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INTEGRABILITY AND CONVERGENCE OF MODIFIED SUM UNDER SOME GENERALIZED COEFFICIENT CLASS IN L^1-METRIC

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Abstract. The class $S_r^{'}$, a generalization of the class $S^{'}$ of Fourier coefficients is used to investigate the integrability and convergence of modified sum

$$h_\eta(y) = \frac{1}{2}\sum_{\mathfrak{m}=0}^{\eta}\Delta q_{\mathfrak{m}} + \sum_{\mathfrak{m}=1}^{\eta}\sum_{\jmath=\mathfrak{m}}^{\eta}\Delta q_{\jmath}\cos \mathfrak{m}y$$

under L^1-metric. The criterion for L^1-Convergence of r^{th} derivative of modified sum $h_\eta(y)$ is also obtained under the same class of Fourier coefficients.

1. Introduction

The challenge of L^1-convergence of trigonometric series through coefficients entails determining the features of coefficients which make a expansion

$$f^*(y) = \frac{q_0}{2} + \sum_{\kappa=1}^{\infty} q_\kappa \cos \kappa y \tag{1.1}$$

a Fourier series to its value f^* and $||S_\eta - f^*||_{L^1} = 0(1)$ as $\eta \to \infty$ iff $q_\eta \log \eta = 0(1)$ as $\eta \to \infty$, while S_η represents η^{th} partial sum of series (1.1).
Various researchers have established numerous theorems regarding the L^1-convergence of trigonometric expression (1.1) based upon the different classes of coefficient sequence $\{q_\eta\}$. Some of the coefficients classes are written below:

Definition 1.1. A sequence $\{q_\nu\}$ follows **convex** class if

$$\Delta^2 q_\nu \geq 0,$$

where $\Delta q_\nu = q_\nu - q_{\nu+1}$ and $\Delta^2 q_\nu = q_\nu - 2q_{\nu+1} + q_{\nu+2}$.

Definition 1.2. [1] A sequence $\{q_\nu\}$ follows **quasi-convex** class if

$$\sum_{\nu=1}^{\infty}(\nu+1)|\Delta^2 q_\nu| < \infty.$$

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Definition 1.3. [12, 14] A sequence $\{q_\nu\}$ with $q_\nu \to 0$ as $\nu \to \infty$, follows the **class S** if $\exists$ a sequence $\{Q_\nu\}$ satisfying

(i) $Q_\nu \geq Q_{\nu+1}$,
(ii) $\sum_{\nu=0}^{\infty} Q_\nu < \infty$,
(iii) $|\Delta q_\nu| \leq Q_\nu, \quad \forall \quad \nu.$

After that Shah and Boas[10, 2] defined the concept of quasi-monotone sequence as

Definition 1.4. [2, 10] A positive number sequence $\{q_\nu\}$ is called **quasi-monotone** if

$$q_{\nu+1} \leq q_\nu \left(1 + \frac{\alpha}{\nu}\right), \forall \quad \nu > \nu_0(\alpha)$$

.

By elaborating the concept of quasi-monotone sequence, Singh and Sharma [13] gave a class S' as

Definition 1.5. [13] A sequence $\{q_\nu\}$ follows **class** S' , if $\exists$ a sequence $\{Q_\nu\}$ satisfying

(i) $q_\nu \to 0$ as $\nu \to \infty$,
(ii) $\{Q_\nu\}$ is quasi-monotone,
(iii) $\sum_{\nu=0}^{\infty} Q_\nu < \infty$,
(iv) $|\Delta q_\nu| \leq Q_\nu, \quad \forall \quad \nu.$

Telyakovskii[14] established the theorem regarding the L^1-convergence of expression(1.1) for the coefficient class S. Moving ahead, modified cosine sums

$$h_\eta(y) = \frac{1}{2}\sum_{\kappa=0}^{\eta} \Delta q_\kappa + \sum_{\kappa=1}^{\eta}\sum_{j=\kappa}^{\eta} \Delta q_j \cos \kappa y$$

was introduced by Rees and Stanojević[9] and they had derived the criterion for integrability and convergence of series(1.1) in L^1-norm.
Regarding the convergence of $h_\eta(y)$ in L^1-metric, following theorem was proved by Garrett and Stanojević[5]:

