Gulf Journal of Mathematics
Vol 16, Issue 2 (2024) 12-26

ISSN: 2309-4966
https://doi.org/10.56947/gjom.v16i2.1841

THE BEREZIN TRANSFORM AND TOEPLITZ OPERATORS ON THE BERGMAN SPACE OVER THE QUARTER PLANE

HOCINE GUEDIRI

Abstract. In this note, we introduce Bergman spaces on the complex upper-right quarter-plane (the first quadrant), and establish some of their fundamental properties. Moreover, we study the associated Berezin transform, and we deduce characterizations of bounded as well as compact Toeplitz operators with symbols that are either harmonic or continuous.

1. Introduction

In the setting of the complex unit disk, the class of Toeplitz operators on the Bergman spaces has become very popular. A whole theory devoted to this celebrated class of concrete operators, which are acting on one of the most important reproducing kernel Hilbert spaces, has been well elaborated during the last five decades. Subsequently, many interesting developments regarding their algebraic and spectral properties in different settings have been performed [1, 3, 4, 14]. For more details, we refer to the monographs of N. Vasilevski [19] and K. Zhu [21].

The study of Toeplitz operators on the Bergman spaces in the setting of general domains of the complex plane has been attracting more attention in the literature. Interesting results related to their boundedness, compactness, commutativity, products and spectra can be found in recent literature. See for instance N. Vasilevski [19], F. Alshormani and H. Guediri [3], as well as [11, 12, 13, 14, 17].

Our main task here is three folds. First, we introduce the Bergman spaces in the framework of the complex upper-right quarter plane (the first quadrant). Then, we establish some of their elementary properties, such as structure and completeness. For the particular case of the Hilbert-Bergman space, we determine a corresponding orthonormal basis, and we compute its reproducing kernel. Moreover, we describe the related orthogonal projection. Second, we explore the associated Berezin transform, and we establish some of its fundamental properties, and then we discuss its connection with the invariant Laplacian. Third, we deal with Toeplitz operators with either harmonic or continuous defining symbols in the framework of the quarter plane. In particular, we characterize bounded and compact Toeplitz operators among these two classes.

Date: Received: Feb 18, 2024; Accepted: Mar 17, 2024.

* Corresponding author.

2020 *Mathematics Subject Classification.* Primary 47B35; Secondary 32A36.

Key words and phrases. Bergman space of the quarter plane, Toeplitz operators, Berezin transform.

2. The Bergman space on the quarter plane

Let $\mathcal{Q}$ be the open complex upper-right quarter-plane (in short, we say: quarter-plane or first quadrant) in $\mathbb{C}$, which is defined by

$$\mathcal{Q} := \{z = x + iy \in \mathbb{C}, \quad \operatorname{Re} z = x > 0,\ \operatorname{Im} z = y > 0\}.$$

Combining the conformal map $\psi(z) = z^2$, sending the quarter-plane $\mathcal{Q}$ into the complex upper half-plane $\Pi^+ := \{z = x + iy \in \mathbb{C}, \quad \operatorname{Im} z = y > 0\}$, with the Cayley transform $\phi(z) = \frac{z-i}{z+i}$, mapping the latter to the unit disk $\mathbb{D} := \{z \in \mathbb{C}, |z| < 1\}$, we obtain the following biholomorphic mapping between $\mathcal{Q}$ and $\mathbb{D}$:

$$\Phi(z) = \phi \circ \psi(z) = \frac{z^2 - i}{z^2 + i}, \tag{2.1}$$

which satisfies $\Phi(\frac{1+i}{\sqrt{2}}) = 0$, and the boundary value $\Phi(0) = -1$, as well as:

$$|\Phi'(z)|^2 := \frac{16|z|^2}{|z^2 + i|^4}, \qquad \Phi^{-1}(w) = \sqrt{i\frac{1+w}{1-w}}, \tag{2.2}$$

where, in the latter, we consider the principal branch for the complex square root. This Möbius transform enables us to carry certain properties of analytic functions from $\mathcal{Q}$ to $\mathbb{D}$, and vice versa. For instance, we have:

$$\text{If } f \in L^p(\mathcal{Q}),\ \text{with } 1 \le p \le \infty,\ \text{then } f \circ \Phi^{-1} \in L^p(\mathbb{D}). \tag{2.3}$$

The general form of the quarter-plane automorphism reads as

$$\varphi(w) = \sqrt{\frac{aw^2 + b}{cw^2 + d}},\ w \in \mathcal{Q},\ \text{where } a, b, c, d \in \mathbb{R},\ ad - bc > 0, \tag{2.4}$$

which reduces to

$$\varphi_z(w) = \sqrt{\operatorname{Im} z^2 w^2 + \operatorname{Re} z^2},\ z \in \mathcal{Q},\ \text{for all } w \in \mathcal{Q}, \tag{2.5}$$

where again the principal branch for the complex square root has been chosen. The group of all automorphisms of the quarter-plane $\mathcal{Q}$ is denoted by $\operatorname{Aut}(\mathcal{Q})$.

