

PROLATE SPHEROIDAL WAVELETS AND MULTIDIMENSIONAL CHROMATIC SERIES EXPANSIONS

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ABSTRACT. Chromatic series were originally introduced for bandlimited functions. The n^{th} chromatic derivative of an analytic function is a linear combination of k^{th} ordinary derivatives with $0 \leq k \leq n$, where the coefficients of linear combination are based on a suitable system of orthogonal polynomials. Chromatic derivative and series expansions of bandlimited functions have been used as a replacement for Taylor's series and they have been shown to be more useful in practical signal processing applications than Taylor series. In this paper we have shown that the theory can be extended to prolate spheroidal wavelet series that than combine chromatic series with sampling series in higher dimensions. The multidimensional case has much richer structure than in the univariate case and will find more applications in image processing and analysis.

1. INTRODUCTION

The theory of chromatic derivatives and series expansions have been introduced by A. Ignjatovic in [6,7] as an alternative representation to Taylor series and they have been shown to be more useful in practical applications than Taylor series (see [3,4,5,6,7,8,9,10]).

An entire function f has a Taylor series expansion of the form $f(t) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} t^n$. This series clearly is not globally convergent since it uses the values of f and all its derivatives at a single point. Unlike the sampling series representation [13].

$$f(t) = \sum_{n=-\infty}^{\infty} f(n) \frac{\sin \pi(t-n)}{\pi(t-n)} \quad (1.1)$$

of any function $f \in PW_{\pi}$ (The Paley-Wiener space of functions bandlimited to $[-\pi, \pi]$) plays an important role in signal processing. The expansion (1.1) may be viewed as a global expansion because it uses function values at infinitely many points uniformly distributed on the real line. The Taylor series has very limited applications because a truncated Taylor series is a polynomial and not bandlimited. The chromatic derivatives are basically related to the Fourier transformation. The main idea is that, under the Fourier transformation, differentiation in

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the time domain corresponds to multiplication by powers of w in the frequency domain. Indeed, the n^{th} chromatic derivative $K^n[f](t_o)$ of an analytic function $f(t)$ at t_o is a linear combination of the ordinary derivatives $f^{(k)}(t_o)$, $0 \leq k \leq n$, where the coefficients of the combination are based on systems of orthogonal polynomials.

The Ignjatovic theory is a local theory and requires knowledge of the function (signal) only in a neighborhood of the origin, may be viewed as an alternative to the Shannon sampling theorem which is used to approximate bandlimited functions, but involves the values of the function at infinitely many different points. Interestingly, in contrast to the Taylor series, the partial sums of the sampling series converges uniformly, but are not local since they require knowledge of the function at all the integers.

The second approach [12] to study the local theory is the use of prolate spheroidal wave functions (PSWFs). These do not strictly lead to chromatic derivatives since they are not polynomials. However, They are entire functions whose expansion converges very rapidly and can be approximated by polynomials of low degree. These functions are solutions of an integral equation and solve the problem of finding the bandlimited function of unit total energy whose energy on a finite interval is maximized. In common with polynomials they can be extended to the entire real line and even to the complex plane. However, in contrast to polynomials, they are bounded on the real line and are entire functions of exponential type in the complex plane. These functions forms an orthogonal basis both of the space of bandlimited functions and of L^2 on the concentration interval. The aim of this paper is to show how both approaches may be combined in higher dimensions.

2. CONSTRUCTION OF CHROMATIC DERIVATIVES

Let $W(w)$ be a non-negative weight function such that all of its moments are finite, i.e., such that

$$\mu_n = \int_{-\infty}^{\infty} w^n W(w) dw < \infty.$$

Let $\{P_n(w)\}$ be the family of polynomials orthonormal with respect to $W(w)$:

$$\int_{-\infty}^{\infty} P_n(w) P_m(w) W(w) dw = \delta(m - n),$$

and let $K^n(f) = P_n\left(-i\frac{d}{dt}\right)(f)$ be the corresponding linear differential operators obtained from $P_n(w)$ by replacing w^k ($0 \leq k \leq n$) with $\left(-i\frac{d}{dt}\right)^k$. These differential operators are called chromatic derivatives associated with the family of orthogonal polynomials $\{P_n(w)\}$ because they preserve the spectral features of a bandlimited signal.