Theorem 1.6. *If null sequence $\{q_\eta\}$ follows definition1.2, then $h_\eta(y)$ converges to $f^*(y)$in L^1-metric.*

Ram [8] demonstrated that the conclusion of Theorem1.6 is likewise implied by coefficient class S. After that, Singh and Sharma[13] considered the coefficients from class S' to prove Theorem1.6 and the conclusions given by Ram [8]. They established the result given below:

Theorem 1.7. *If $\{q_\eta\} \in S'$, then $h_\eta(y)$ converges to $f^*(y)$ in L^1-metric.*

Tomovskii[15] elaborated the class S as follows:

Definition 1.8. [15] A sequence $\{q_\nu\}$ follows class S_r, $r \in \mathbb{N}U\{0\}$ If $\exists$ a sequence $\{Q_\nu\}$ s.t.

(i) $q_\nu \to 0$ as $\nu \to \infty$,
(ii) $Q_\nu \downarrow 0, \nu \to \infty$,
(iii) $\sum_{\nu=0}^{\infty} \nu^r Q_\nu < \infty$,
(iv) $|\Delta q_\nu| \leq Q_\nu \quad \forall \quad \nu$.

Kaur and Bhatia [6] had extended Theorem1.7 to the wider class of coefficient sequences S_r, where $r \in \mathbb{N}U\{0\}$.

Motive of this research article is to generalize Theorem1.7 and results given by Kaur and Bhatia [6] for the extended class $S_r^{'}$, where $r \in \mathbb{N}U\{0\}$, of coefficient sequence. Here, Coefficient class $S_r^{'}$ is discribed as:

Definition 1.9. A sequence $\{q_\nu\}$ follows class $S_r^{'}$, $r \in \mathbb{N}U\{0\}$ if $\exists$ a sequence Q_ν s.t.

(i) $q_\nu \to 0$ as $\nu \to \infty$,
(ii) $\{Q_\nu\}$ is quasi-monotone,
(iii) $\sum_{\nu=0}^{\infty} \nu^r Q_\nu < \infty$,
(iv) $|\Delta q_\nu| \leq Q_\nu \quad \forall \quad \nu$.

If value of r is taken to be 0, then the class $S_r^{'}$ corresponds to coefficient class $S^{'}$.

Clearly $S_{r+1}^{'} \subset S_r^{'}$ as $\sum_{\nu=0}^{\infty} \nu^r Q_\eta < \sum_{\nu=0}^{\infty} \nu^{r+1} Q_\nu < \infty$.

But the converse need not to be true. To illustrate it with the help of an example, consider $\Delta q_\nu = \dfrac{1}{\nu^{r+2}}, \quad r = 0, 1, 2, ...$

So $q_\nu = \sum_{\kappa=\nu}^{\infty} \Delta q_\kappa = \sum_{\kappa=\nu}^{\infty} \dfrac{1}{\kappa^{r+2}} \to 0 \quad \text{as} \quad \nu \to \infty$.

Now define $Q_\nu = \max_{\imath \geq \nu} |\Delta q_\imath|$

i.e. $|\Delta q_\nu| = Q_\nu \quad \because |\Delta q_\nu|$ is monotonically decreasing sequence.

$Q_{\nu+1} = |\Delta q_{\nu+1}| = \dfrac{1}{(\nu+1)^{r+2}} < \dfrac{1}{\nu^{r+2}} < Q_\nu < Q_\nu \left(1 + \dfrac{\alpha}{\nu}\right)$, for some positive α,

Thus, $\{Q_\nu\}$ is quasi-monotone sequence.

$\sum_{\nu=1}^{\infty} \nu^r Q_\nu = \sum_{\nu=1}^{\infty} \dfrac{1}{\nu^2} < \infty$, that shows $\{q_\nu\} \in S_r^{'}$.

But $\sum_{\nu=1}^{\infty} \nu^{r+1} Q_\nu = \sum_{\nu=1}^{\infty} \dfrac{1}{\nu}$, a divergent series, which shows $\{q_\nu\} \notin S_{r+1}^{'}$, which implies $S_r^{'} \nsubseteq S_{r+1}^{'}$

Here, we have cosidered the L^1-convergence of single sequences. Various other types of convergences for the sequences are given by Nuray [7] and Choudhury and Granados [3].