Let $dA(z) = \frac{1}{\pi} dxdy$ denote the Lebesgue area measure over $\mathcal{Q}$. For $1 \le p < \infty$, denote by $L^p(\mathcal{Q}, dA) = L^p(\mathcal{Q})$ the Lebesgue space of p-integrable functions over $\mathcal{Q}$, which is the Banach space of Lebesgue measurable functions f on $\mathcal{Q}$ satisfying

$$\|f\|_p = \left(\int_{\mathcal{Q}} |f(z)|^p \, dA(z)\right)^{\frac{1}{p}} < \infty,$$

while $L^\infty(\mathcal{Q})$ denotes the space of essentially bounded functions such that

$$\|f\|_\infty = \operatorname{ess.sup}\{|f(z)|, z \in \mathcal{Q}\} < \infty.$$

Definition 2.1. For $1 \le p < \infty$, the Bergman space $L^p_a(\mathcal{Q}) := L^p(\mathcal{Q}) \cap \mathscr{H}(\mathcal{Q})$ is the subspace of $L^p(\mathcal{Q})$ consisting of holomorphic functions on $\mathcal{Q}$, while the algebra of all bounded analytic functions on $\mathcal{Q}$ is denoted by $\mathcal{H}^\infty(\mathcal{Q}) := L^\infty(\mathcal{Q}) \cap \mathscr{H}(\mathcal{Q})$, where $\mathscr{H}(\mathcal{Q})$ is the space of all holomorphic functions on $\mathcal{Q}$. Moreover, denote by $\mathscr{B}(L^p_a(\mathcal{Q}))$, the algebra of all bounded linear operators on $L^p_a(\mathcal{Q})$.

For more details on the theory of Bergman spaces and related function spaces of analytic functions, we refer to the monographs [6, 9, 10, 15].

The following estimate describes the growth of $L_a^p(\mathcal{Q})$ functions near the boundary of the quarter-plane $\mathcal{Q}$:

Lemma 2.2. *For $z \in \mathcal{Q}$, set $\operatorname{dist}(z, \partial\mathcal{Q})) := \min\{|z-w|,\ w \in \partial\mathcal{Q}\}$. For any $f \in L_a^p(\mathcal{Q})$, with $1 \le p < \infty$, the following estimate holds:*

$$|f(z)| \le \frac{1}{(\operatorname{dist}(z, \partial\mathcal{Q}))^{\frac{2}{p}}} \|f\|_p \,. \tag{2.6}$$

Proof. Let $\mathcal{D}(z, \delta)$ be the disk centred at $z \in \mathcal{Q}$ with radius $\delta = \operatorname{dist}(z, \partial\mathcal{Q})$. Combining the area mean value property with Jensen's inequality, we obtain

$$|f(z)|^p \le \frac{1}{|\mathcal{D}(z,\delta)|} \int_{\mathcal{D}(z,\delta)} |f(w)|^p \, dA(w) \le \frac{1}{\delta^2} \int_{\mathcal{Q}} |f(w)|^p \, dA(w).$$ □

It follows that for every compact subset $\mathcal{K} \Subset \mathcal{Q}$, there is a constant $C_{\mathcal{K}}$ such that

$$\sup_{z \in \mathcal{K}} |f(z)| \le C_{\mathcal{K}} \|f\|_p \,, \quad \text{for all } \ f \in L_a^p(\mathcal{Q}). \tag{2.7}$$

In particular, Lemma 2.2 implies that the Bergman spaces $L_a^p(\mathcal{Q})$, $1 \le p < \infty$, are in fact Banach spaces:

Proposition 2.3. *For $1 \le p < \infty$, the Bergman space $L_a^p(\mathcal{Q})$ is a closed subspace of $L^p(\mathcal{Q})$.*

Proof. Let $\{f_n\} \subset L_a^p(\mathcal{Q})$ be a Cauchy sequence such that $\|f_n - f\|_p \longrightarrow 0$, as $n \longrightarrow \infty$, for some $f \in L^p(\mathcal{Q})$. By (2.7), it is a uniform Cauchy sequence on each compact subset $\mathcal{K}$ in $\mathcal{Q}$, namely that $|f_n(z) - f_m(z)| \le C_{\mathcal{K}} \|f_n - f_m\|_p \,, \forall z \in \mathcal{K}$. It follows that $\lim_{n,m\to\infty} \sup |f_n(z) - f_m(z)| = 0$. Hence, $f_n \longrightarrow f$ uniformly on each compact subset of $\mathcal{Q}$. Thus, f must be in $L_a^p(\mathcal{Q})$, completing the proof. □

Now, we confine ourselves to the most important case, namely when $p = 2$. Since $L_a^2(\mathcal{Q})$ is a closed subspace of $L^2(\mathcal{Q})$, then it is a Hilbert subspace with induced inner product given by:

$$\langle f, g \rangle = \int_{\mathcal{Q}} f(z)\overline{g(z)} dA(z), \ \forall f, g \in L_a^2(\mathcal{Q}).$$

Another important consequence of Lemma 2.2 asserts that the point evaluation functional is bounded linear on $L_a^p(\mathcal{Q})$. Thus, by the Riesz representation theorem, there is a unique kernel function, denoted $K_z \in L_a^2(\mathcal{Q})$, such that the following useful reproducing property holds:

$$f(z) = \langle f, K_z \rangle \,, \forall f \in L_a^2(\mathcal{Q}), \tag{2.8}$$

The function K_z is called the reproducing kernel of the Bergman space $L_a^2(\mathcal{Q})$. Later on, we will establish an explicit formula for K_z, namely the formula (2.10).