Let $\varphi(z)$ be the Fourier transform of the weight function $W(w)$,

$$\varphi(w) = \int_{-\infty}^{\infty} e^{iws} W(s) ds.$$

Since $\varphi^{(n)}(0) = i^n \mu_n$, $\varphi(z)$ is analytic around the origin. It is shown in [2] that $\limsup(\mu_n/n!)^{1/n} < \infty$ if and only if

$$\int_{-\infty}^{\infty} e^{c|w|} W(w) dw < \infty$$

for some $c > 0$ and in this case $\varphi(z)$ is analytic in a strip $S_d(c/2) = \{z : \text{Im}(z) < c/2\}$.

The chromatic derivative is a generalization of ordinary derivative and leads to a series referred to as the chromatic expansion of a function $f \in C^\infty$ as

$$f(z) \sim \sum_{n=0}^{\infty} K^n(f)(u) K^n(\varphi)(z - u). \quad (2.1)$$

If $f(z)$ is analytic in the strip $S_d(c/2)$ and $\sum_{n=0}^{\infty} |K^n(f)(0)|^2$ converges, then for all $u \in R$, the series (2.1) converges to $f(z)$ uniformly in every strip $\{z : |\text{Im}(z)| < c/2 - \varepsilon\}$, for every $\varepsilon > 0$.

3. ORTHOGONAL POLYNOMIALS AND MULTIDIMENSIONAL CHROMATIC DERIVATIVES

Notations:

$$\begin{aligned} \alpha &= (\alpha_1, \dots, \alpha_d) \in N_0^d, \alpha! = \alpha_1! \alpha_2! \dots \alpha_d!, |\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_d, \\ \delta_{\alpha, \beta} &= \delta_{\alpha_1 \beta_1} \dots \delta_{\alpha_d \beta_d}, x = (x_1, \dots, x_d) \in R^d, x^\alpha = x_1^{\alpha_1} \dots x_d^{\alpha_d}, \end{aligned}$$

Here N_0 denote the set of non-negative integers and $|\alpha|$ is the degree of monomial x^α .

A polynomial P in d variables is a linear combination of the form

$$P(x) = \sum_{\alpha} c_{\alpha} x^{\alpha}, c_{\alpha} \text{ are complex numbers.}$$

We denote the set of all polynomials in d variables by $\prod^d$ and the set of all polynomials of degree at most n by $\prod_n^d$. The set of all homogeneous polynomials of degree n will be denoted by P_n^d

$$P_n^d = \left\{ P : P(x) = \sum_{|\alpha|=n} c_{\alpha} x^{\alpha} \right\}, \left\{ \text{dimension of } P_n^d \text{ is } r_n^d = \binom{n+d-1}{n} \right\}.$$

Every polynomial in d variables can be written as a linear combination of homogeneous polynomials

$$P(x) = \sum_{k=0}^n \sum_{|\alpha|=k} c_{\alpha} x^{\alpha}.$$

In one variable, monomials are ordered according to their degrees as $1, x, x^2, \dots$; but in several variables such order does not exist. To overcome this problem we shall use the lexicographic order, i.e., $\alpha > \beta$ if the first nonzero entry in the difference $\alpha - \beta = (\alpha_1 - \beta_1, \alpha_2 - \beta_2, \dots, \alpha_d - \beta_d)$ is positive.

Definition 3.1. If $\langle, \rangle$ is an inner product on $\prod^d$, a polynomial P is orthogonal to a polynomial Q if $\langle P, Q \rangle = 0$. A polynomial P is orthogonal to a polynomial if P is orthogonal to all polynomials of lower degree, i.e., $\langle P, Q \rangle = 0$ for all $Q \in \prod^d$ with $\deg Q < \deg P$.

We define a linear functional on $\prod^d$ by

$$L(x^\alpha) = s_\alpha, \alpha \in N_o^d$$

where $s = (s_\alpha) : N_o^d \rightarrow R$ is a multi-sequence. The elements of the set $\{\alpha \in N_o^d : |\alpha| = n\}$ be arranged as $\alpha^{(1)}, \alpha^{(2)}, \dots, \alpha^{(r_n)}$ according to the lexicographic order.

Define the vector moments $s_k = L(x^k); x^k = x^k = (x^\alpha)_{|\alpha|=k} = (x^{\alpha(j)})_{j=1}^{r_k}$ is a column vector whose elements are the monomials x^α for $|\alpha| = k$, arranged in lexicographic order.

Also, we define the matrix

$$M_{n,d} = (S_{(k)+(j)})_{k,j=0}^n \text{ and } \Delta_{n,d} = \det M_{n,d}$$

The elements of moment matrix $M_{n,d}$ are $L(x^{\alpha+\beta})$ for $|\alpha| \leq n, |\beta| \leq n$. If $\{P_\alpha\}_{|\alpha|=n}$ is a sequence of polynomials in $\prod_n^d$ we get the column polynomial vector $P_n = (P_{\alpha(1)}, \dots, P_{\alpha(r_n)})^T$, where $\alpha^{(1)}, \dots, \alpha^{(r_n)}$ is the lexicographic order in $\{\alpha \in N_o^d : |\alpha| = n\}$.