2. Main Results

Here, in this research paper, following two theorems will be proved:

Theorem 2.1. *If $\{q_\eta\} \in S_r^{'}$, then $||f^* - h_\eta||_{L^1} = 0(1), \eta \to \infty$.*

Theorem 2.2. *If* $\{q_\eta\} \in S_r^{'}$, *then* $||f^{*r} - h_\eta^r||_{L^1} = 0(1), \eta \to \infty$, *where* $h_\eta^r(y)$ *represents the* r^{th} *derivative of* $h_\eta(y)$.

3. Lemmas

To prove the aforementioned theorems, the following lemmas are needed:

Lemma 3.1. *Let* $\{Y_\ell\}$ *be quasi-monotone sequence of constants and* $\sum \ell^r Y_\ell < \infty$, *then* $\ell^{r+1} Y_\ell = 0(1), \quad r \in \mathbb{N}U\{0\}$.

Proof.

$$\begin{aligned}\ell^{r+1}Y_{2\eta} &= \ell^r \ell Y_{2\ell}, \\ &= \ell^r \left(Y_{2\ell} + Y_{2\ell} + Y_{2\ell} + ... + Y_{2\ell}\right).\end{aligned}$$

As $\{Y_\ell\}$ be quasi-monotone sequence of constants, So
$Y_{2\ell} \leq Y_{2\ell-1}\left(1 + \frac{\alpha}{2\ell-1}\right) \leq Y_{2\ell-1}\left(1+\frac{\alpha}{\ell}\right) \quad \forall \quad \ell \geq 1.$
Similarly, we can write $Y_{2\ell} \leq Y_{2\ell-2}\left(1+\frac{\alpha}{\ell}\right)^2 \quad \forall \quad \ell \geq 1$
and $Y_{2\ell} \leq Y_{\ell+1}\left(1+\frac{\alpha}{\ell}\right)^{\ell-1} \forall \quad \ell \geq 1.$
Therefore,

$$\begin{aligned}\ell^{r+1}Y_{2\ell} &= \ell^r(Y_{2\ell} + Y_{2\ell} + Y_{2\ell} + ... + Y_{2\ell}), \\ &\leq \ell^r\left(Y_{\ell+1}\left(1+\frac{\alpha}{\ell}\right)^{\ell-1} + Y_{\ell+2}\left(1+\frac{\alpha}{\ell}\right)^{\ell-2} + ... + Y_{2\ell-1}\left(1+\frac{\alpha}{\ell}\right) + Y_{2\ell}\right).\end{aligned}$$

By expanding through binomial expansions, we get

$$\ell^{r+1}Y_{2\ell} < e^\alpha \sum_{\imath=\ell+1}^{\infty} \imath^r Y_\imath.$$

Therefore,

$$(2\ell)^{r+1}Y_{2\ell} \leq 2^{r+1}e^\alpha \sum_{\imath=\ell+1}^{\infty} \imath^r Y_\imath \to 0(1) \quad \text{as} \quad \eta \to \infty.$$

Generalizing the above expression to the next term, we get

$$\begin{aligned}(2\ell+1)^{r+1}Y_{2\ell+1} &\leq \left(2+\frac{1}{\ell}\right)^{r+1} \ell^{r+1}Y_{2\ell+1}, \\ &\leq \left(2+\frac{1}{\ell}\right)^{r+1} \ell^{r+1}Y_{2\ell}\left(1+\frac{\alpha}{2\ell}\right), \\ &\to 0(1) \quad \text{as} \quad \ell \to \infty.\end{aligned}$$

Therefore, $\ell^{r+1}Y_\ell = 0(1) \quad \text{as} \quad \ell \to \infty.$

□

Lemma 3.2. *Let* $\{Y_\mathfrak{m}\}$ *be a quasi-monotone sequence and* $\sum\limits_{\mathfrak{m}=1}^{\infty} \mathfrak{m}^r Y_\mathfrak{m} < \infty$ *for* $r \in \mathbb{Z}^+$, *then* $\sum\limits_{\mathfrak{m}=1}^{\infty} \mathfrak{m}^{r+1}(\Delta Y_\mathfrak{m}) < \infty$.