In the following assertion, we provide an orthonormal basis to the separable Hilbert space $L_a^2(\mathcal{Q})$:

Proposition 2.4. *The following system constitutes an orthonormal basis of the Bergman space* $L_a^2(\mathcal{Q})$*:*

$$e_n(z) = \sqrt{16(n+1)}\frac{z(z^2-i)^n}{(z^2+i)^{n+2}}, \quad n = 0, 1, 2, \ldots ., \ z \in \mathcal{Q}. \tag{2.9}$$

Proof. By explicit computations, we have

$$\langle e_n, e_m \rangle = \sqrt{n+1}\sqrt{m+1}\int_{\mathcal{Q}} \left(\frac{z^2-i}{z^2+i}\right)^n \overline{\left(\frac{z^2-i}{z^2+i}\right)^m} \frac{16|z|^2 dA(z)}{|z^2+i|^4}.$$

Setting $w = \Phi(z) = \frac{z^2-i}{z^2+i}$ as in (2.1), we see that $dA(w) = |\Phi'(z)|^2 dA(z) = \frac{16|z|^2 dA(z)}{|z^2+i|^4}$, as in (2.2). Thus, we obtain

$$\begin{aligned} \langle e_n, e_m \rangle &= \sqrt{n+1}\sqrt{m+1}\int_{\mathbb{D}} w^n \overline{w}^m dA(w), \\ &= \frac{\sqrt{n+1}\sqrt{m+1}}{\pi}\int_0^1 \int_0^{2\pi} r^{n+m+1} e^{i(n-m)\theta} d\theta dr = \delta_{nm}, \end{aligned}$$

with δ_{nm} being the Krönecker δ-symbol. This shows that the set $\{e_n\}_{n\in\mathbb{N}}$ is orthonormal in $L_a^2(\mathcal{Q})$.
Next, we have to show that the system $\{e_n\}_{n\in\mathbb{N}}$ is total. For, it suffices to show that $\{e_n\}^{\perp} = \{0\}$. Let $f \in \{e_n(z)\}^{\perp}$, then $\langle e_n, f\rangle = 0$ for all $n \geq 0$. So, we obtain, by the same change of variables $z = \Phi^{-1}(w)$, that:

$$\langle e_n, f \rangle = \int_{\mathcal{Q}} 4\sqrt{n+1}\left(\frac{z^2-i}{z^2+i}\right)^n \frac{z}{(z^2+i)^2}\overline{f(z)} dA(z) = \langle a_n, g\rangle = 0,$$

where $a_n(w) = \sqrt{n+1}w^n$ is the standard orthonormal basis of the Bergman space $L_a^2(\mathbb{D})$ and $g(w) = \frac{\overline{((\Phi^{-1}(w)^2+i))^2}}{16\overline{\Phi^{-1}(w)}} f(\Phi^{-1}(w))$, which is in $L_a^2(\mathbb{D})$, because it is analytic and it satisfies

$$\int_{\mathbb{D}} |g(w)|^2 dA(w) = \int_{\mathcal{Q}} |f(z)|^2 dA(z) < \infty.$$

We infer that $g(w) = 0$, for all $w \in \mathbb{D}$, which implies that $f(z) = 0$, for all $z \in \mathcal{Q}$, and completes the proof. □

Besides the density of the system $\{e_n\}$ in $L_a^2(\mathcal{Q})$, we have also the density of a certain collections of analytic functions in $L_a^p(\mathcal{Q})$. The following assertion has been proven in [11, 12] for the Bergman space $L_a^2(\Pi^+)$ on the upper half-plane Π^+. A quarter-plane analogue of it remains valid for $L_a^p(\mathcal{Q})$, $1 \leq p < \infty$.

Proposition 2.5. *For any* $1 \leq p < \infty$*, the spaces* $L_a^p(\mathcal{Q}) \cap \mathcal{H}^\infty(\mathcal{Q})$ *and* $L_a^p(\mathcal{Q}) \cap L_a^2(\mathcal{Q})$ *are dense in* $L_a^p(\mathcal{Q})$*.*

In general, the Bergman kernel can be calculated by various techniques, see [6, 10, 19, 21]. The following method consists in obtaining the explicit formula of the reproducing kernel of $L_a^2(\mathcal{Q})$ by making use of its orthonormal basis (2.9). More precisely, the reproducing kernel of the Bergman space $L_a^2(\mathcal{Q})$ can be represented as follows:

$$\begin{aligned}K(w,z) &= \sum_{n=0}^{\infty} e_n(w)\overline{e_n(z)} = \sum_{n=0}^{\infty}(n+1)\left(\frac{w^2-i}{w^2+i}\right)^n \frac{4z}{(w^2+i)^2}\overline{\left(\frac{z^2-i}{z^2+i}\right)}^n \frac{4\overline{w}}{\overline{(z^2+i)}^2},\\ &= \sum_{n=0}^{\infty}(n+1)\left(\Phi(w)\overline{\Phi(z)}\right)^n \frac{16z\overline{w}}{(w^2+i)^2\overline{(z^2+i)}^2},\\ &= \frac{1}{\left(1-\Phi(w)\overline{\Phi(z)}\right)^2}\frac{16z\overline{w}}{(w^2+i)^2\overline{(z^2+i)}^2} = \frac{-4w\overline{z}}{(w^2-\overline{z}^2)^2}.\end{aligned}$$

Thus, the compact form of the Bergman reproducing kernel of $L_a^2(\mathcal{Q})$ reads as (see also [16] for an alternative method):

$$K(w,z) = K_z(w) = \frac{-4w\overline{z}}{(w^2-\overline{z}^2)^2}, \quad z,w\in\mathcal{Q}. \tag{2.10}$$

Hence the normalized reproducing kernel of $L_a^2(\mathcal{Q})$ reads as

$$k_z(w) = \frac{K_z(w)}{\|K_z\|_2} = \frac{K_z(w)}{\sqrt{K_z(z)}} = \frac{-8\ w\ \overline{z}(\operatorname{Im} z\operatorname{Re} z)}{|z|(\overline{z}^2-w^2)^2}. \tag{2.11}$$