Let L be a moment functional. A sequence of polynomials $\{P_\alpha\}_{|\alpha|=n}$ in $\prod_n^d$ is said to be orthogonal with respect to L if

$$L(x^m P_n^T) = 0 \text{ for } n > m, \text{ and } L(x^n P_n^T) = s_n,$$

where s_n is an invertible matrix of size $r_n^d \times r_n^d$.

Definition 3.2. A moment linear functional L is said to be positive definite if $L(p^2) > 0$ for all $p \in \prod^d, p \neq 0$. The associated sequence $\{s_\alpha\}$ with L will be also called positive definite.

Let M denote the set of non-negative Borel measures on R^d having moments of all orders. We assume that $\mu \in M$ is absolutely continuous so that $d\mu = W(x)dx$, with W being non-negative so that $L_W(f) = \int f(x)W(x)dx$ is positive definite. We also assume that for some $c > 0$

$$\int_{R^d} e^{c\|x\|} W(x)dx < \infty, \quad (3.1)$$

so that polynomials are dense in $L^2(\mu)$ [1,p.74]. Condition (3.1) is satisfied, if μ is compactly supported.

If $P(x_1, \dots, x_d)$ is a polynomial in $x_1, \dots, x_d$ then

$$P\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_d}\right) \equiv P\left(\frac{\partial}{\partial x}\right)$$

and

$$\frac{\partial^{|\alpha|}}{\partial x^\alpha} \equiv \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_d^{\alpha_d}}.$$

Let $\{P_\alpha^n\}_{n=0}^\infty$ denotes the sequence of orthonormal polynomials with respect to L_w . Let $L_W^2(R^d)$ denote the space of all square integrable functions with respect to weight functions W .

Let $z = (z_1, z_2, \dots, z_d) \in C^d$ and define the inner product

$$\langle z, w \rangle = \sum_{k=1}^d z_k \overline{w_k}, \quad z, w \in C^d \quad \text{so that} \quad \|z\|^2 = \sum_{k=1}^d |z_k|^2.$$

If $z_k = x_k + iy_k$, $\|Rz\|^2 = \sum_{k=1}^d x_k^2$, $\|Im(z)\|^2 = \sum_{k=1}^d y_k^2$ and for every real $a > 0$, $S_d(a) = \{z \in C^d : \|Im(z)\| < a\}$.

Definition 3.3. Let $f : C^d \rightarrow C_t$; the n^{th} chromatic derivative $K_\alpha^n(f)$ of $f(z)$ with respect to the polynomials $\{P_\alpha^n\}$ is defined as

$$K^n(f) = \kappa_n \left(-i \frac{\partial}{\partial z} \right) (f),$$

where κ_n is the column vector.

It is clear from Definition 3.3 that n^{th} chromatic derivative of f is a column vector with r_n^d components, with each component a linear combination of partial derivatives and if $w \in R^d$ is fixed, then

$$K_\alpha^n(e^{i\langle w, z \rangle}) = P_\alpha^n(w) e^{i\langle w, z \rangle}.$$

Now let $\varphi_m(w) \in L_W^2(R^d)$ with a Fourier transform given by $\widehat{\varphi_m}(w) = \varphi_m(w) e^{i\langle w, u \rangle}$. Define a function $f\varphi_m : S_d(c/2) \rightarrow C$ by a $W(w)$ -weighted inverse Fourier transform of φ_m :

$$f\varphi_m(z) = \int_{R^d} \varphi_m(w) e^{i\langle w, z \rangle} W(w) dw. \quad (3.2)$$

Then $f\varphi_m(z)$ is analytic on $S_d(c/2)$ and for all n and $z \in S_d(c/2)$

$$K^n[f\varphi_m](z) = \int_{R^d} \kappa_n(w) \varphi_m(w) e^{i\langle w, z \rangle} W(w) dw.$$

It has been shown that [2] if condition (3.1) holds, then $\{\kappa_n(w)\}$ is a complete system in $L_W^2(R^d)$ and the expansion in terms of these polynomials of a appropriate function $\varphi_m(w) e^{i\langle w, u \rangle}$ converges:

$$\varphi_m(w) e^{i\langle w, u \rangle} = \sum_{n=0}^{\infty} [K^n[f\varphi_m](u)]^T \kappa_n(w)$$

for some fixed $u \in S_d(c/2)$ and $\varphi_m(w) e^{i\langle w, u \rangle} \in L_W^2(R^d)$.