Proof. Consider

$$\sum_{\mathfrak{m}=1}^{\eta-1} \mathfrak{m}^{r+1}(\Delta Y_{\mathfrak{m}}) = \sum_{\mathfrak{m}=1}^{\eta-1} \mathfrak{m}^{r+1} Y_{\mathfrak{m}} - \sum_{\mathfrak{m}=1}^{\eta} (\mathfrak{m}-1)^{r+1} Y_{\mathfrak{m}},$$
$$= \sum_{\mathfrak{m}=1}^{\eta} [\mathfrak{m}^{r+1} - (\mathfrak{m}-1)^{r+1}] Y_{\mathfrak{m}} - \eta^{r+1} Y_{\eta},$$
$$\leq \left(\sum_{\mathfrak{m}=1}^{\infty} \mathfrak{m}^{r} Y_{\mathfrak{m}} \right) - \eta^{r+1} Y_{\eta}.$$

Proof of Lemma3.2 follows from the given statement and Lemma3.1. □

Lemma 3.3. [4] *Let* $|b^*_\kappa| \leq 1$, *then*

$$\int_0^\pi \left| \sum_{\kappa=0}^{m} b^*_\kappa D_\kappa(y) \right| dy \leq P(m+1)$$

as P *is +ve constant.*

Lemma 3.4. [15] *Let* $|b^*_\kappa| \leq 1$,*where* $\{b^*_\kappa\} \in \Re \quad \forall \quad \kappa$. *Then* $\exists$ *a constant* P *s.t.*

$$\int_0^\pi \left| \sum_{\kappa=0}^{m} b^*_\kappa D^r_\kappa(y) \right| dy \leq P(m+1)^{r+1}$$

for any $\eta \geq 0$ *and* $r = 0, 1, 2, 3, \ldots$

Lemma 3.5. [11]

$$||D^r_\eta(y)||_{L^1} = 0(\eta^r \log \eta), \quad r \in \mathbb{N}U\{0\},$$

where $D^r_\eta(y)$ *represents* r^{th} *differential of Dirichlet Kernel.*

4. Proofs

Explanation for Theorem 2.1:

$$f^*(y) = \lim_{\eta \to \infty} \left[\frac{q_0}{2} + \sum_{\kappa=1}^{\eta} q_\kappa \cos \kappa y \right],$$

After implementing Abel's Tranformation,

$$= \lim_{\eta \to \infty} \left[\sum_{\kappa=0}^{\eta-1} \Delta q_\kappa D_\kappa(y) + q_\eta D_\eta(y) \right],$$
$$= \sum_{\kappa=0}^{\infty} \Delta q_\kappa D_\kappa(y).$$

$\because \lim_{\eta\to\infty} q_\eta D_\eta(y) = 0 \quad \text{If} \quad y \neq 0.$

$$\begin{aligned}
h_\eta(y) &= \frac{1}{2}\sum_{\kappa=0}^{\eta}\Delta q_\kappa + \sum_{\kappa=1}^{\eta}\sum_{j=\kappa}^{\eta}\Delta q_j \cos\kappa y,\\
&= \frac{q_0}{2} + \sum_{\kappa=1}^{\eta} q_\kappa \cos\kappa y - q_{\eta+1}D_\eta(y),\\
&= \sum_{\kappa=0}^{\eta}\Delta q_\kappa D_\kappa(y).
\end{aligned}$$

Therefore, $f^*(y) - h_\eta(y) = \sum\limits_{\kappa=\eta+1}^{\infty} \varDelta q_\kappa D_\kappa(y)$.