Observe that the Bergman kernel function $K(w,z) = K_z(w)$ is holomorphic in w and conjugate-holomorphic in z. Observe also that the Bergman kernel function obeys a complex-symmetry property

$$\overline{K(z,w)} = K(w,z).$$

By direct calculations, we observe the following:

Proposition 2.6. *For the reproducing kernel given by (2.10), we have*

(1) *K_z is a bounded analytic function in $\mathcal{Q}$, i.e. $K_z\in\mathcal{H}^\infty(\mathcal{Q})$.*
(2) *K_z is p-integrable on $\mathcal{Q}$, for $1<p<\infty$, i.e. $K_z\in L_a^p(\mathcal{Q})$.*
(3) *K_z is not integrable on $\mathcal{Q}$, i.e. $K_z\notin L_a^1(\mathcal{Q})$.*

The explicit formula (2.10) of the kernel function of $L_a^2(\mathcal{Q})$ yields an integral form of the reproducing property (2.8) as follows:

$$f(z) = \langle f, K_z\rangle = \int_{\mathcal{Q}} f(w)\overline{K_z(w)}dA(w) = \int_{\mathcal{Q}}\frac{-4z\overline{w}}{(z^2-\overline{w}^2)^2}f(w)dA(w). \tag{2.12}$$

This normalized reproducing kernel k_z possesses a weak convergence character as the next proposition shows:

Proposition 2.7. *The normalized reproducing kernel k_z, $z\in\mathcal{Q}$, converges weakly to zero in $L_a^2(\mathcal{Q})$ as $z\longrightarrow\partial\mathcal{Q}$.*

Proof. For each $f\in L_a^2(\mathcal{Q})\cap\mathcal{H}^\infty(\mathcal{Q})$, by the reproducing property (2.12), we have

$$\langle f,k_z\rangle = \frac{2(\operatorname{Im} z\operatorname{Re} z)}{|z|}\langle f,K_z\rangle = \frac{2(\operatorname{Im} z\operatorname{Re} z)}{|z|}f(z).$$

It follows that $\langle f, k_z\rangle \longrightarrow 0$ as $z \longrightarrow \partial\mathcal{Q}$ for each $f \in L_a^2(\mathcal{Q}) \cap \mathcal{H}^\infty(\mathcal{Q})$. By the density of $L_a^2(\mathcal{Q}) \cap \mathcal{H}^\infty(\mathcal{Q})$ in $L_a^2(\mathcal{Q})$, stated in Proposition 2.5, the result follows. □

Next, we extend the reproducing property (2.12), stated first for the "Hilbertian" Bergman space $L_a^2(\mathcal{Q})$, to the "Banach" Bergman spaces $L_a^p(\mathcal{Q})$, for $1 \le p < \infty$. Proposition 2.6 implies that the integral in (2.12) makes sense for each function f in $L_a^p(\mathcal{Q})$, $1 \le p < \infty$. By the density of $L_a^p(\mathcal{Q}) \cap L_a^2(\mathcal{Q})$ in $L_a^p(\mathcal{Q})$, stated in Proposition 2.5, the reproducing property (2.12) holds also for $L_a^p(\mathcal{Q})$, $1 \le p < \infty$. More precisely, we have:

$$f(z) = \int_{\mathcal{Q}} \frac{-4z\overline{w}}{(z^2 - \overline{w}^2)^2} f(w)dA(w), \ \forall f \in L_a^p(\mathcal{Q}), \ 1 \le p < \infty. \tag{2.13}$$

The fact that $L_a^2(\mathcal{Q})$ is closed in $L^2(\mathcal{Q})$, ensures the existence of an orthogonal projection $\mathcal{P}$ from $L^2(\mathcal{Q})$ onto $L_a^2(\mathcal{Q})$, namely the Bergman projection, which has the following integral representation, owing to the reproducing property (2.12):

$$\mathcal{P}(f)(z) = \langle f, K_z\rangle = \int_{\mathcal{Q}} \frac{-4z\overline{w}}{(z^2 - \overline{w}^2)^2} f(w)dA(w), \forall f \in L^2(\mathcal{Q}), \forall z \in \mathcal{Q}. \tag{2.14}$$

It follows form (2.14) and Proposition 2.6, that $\mathcal{P}$ can be extended to $L^p(\mathcal{Q})$, $1 \le p < \infty$, and its compression to $L_a^p(\mathcal{Q})$ coincides with the identity operator, namely that

$$\mathcal{P}(f) = f, \quad \forall f \in L_a^p(\mathcal{Q}).$$

Moreover, by a standard way, (for instance arguing in the same manner as in Theorem 2.4 of [13]), we obtain

Theorem 2.8. *For $1 < p < \infty$, the Bergman projection $\mathcal{P}$ is a bounded linear operator from $L^p(\mathcal{Q})$ onto $L_a^p(\mathcal{Q})$.*

Remark 2.9. The result of the latter Theorem 2.8 fails for the case $p = 1$. In other words, the Bergman projection $\mathcal{P}$ is not bounded on $L^1(\mathcal{Q})$.

3. The Berezin transform

The Berezin transform has become a powerful tool in the study of operators on reproducing kernel Hilbert spaces, namely that it is very related to compactness and to harmonic functions. This fact is more apparent in the study of compactness and products of Toeplitz operators on Bergman spaces. For more details on the Berezin transform, we refer to [18, 20] and the monographs [10, 21]. In this section, we introduce the Berezin transform in the present context on the quarter plane, and establish some of its fundamental properties closely related to our purpose here.