Definition 3.4. For $z \in S_d(c/2)$ set

$$\Psi(z) = \int_{R^d} e^{i\langle w, z \rangle} W(w) dw$$

and

$$K_\alpha^n(\Psi(z)) = \Psi_\alpha^n(z) = \int_{R^d} P_\alpha^n(w) e^{i\langle w, z \rangle} W(w) dw$$

Let Λ_W^2 denote the vector space of functions $f : S_d(c/2) \rightarrow C$ which are analytic on $S_d(c/2)$ and satisfy $\sum_{n=0}^{\infty} \sum_{|\alpha|=n} |K_\alpha^n[f\varphi_m]|^2 < \infty$. The mapping

$$f(z) = \varphi_{mf}(w) = \sum_{n=0}^{\infty} [K^n[f](0)]^T \kappa_n(w) \quad (3.3)$$

is an isomorphism between the vector spaces Λ_W^2 and $L_W^2(R^d)$, and its inverse is given by (3.2). For $f \in \Lambda_W^2$ we denote $\varphi_{mf}(w)$ by $F_W[f](w)$ then for $z \in S_d(c/2)$, we have

$$f(z) = \int_{R^d} F_W[f](w) e^{i\langle w, z \rangle} W(w) dw. \quad (3.4)$$

Then formally we find the chromatic series expansion of a function $f \in C^\infty$ as

$$f(z) \sim \sum_{n=0}^{\infty} [K^n(f)(u)]^T \Psi_n(z - u)$$

where $\Psi_n(z - u) = (\Psi_\alpha^n(z - u))_{|\alpha|=n}$. It has proved in [2] if $f \in \Lambda_W^2$ then for all $u \in R^d$ and $\varepsilon > 0$, the chromatic series of $f(z)$ converges to $f(z)$ uniformly on $S(c/2 - \varepsilon)$.

Combining (3.3) and (3.4) we get the chromatic expansion evaluated at 0 as

$$f(z) = \sum_{n=0}^{\infty} [K^n[f](0)]^T \Psi_n(z).$$

This method works for all the classical orthogonal polynomials, but is most desirable when a closed form can be found for $f\varphi_m(z)$.

4. PROLATE SPHEROIDAL WAVE FUNCTIONS AND CHROMATIC SERIES EXPANSIONS

The prolate spheroidal wave functions (PSWFs) have a combination of local and global properties that might make them a suitable generalization of chromatic expansions. PSWFs are characterized as:

- (i) the eigen functions of a differential operator arising from a Helmholtz equation on a prolate spheroid:

$$(\tau^2 - t^2) \frac{d^2 \phi_{n,\sigma,\tau}}{dt^2} - 2t \frac{d\phi_{n,\sigma,\tau}}{dt} - \sigma^2 t^2 \phi_{n,\sigma,\tau} = \mu_{n,\sigma,\tau} \phi_{n,\sigma,\tau} \quad (4.1)$$

- (1) the maximum energy concentration of a σ -bandlimited function on the interval $[-\tau, \tau]$; that is $\phi_{0,\sigma,\tau}$ is the function of total energy 1 ($= \|\phi_{0,\sigma,\tau}\|^2$) such that $\int_{-\tau}^{\tau} (f(t))^2 dt$ is maximized, $\phi_{1,\sigma,\tau}$ is the function with the maximum energy concentration among those functions orthogonal to $\phi_{0,\sigma,\tau}$ etc.. The eigen functions $\{\phi_{n,\sigma,\tau}\}$ are not polynomials, although they share many of their properties. These solutions are the PSWFs and constitute an orthogonal basis of $L^2[-\tau, \tau]$.

Let us take the orthogonal system consisting of their Fourier transform $\{\widehat{P}_n(w) \text{ on } [-\pi, \pi]\}$. Then, for any $\varphi_m(w)e^{i\langle w, u \rangle} \in L^2[-\pi, \pi]$ we have

$$\varphi_m(w)e^{i\langle w, u \rangle} = \sum_{n=0}^{\infty} \left\{ \int_{-\pi}^{\pi} \varphi_m(w)e^{i\langle w, z \rangle} W(w) \widehat{P}_n(w) \right\} \widehat{P}_n(w)$$

or

$$\langle \varphi_m(w)e^{i\langle w, u \rangle}, \widehat{P}_n(w) \rangle_W = K^n[f\varphi_m](u)$$

or

$$\varphi_m(w)e^{i\langle w, u \rangle} = \sum_{n=0}^{\infty} [K^n[f\varphi_m](u)]^T \widehat{P}_n(w),$$

then for any $\varphi_m(w) \in L^2[-\pi, \pi]$, we have

$$f(z) = \sum_{n=0}^{\infty} K^n[f\varphi_m](0) f_{\varphi_m}(z.) \quad (4.2)$$

Since the $[f\varphi_m]$ are entire functions, the chromatic derivative is given by a power series which converges very rapidly but only locally. A better choice would be to approximate the $f[\varphi_m]$ by Legendre polynomials on the interval $[-\pi, \pi]$.