$$\begin{aligned}
&\int_0^\pi |f^*(y) - h_\eta(y)|dy\\
&= \int_0^\pi \left|\sum_{\kappa=\eta+1}^{\infty} Q_\kappa \frac{\Delta q_\kappa}{Q_\kappa} D_\kappa(y)\right| dy,\\
&= \lim_{M\to\infty}\int_0^\pi \left|\sum_{\kappa=\eta+1}^{M-1}\Delta Q_\kappa \sum_{\nu=1}^{\kappa} D_\nu(y)\frac{\Delta q_\nu}{Q_\nu} + Q_M\sum_{\kappa=1}^{M} D_\kappa(y)\frac{\Delta q_\kappa}{Q_\kappa} - Q_{\eta+1}\sum_{\kappa=1}^{\eta} D_\kappa(y)\frac{\Delta q_\kappa}{Q_\kappa}\right| dy,\\
&< \lim_{M\to\infty}\left[\sum_{\kappa=\eta+1}^{M-1}\Delta Q_\kappa \frac{\kappa^r}{\kappa^r}\int_0^\pi\left|\sum_{\nu=1}^{\kappa} D_\nu(y)\frac{\Delta q_\nu}{Q_\nu}\right| dy + Q_M\int_0^\pi\left|\sum_{\kappa=1}^{M} D_\kappa(y)\frac{\Delta q_\kappa}{Q_\kappa}\right| dy\right]\\
&+ Q_{\eta+1}\int_0^\pi\left|\sum_{\kappa=1}^{\eta} D_\kappa(y)\frac{\Delta q_\kappa}{Q_\kappa}\right| dy,
\end{aligned}$$

since $\left|\dfrac{\Delta q_\kappa}{Q_\kappa}\right| < 1$, therefore,

$$\begin{aligned}
\int_0^\pi |f^*(y) - h_\eta(y)|dy &< \lim_{\eta\to\infty}\left[\frac{1}{(\eta+1)^r}\sum_{\kappa=\eta+1}^{M-1}\kappa^r|\Delta Q_\kappa|\int_0^\pi\left|\sum_{\nu=1}^{\kappa} D_\nu(y)\frac{\Delta q_\nu}{Q_\nu}\right| dy + Q_{\eta+1}(\eta+1)\right]\\
&+ Q_{M+1}(M+1),\\
&\le C\sum_{\kappa=\eta+1}^{\infty}(\kappa+1)^{r+1}\Delta q_\kappa, \quad \text{where C is arbitrary constant}\\
&< \infty,
\end{aligned}$$

(By applying Lemma 3.1 for r = 0, Lemmas 3.2 and Lemma 3.3).

Corollary 4.1. *If the coefficient sequence of cosine series(1.1) is from the class S_r', $r \in \mathbb{N}U\{0\}$, then*

$$||f^* - S_\eta||_{L^1} = o(1), \eta\to\infty \quad \textit{Iff} \quad |q_{\eta+1}|\log\eta = o(1), \eta\to\infty.$$

Proof. Consider

$$\begin{aligned}||f^* - S_\eta||_{L^1} &= ||f^* - S_\eta + h_\eta - h_\eta||_{L^1},\\ &\leq ||f^* - h_\eta||_{L^1} + ||S_\eta - h_\eta||_{L^1},\\ &= ||f^* - h_\eta||_{L^1} + ||q_{\eta+1}D_\eta(y)||_{L^1},\\ &\leq ||f^* - h_\eta||_{L^1} + |q_{\eta+1}| \int_0^\pi |D_\eta(y)|dy.\end{aligned}$$

Thus, proof concludes by Theorem 2.1 and Lemma3.5. □

Remark 4.2. For r $= 0$, results are same as the results under the class $S^{'}$ [13].

Proof of the Theorem 2.2.

Proof.

$$\begin{aligned}h_\eta(y) &= \frac{1}{2}\sum_{\kappa=0}^{\eta}\Delta q_\kappa + \sum_{\kappa=1}^{\eta}\sum_{j=\kappa}^{\eta}\Delta q_j \cos\kappa y,\\ &= \frac{q_0}{2} + \sum_{\kappa=1}^{\eta} q_\kappa \cos\kappa y - q_{\eta+1}D_\eta(y),\\ &= S_\eta(y) - q_{\eta+1}D_\eta(y).\end{aligned}$$