Definition 3.1. The Berezin symbol of a bounded linear operator $A \in \mathscr{B}(L_a^2(\mathcal{Q}))$ is defined to be the infinitely differentiable function

$$\mathcal{B}(A)(z) = \widetilde{A}(z) := \langle Ak_z, k_z\rangle, \tag{3.1}$$

where k_z is the normalized reproducing kernel of $L_a^2(\mathcal{Q})$ given by (2.11).

Observe that $\tilde{A} \in L^\infty(\mathcal{Q})$. Indeed, by (3.1) together with the Cauchy-Schwarz inequality we see that

$$|\tilde{A}(z)| = |\langle Ak_z, k_z \rangle| \leq \|A\| \|k_z\|_2^2 = \|A\|,$$

whence $\|\tilde{A}\|_\infty \leq \|A\|$.

Similarly, for any function $f \in L^p(\mathcal{Q})$, $1 \leq p \leq \infty$, and for any $z \in \mathcal{Q}$, the Berezin symbol of f is defined by

$$\mathcal{B}(f)(z) = \widetilde{f}(z) := \langle f k_z, k_z \rangle = \int_{\mathcal{Q}} \frac{64|w|^2(\operatorname{Im} z \operatorname{Re} z)^2}{|z^2 - \overline{w}^2|^4} f(w) dA(w). \tag{3.2}$$

The Berezin transform provides useful information about the compactness of operators [4, 5, 11, 18, 20, 21]. In particular, we have

Theorem 3.2. *If A is compact on $L^2_a(\mathcal{Q})$, then $\tilde{A}(z) \longrightarrow 0$ as $z \longrightarrow \partial\mathcal{Q}$.*

Proof. For a compact operator $A \in \mathscr{B}(L^2_a(\mathcal{Q}))$, observe that

$$|\tilde{A}(z)| = |\langle Ak_z, k_z \rangle| \leq \|Ak_z\|_2 \|k_z\|_2 = \|Ak_z\|_2 \longrightarrow 0 \quad \text{as} \quad z \longrightarrow \partial\mathcal{Q},$$

because $k_z \longrightarrow 0$ weakly as $z \longrightarrow \partial\mathcal{Q}$ by Proposition 2.7. □

The converse of the latter fact is known to be not true in general. However, it does hold for Toeplitz operators [4, 5].

The following assertion uses the Schur test for integral operators [21], and guarantees the boundedness of the Berezin transform:

Proposition 3.3. *The Berezin transform (3.2) is a bounded linear integral operator on $L^p(\mathcal{Q})$, for $1 < p < \infty$.*

Proof. Applying Schur's test, (see Sections 3.3 and 3.4 of [21]), for the particular choice of the function $h(z) = |z|^{\frac{1}{pq}} (\operatorname{Im} z \operatorname{Re} z)^{-\frac{1}{pq}}$, we get the result. □

The following two assertions are standard, see for example K. Stroethoff [18]:

Lemma 3.4. *Let $\Omega \subset \mathbb{C}$ be a domain, and let $F(z, w)$ be a holomorphic function on $\Omega \times \overline{\Omega}$, where $\overline{\Omega} = \{\overline{z} : z \in \Omega\}$. If $F(z, \overline{z}) = 0$ for all $z \in \Omega$, then, F vanishes identically on $\Omega \times \overline{\Omega}$.*

Proposition 3.5. *Let $A \in \mathscr{B}(L^2_a(\mathcal{Q}))$. Then, $\widetilde{A}(z) = 0$, $\forall z \in \mathcal{Q}$, if and only if $A \equiv 0$. In other words, the Berezin transform is injective on $\mathscr{B}(L^2_a(\mathcal{Q}))$.*

Proof. If $A = 0$ then immediately $\widetilde{A}(z) = 0$, for all $z \in \mathcal{Q}$. For the converse, suppose that $\widetilde{A}(z) = 0$, for all $z \in \mathcal{Q}$, and consider the function

$$F(z, w) = \langle AK_{\overline{w}}, K_z \rangle = (AK_{\overline{w}})(z), \ \forall (z, w) \in \mathcal{Q} \times \overline{\mathcal{Q}}.$$

The reproducing property guarantees that F is analytic in z. Furthermore, using the fact that

$$F(z, w) = \langle K_{\overline{w}}, A^* K_z \rangle = \overline{(A^* K_z)(\overline{w})}, \ \forall (z, w) \in \mathcal{Q} \times \overline{\mathcal{Q}},$$

and owing to the analyticity of the function $w \longrightarrow \overline{g(\overline{w})}$ on $\overline{\mathcal{Q}}$ for any function g analytic in $\mathcal{Q}$, we see that F is analytic in w as well. On the other hand, by assumption, we have $F(z,\bar z) = \|k_z\|_2^2 \widetilde{A}(z) = 0$, for all $z \in \mathcal{Q}$. Thus, by Lemma 3.4, we infer that $F(z,w) = 0$ on $\mathcal{Q} \times \overline{\mathcal{Q}}$. In other words, $(AK_w)(z) = 0$, for all $z, w \in \mathcal{Q}$, and thus $AK_w = 0$ for all $w \in \mathcal{Q}$. Finally, for $f \in L_a^2(\mathcal{Q})$ being arbitrary, we see that

$$(A^*f)(w) = \langle A^*f, K_w\rangle = \langle f, AK_w\rangle = 0, \ \forall w \in \mathcal{Q}.$$