5. PROLATE SPHEROIDAL WAVELETS AND CHROMATIC SERIES

The first spheroidal wave function φ_0 , the one with maximum concentration on $[-\tau, \tau]$, may be used to generate a basis of B_π by the expedient of taking its integer translates. The set of all such translates $\varphi_0(t - n)$ has been shown to be a Riesz basis of B_π [11]. In reference of wavelet theory, φ_0 is a scaling function or father wavelet and the closed linear span of $\{\varphi_0(t - n)\}$ is the space denoted as V_0 , in this case, our Paley-Wiener space B_π . A Riesz basis has a dual basis $\{\widetilde{\varphi}_0(t - n)\}$ which together with $\{\varphi_0(t - n)\}$ constitute a bi-orthogonal system.

Now we can find chromatic series by using the generalization to (4.1).

The coefficients will be given by

$$\begin{aligned} & \int_{R^d} \widehat{\phi}_0(\cdot - n) \varphi_m(w) e^{i\langle w, u \rangle} W(w) dw \\ &= \int_{R^d} \widehat{\phi}_0(w) e^{-in\langle w, z \rangle} \varphi_m(w) e^{i\langle w, u \rangle} W(w) dw \\ &= K^n[f\phi_0](n). \end{aligned}$$

Hence the chromatic series in this case is

$$f(z) = \sum_{n=0}^{\infty} K^n[f\phi_0](n) [f\widetilde{\phi}_0](z - n). \quad (5.1)$$

Thus we study two possible approaches in multidimensions based on PSWFs. The first involves the standard expansion in PSWF given by (4.1) which uses only local values for the series while the second given by (5.1) involves global values. If we combine the two, we have an over determined system, but can use some terms from (4.1) to get good local approximation and then some from (5.1) to

get global approximation. If f is highly concentrated on $[-\tau, \tau]$, the former series works better, if not, the latter works better.

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REFERENCES

1. C. Dunkl and Y. Xu, *Orthogonal Polynomials of Several Variables*, Encyclopedia of Mathematics, Cambridge University Press, 2001.
2. A. Ignjatovic and A.I. Zayed, *Multi dimensional Chromatic derivative and Series expansions*, Preprint.
3. A. Ignjatovic, *Numerical Differentiation and Signal Processing*, Kromos Technology technical report, Los Altos, California, 2001.
4. A. Ignjatovic, *Numerical Differentiation and Signal Processing*, Proc. International Conference on Information, Communications and Signal Processing (ICICS), Singapore, 2001.
5. A. Ignjatovic, *Local approximations based on orthogonal differential operators*, J. Fourier Analysis and Applications, Vol. 13, No.3, 2007.
6. A. Ignjatovic, *Signal Processor with Local signal Behavior*, US Patent 6115726 filed October 1997, issued September 2000.
7. A. Ignjatovic and N. Carlin, *Method and system of acquiring local signal behavior parameters for representing and processing a signal*, US Patent 6313778 filed July 1999, issued November 6, 2001.
8. M. Cushman and T. Herron, *The General Theory of Chromatic Derivative*, Kromos Technology report, Los Altos, California, 2001.
9. P.P. Vaidyanathan, A. Ignjatovic and S. Narsimbha, *New Sampling Expansions of Bandlimited Signals Based on Chromatic Derivatives*, Proc. 35th Asilomar Conference on Signals, Systems and Computers, Monterey California, 2001.
10. G.G. Walter and X. Shen, *Wavelets based on prolate spheroidal wave functions*, J. Fourier Analysis and Applications, **10**(2004), 1-26.
11. G.G. Walter and X. Shen, *A Sampling Expansion for non bandlimited Signals in Chromatic Derivatives*, IEEE Transactions on Signal Processing, Vol. 53(2005), 1291-1298.
12. G.G. Walter, *Chromatic series with prolate spheroidal wave functions*, J. Integral Equations and Applications, **20**, No.2(2008), 263-280.
13. A. Zayed, *Advances in Shannon's sampling Theory*, CRC Press Boca Raton, Florida, 1993.

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