Taking the r^{th} derivative on both sides, we get

$$h_\eta^r(y) = S_\eta^r(y) - q_{\eta+1}D_\eta^r(y)$$

Clearly,

$$f^{*r}(y) - h_\eta^r(y) = \sum_{\kappa=\eta+1}^{\infty} q_\kappa \kappa^r \cos\left(\kappa y + r\frac{\pi}{2}\right) + q_{\eta+1}D_\eta^r(y)$$

By implementing Abel's transformation,

$$f^{*r}(y) - h_\eta^r(y) = \sum_{\kappa=\eta+1}^{\infty} \Delta q_\kappa D_\kappa^r(y)$$

Next,

$$\begin{aligned}&\int_0^\pi |f^{*r}(y) - h_\eta^r(y)|dy\\ &= \int_0^\pi \left|\sum_{\kappa=\eta+1}^{\infty} Q_\kappa \frac{\Delta q_\kappa}{Q_\kappa} D_\kappa^r(y)\right| dy,\\ &= \lim_{M\to\infty}\int_0^\pi \left|\sum_{\kappa=\eta+1}^{M-1}\Delta Q_\kappa \sum_{\nu=1}^{\kappa} D_\nu^r(y)\frac{\Delta q_\nu}{Q_\nu} + Q_M\sum_{\kappa=1}^{M} D_\kappa^r(y)\frac{\Delta q_\kappa}{Q_\kappa} - Q_{\eta+1}\sum_{\kappa=1}^{\eta} D_\kappa^r(y)\frac{\Delta q_\kappa}{Q_\kappa}\right| dy,\\ &< \lim_{M\to\infty}\left[\sum_{\kappa=\eta+1}^{M-1}\Delta Q_\kappa \int_0^\pi\left|\sum_{\nu=1}^{\kappa} D_\nu^r(y)\frac{\Delta q_\nu}{Q_\nu}\right| dy + Q_M\int_0^\pi\left|\sum_{\kappa=1}^{M} D_\kappa^r(y)\frac{\Delta q_\kappa}{Q_\kappa}\right| dy\right]\\ &+ Q_{\eta+1}\int_0^\pi\left|\sum_{\kappa=1}^{\eta} D_\kappa^r(y)\frac{\Delta q_\kappa}{Q_\kappa}\right| dy.\end{aligned}$$

Since $\left|\dfrac{\Delta q_\kappa}{Q_\kappa}\right| < 1$,
Therefore,

$$\int_0^\pi |f^{*r}(y) - h_\eta^r(y)|dy < \lim_{M\to\infty}\left[\sum_{\kappa=\eta+1}^{M-1} |\Delta Q_\kappa| \int_0^\pi \left|\sum_{\nu=1}^{\kappa} D_\nu^r(y)\frac{\Delta q_\nu}{Q_\nu}\right| dy + Q_M(M+1)^r + Q_{\eta+1}(\eta+1)^r\right],$$
$$\leq C\sum_{\kappa=\eta+1}^{\infty}(\kappa+1)^{r+1}\Delta q_\kappa, \quad \text{where C is arbitrary constant}$$
$$< \infty.$$

(By implementing Lemmas 3.1, 3.2 and 3.4)

□

Corollary 4.3. *If coefficient sequence* $\{q_\eta\} \in S_r'$, $r \in \mathbb{N}U\{0\}$, *then*

$$||f^{*r} - S_\eta^r||_{L^1} = 0(1), \eta \to \infty \quad \textit{Iff} \quad |q_{\eta+1}|\eta^r \log\eta = 0(1), \quad \eta\to\infty.$$

Proof. Consider

$$\begin{aligned}||f^{*r} - S_\eta^r||_{L^1} &= ||f^{*r} - S_\eta^r + h_\eta^r - h_\eta^r||_{L^1},\\ &\leq ||f^{*r} - h_\eta^r||_{L^1} + ||S_\eta^r - h_\eta^r||_{L^1},\\ &= ||f^{*r} - h_\eta^r||_{L^1} + ||q_{\eta+1}D_\eta^r(y)||_{L^1},\\ &\leq ||f^{*r} - h_\eta^r||_{L^1} + |q_{\eta+1}|\int_0^\pi |D_\eta^r(y)|dy.\end{aligned}$$

By implementing Theorem 2.2 and Lemma 3.5, we get the result of corollary 4.3.

□

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