Therefore, we obtain $A^* = 0$; yielding $A = 0$. □

The following property asserts that the Berezin transform is Möbius invariant:

Proposition 3.6. *Let $1 \le p \le \infty$. Then, for any $f \in L^p(\mathcal{Q})$ and any quarter plane automorphism $\varphi \in Aut(\mathcal{Q})$, given by (2.4) or (2.5), we have*

$$\mathcal{B}(f \circ \varphi) = \mathcal{B}(f) \circ \varphi. \tag{3.3}$$

Proof. We know that any $\varphi \in \mathrm{Aut}(\mathcal{Q})$, has the form (2.5). Thus, we may perform the variable change $\xi = \varphi_z(\zeta) = \sqrt{\operatorname{Im} z^2 \zeta^2 + \operatorname{Re} z^2}$, for some $z \in \mathcal{Q}$. Direct computations lead to the following identities:

$$(\operatorname{Im} z^2 \operatorname{Im} w^2)^2 = 4(\operatorname{Im} \varphi_z(w) \operatorname{Re} \varphi_z(w))^2 \ , \ (\operatorname{Im} w \operatorname{Re} w)^2 = \frac{1}{4}(\operatorname{Im} w^2)^2,$$

$$|w^2 - \overline{\zeta}^2|^4 = \frac{|\varphi_z^2(w) - \overline{\xi}^2|^4}{(\operatorname{Im} z^2)^4} \ , \ |\zeta|^2 dA(\zeta) = \frac{|\xi|^2 dA(\xi)}{(\operatorname{Im} z^2)^2}.$$

Inserting these to the integral formula (3.2) of the Berezin transform, we obtain

$$\begin{aligned}
\mathcal{B}(f \circ \varphi_z)(w) &= \int_{\mathcal{Q}} \frac{64|\zeta|^2 (\operatorname{Im} w \operatorname{Re} w)^2}{|w^2 - \overline{\zeta}^2|^4} (f \circ \varphi_z)(\zeta) dA(\zeta), \\
&= \int_{\mathcal{Q}} \frac{64|\xi|^2 \frac{(\operatorname{Im} w^2)^2}{4} (\operatorname{Im} z^2)^4}{|\varphi_z^2(w) - \overline{\xi}^2|^4 (\operatorname{Im} z^2)^2} f(\xi) dA(\xi), \\
&= \int_{\mathcal{Q}} \frac{64|\xi|^2 (\operatorname{Im} \varphi_z(w) \operatorname{Re} \varphi_z(w))^2}{|\varphi_z^2(w) - \overline{\xi}^2|^4} f(\xi) dA(\xi), \\
&= (\mathcal{B}f)(\varphi_z(w)).
\end{aligned}$$

□

The quarter-plane automorphism group provides an alternative formula for the Berezin transformation:

Proposition 3.7. *The Berezin transform can be rewritten as*

$$\mathcal{B}f(z) = \int_{\mathcal{Q}} (f \circ \varphi_z)(w) \frac{16|w|^2 dA(w)}{|i + w^2|^4}, \ \forall f \in L^p(\mathcal{Q}), 1 \le p \le \infty, \tag{3.4}$$

with φ_z given by (2.5).

Proof. Let $f \in L^p(\mathcal{Q}), 1 \le p \le \infty$. Then, evaluating the Berezin transform (3.2) at the point $\frac{1+i}{\sqrt{2}} \in \mathcal{Q}$, we obtain

$$\mathcal{B}f\left(\frac{1+i}{\sqrt{2}}\right) = \int_{\mathcal{Q}} \frac{16|w|^2}{|i + w^2|^4} f(w) dA(w).$$

Observing that $\varphi_z(\frac{1+i}{\sqrt{2}}) = z$, and combining the latter, (applied to the function $f \circ \varphi_z$), with Proposition 3.6, we obtain

$$\mathcal{B}f(z) = (\mathcal{B}f)\left(\varphi_z\left(\frac{1+i}{\sqrt{2}}\right)\right) = \mathcal{B}(f\circ\varphi_z)\left(\frac{1+i}{\sqrt{2}}\right) = \int_{\mathcal{Q}} \frac{16|w|^2(f\circ\varphi_z)(w)}{|i+w^2|^4}dA(w).$$

□

Next, let $\widehat{\mathcal{Q}} := \overline{\mathcal{Q}} \cup \{\infty\}$ denote the one-point compactification of the "Euclidean" closure $\overline{\mathcal{Q}}$ of the quarter-plane $\mathcal{Q}$. Denote by $\mathscr{C}(\widehat{\mathcal{Q}})$ the space of all continuous functions on $\widehat{\mathcal{Q}}$, and let $\mathscr{C}_0(\widehat{\mathcal{Q}})$ be its subspace constituted of functions vanishing on the boundary $\partial_\infty \mathcal{Q} := \partial\widehat{\mathcal{Q}} = \widehat{\mathcal{Q}} \setminus \mathcal{Q}$. Note that $z \longrightarrow \partial_\infty \mathcal{Q}$ means that either $z \longrightarrow \partial\mathcal{Q}$ or $|z| \longrightarrow \infty$. Also, observe that if $z \longrightarrow \partial\mathcal{Q}$, then $\operatorname{Im} z^2 \longrightarrow 0$.

