f) = -\min_K(-f). \tag{4.5}$$

$$\max(x,y) = \frac{x+y+|x-y|}{2} \quad \text{and} \quad \min(x,y) = \frac{x+y-|x-y|}{2}. \tag{4.6}$$

□

Solution method of the problem (4.3). The optimization problem (4.3) is non differentiable (at least at 0) cause of the presence of the norm functions and absolute value function (which is also a norm on $\mathbb{R}$). Then the proper approach is the using of subgradient one or the Derivative Free Optimization (DFO) methods. In addition, in order to speed up the convergence rate of quadratic optimization problem, see the paper [24].

5. Conclusion

In this work, we have studied the tractability of the robust counterpart of problem (1.2) by using a r-norm. We have shown that stability radius optimization approach can be used to take account uncertainties in optimization models like robust optimization and stochastic optimization. In stability radius optimization, we search solutions which are feasible and their optimal values do not overstep the line of a certain threshold. In this work, we have demonstrated that the stability radius analysis makes it is possible to get a tractable robust counterpart of an uncertain quadratic optimization problem. We have also illustrated the optimization of robustness approach with an example and presented some numerical methods which can be used to solve the considered problem.

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[1] Ecole Polytechnique de Ouagadougou (EPO), Ouagadougou, Burkina Faso.
Email address: satafa.sanogo@yahoo.fr

[2] Université Nazi Boni (UNB), Bobo-Dioulasso, Burkina Faso.
Email address: yesmzongo@gmail.com
Email address: mousbarro@yahoo.fr
Email address: traore.sado@yahoo.fr