Proposition 3.8. *If $f \in \mathscr{C}(\widehat{\mathcal{Q}})$, then $\widetilde{f} \in \mathscr{C}(\widehat{\mathcal{Q}})$ and $\widetilde{f} - f \in \mathscr{C}_0(\widehat{\mathcal{Q}})$.*

Proof. Given $w \in \mathcal{Q}$, we see that $\varphi_z(w) = \operatorname{Im} z^2 w^2 + \operatorname{Re} z^2 \longrightarrow \operatorname{Re} z^2$ as $z \longrightarrow \partial\mathcal{Q}$, and $\lim_{|z|\longrightarrow\infty} \varphi_z(w) = \infty$. Using (3.4) and the fact that the measure $\frac{16|w|dA(w)}{|w^2+i|^4}$ is normalized on $\mathcal{Q}$, as well as the dominated convergence theorem, we arrive to

$$\begin{aligned}\lim_{z\longrightarrow\partial_\infty\mathcal{Q}}(\mathcal{B}f(z) - f(z)) &= \lim_{z\longrightarrow\partial_\infty\mathcal{Q}} \int_{\mathcal{Q}} ((f\circ\varphi_z)(w) - f(z)) \frac{16|w|^2 dA(w)}{|w^2+i|^4}, \\ &= \int_{\mathcal{Q}} \lim_{z\longrightarrow\partial_\infty\mathcal{Q}} ((f\circ\varphi_z)(w) - f(z)) \frac{16|w|^2 dA(w)}{|w^2+i|^4} = 0.\end{aligned}$$

It follows that $\mathcal{B}f - f \in \mathcal{C}_0(\widehat{\mathcal{Q}})$, and $\mathcal{B}f \in \mathcal{C}(\widehat{\mathcal{Q}})$. □

Now, we define the invariant Laplacian:

Definition 3.9. For a twice continuously differentiable function f in $\mathcal{Q}$, the invariant Laplacian $\widetilde{\Delta}$, on the quarter plane $\mathcal{Q}$, is defined by

$$\widetilde{\Delta} f(z) = \frac{4(\operatorname{Im} z \operatorname{Re} z)^2}{|z|^2} \Delta f(z), \tag{3.5}$$

where $\Delta = 4\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z} = \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$ is the usual Laplacian.

It is well known that, if f is an analytic function, then $\Delta|f|^2 = |f'|^2$, and that more generally for any $\varphi \in \operatorname{Aut}(\mathcal{Q})$ we have

$$\Delta(f\circ\varphi)(w) = (\Delta f)(\varphi(w))|\varphi'(w)|^2. \tag{3.6}$$

The following result is related to the Möbius invariance property of the invariant Laplacian on the quarter-plane:

Proposition 3.10. *For every $f \in \mathscr{C}^2(\mathcal{Q})$, and for every quarter-plane automorphism φ_z, given by (2.5), we have*

$$\widetilde{\Delta}(f\circ\varphi_z) = (\widetilde{\Delta} f)\circ\varphi_z.$$

Proof. As in the proof of Proposition 3.6, direct computations lead to:

$$(\operatorname{Im} z^2 \operatorname{Im} w^2)^2 = 4(\operatorname{Im}\varphi_z(w)\operatorname{Re}\varphi_z(w))^2\ ,\ 4(\operatorname{Im} w \operatorname{Re} w)^2 = (\operatorname{Im} w^2)^2,$$

$$|\varphi_z'(w)|^2 = \frac{(\operatorname{Im} z^2)^2|w|^2}{|\varphi_z(w)|^2}.$$

Thus, by (3.5) and (3.6), we obtain

$$\begin{aligned}
\widetilde{\Delta}(f\circ\varphi_z)(w) &= \frac{4(\operatorname{Im} w \operatorname{Re} w)^2}{|w|^2}(\Delta f)(\varphi_z(w))|\varphi_z'(w)|^2,\\
&= \frac{(\operatorname{Im} w^2)^2}{|w|^2}\frac{(\operatorname{Im} z^2)^2|w|^2}{|\varphi_z(w)|^2}(\Delta f)(\varphi_z(w)),\\
&= \frac{4(\operatorname{Im}\varphi_z(w)\operatorname{Re}\varphi_z(w))^2}{|\varphi_z(w)|^2}(\Delta f)(\varphi_z(w)),\\
&= (\widetilde{\Delta} f)(\varphi_z(w)).
\end{aligned}$$

□

Propositions 3.6 and 3.10 suggest that the invariant Laplacian commutes with the Berezin transform. In fact, this is a general fact due to M. Engliš [7]:

Proposition 3.11. *Let $1 \le p \le \infty$, and suppose that $f \in \mathscr{C}^2(\mathcal{Q})$, and that both of f and $\widetilde{\Delta} f$ belong to $L^p(\mathcal{Q})$. Then, we have*

$$\widetilde{\Delta}(\mathcal{B}f) = \mathcal{B}(\widetilde{\Delta} f). \tag{3.7}$$

Proof. If the Berezin transform and the invariant Laplacian related to the Bergman space of the unit disk are respectively denoted by $\mathcal{B}_{\mathbb{D}}$ and $\widetilde{\Delta}_{\mathbb{D}}$, (see [1, 7, 10, 21] for details), then for $f \in \mathcal{C}^2(\mathcal{Q})$, by Propositions 3.6 and 3.10, we obtain:

$$\begin{aligned}
\mathcal{B}(\widetilde{\Delta} f) &= \mathcal{B}_{\mathbb{D}}((\widetilde{\Delta} f)\circ\Phi^{-1})\circ\Phi = \mathcal{B}_{\mathbb{D}}(\widetilde{\Delta}_{\mathbb{D}}(f\circ\Phi^{-1}))\circ\Phi,\\
&= \widetilde{\Delta}_{\mathbb{D}}\mathcal{B}_{\mathbb{D}}(f\circ\Phi^{-1})\circ\Phi = \widetilde{\Delta}_{\mathbb{D}}(\mathcal{B}f\circ\Phi^{-1})\circ\Phi = \widetilde{\Delta}(\mathcal{B}f).
\end{aligned}$$

□

A wonderful theorem, due independently to P. Ahern, M. Flores, and W. Rudin [2], and M. Engliš [8], (see also [10, 21] for nice expositions and more comments), asserts that the only integrable functions on $\mathbb{D}$ that are invariant under the action of the Berezin transform are the harmonic ones. Using the above techniques, we can establish the analogue of this result in the setting of the quarter plane $\mathcal{Q}$:

Theorem 3.12. *Let $1 \le p \le \infty$. Then, $f \in L^p(\mathcal{Q})$ is harmonic in $\mathcal{Q}$ if and only if $\widetilde{f} = f$.*

Proof. Suppose that $f \in L^p(\mathcal{Q})$. Then, by Ahern-Flores-Rudin-Engliš theorem and (2.3), we see that: $f\circ\Phi^{-1} \in L^1(\mathbb{D})$ is harmonic if and only if $\mathcal{B}_{\mathbb{D}}(f\circ\Phi^{-1}) = f\circ\Phi^{-1}$. On the other hand, as in Proposition 3.6, we get $\mathcal{B}_{\mathbb{D}}(f\circ\Phi^{-1})\circ\Phi = \mathcal{B}(f)$. It follows that: f is harmonic in $\mathcal{Q}$ if and only if $\mathcal{B}(f) = f$. □

4. Toeplitz operators on the Bergman space

In this section, we characterize bounded and compact Toeplitz operators with either harmonic or continuous symbols. For a function $g \in L^\infty(\mathcal{Q})$, denote by $\mathcal{M}_g$, the pointwise multiplication operator with symbol g on $L^2(\mathcal{Q})$:

$$\begin{aligned} \mathcal{M}_g : L^2(\mathcal{Q}) &\longrightarrow L^2(\mathcal{Q}), \\ f &\longrightarrow \mathcal{M}_g(f) = gf. \end{aligned}$$

Definition 4.1. The Toeplitz operator $T_g : L^2_a(\mathcal{Q}) \longrightarrow L^2_a(\mathcal{Q})$, with symbol $g \in L^\infty(\mathcal{Q})$, is defined by

$$T_g f(z) = (\mathcal{P}\mathcal{M}_g)(f)(z) = \mathcal{P}(gf)(z), \tag{4.1}$$

for all $f \in L^2_a(\mathcal{Q})$.

Using the integral representation (2.14) of the Bergman projection $\mathcal{P}$, we obtain the following useful representation of T_g as an integral operator:

$$T_g f(z) = \int_{\mathcal{Q}} \frac{-4z\overline{w}g(w)}{(z^2 - \overline{w}^2)^2} f(w)dA(w). \tag{4.2}$$

We observe that the integral representation (4.2) of the Toeplitz operator T_g makes sense for any symbol g in $L^2(\mathcal{Q}) \cup L^\infty(\mathcal{Q})$. The density of the space $L^p_a(\mathcal{Q}) \cap \mathcal{H}^\infty(\mathcal{Q})$ in $L^p_a(\mathcal{Q})$, stated in Proposition 2.5, guarantees that the above representation of the Toeplitz operator remains valid for any symbol g in $L^p(\mathcal{Q})$, for $1 \leq p < \infty$.

The next proposition summarizes the most useful elementary properties of Toeplitz operators [1, 19, 21]:

Proposition 4.2. *Given $\alpha, \beta \in \mathbb{C}$, and $f, g \in L^\infty(\mathcal{Q})$, then, we have*

1. *$T_g \in \mathscr{B}(L^2_a(\mathcal{Q}))$, with norm majorized by $\|g\|_\infty$, i.e. $\|T_g\| \leq \|g\|_\infty$.*
2. *$T_g^* = T_{\overline{g}}$, and thus T_g is self-adjoint if and only if g is real-valued.*
3. *$T_g = 0$ if and only if $g = 0$ a.e. in $\mathcal{Q}$.*
4. *$T_{\alpha f + \beta g} = \alpha T_f + \beta T_g$.*
5. *$T_g \geq 0$ if $g \geq 0$.*

Furthermore, if g is analytic, then we have

6. *T_g is one-to-one, provided that g is not identically zero.*
7. *$T_f T_g = T_{fg}$.*
8. *$T_{\overline{g}} T_f = T_{\overline{g}f}$.*

Note that the definition of Toeplitz operators can be extended to unbounded symbols as well, namely that there are unbounded symbols inducing bounded or even compact Toeplitz operators [21]. For instance, if $g \in L^2(\mathcal{Q})$ is compactly supported on $\mathcal{K} \Subset \mathcal{Q}$, then, for any $f \in L^2_a(\mathcal{Q})$, we have by the Cauchy-Schwarz inequality:

$$
\begin{aligned}
|T_g f(z)| &\leq \int_{\mathcal{Q}} \frac{4|z||w||g(w)||f(w)|}{|z^2-\overline{w}^2|^2} dA(w) = \int_{\mathcal{K}} \frac{4|z||w||g(w)||f(w)|}{|z^2-\overline{w}^2|^2} dA(\zeta),\\
&\leq \sup_{z\in\mathcal{K}} |f(z)| \int_{\mathcal{K}} \frac{4|z||w||g(w)|}{|z^2-\overline{w}^2|^2} dA(w),\\
&\leq \sup_{z\in\mathcal{K}} |f(z)| \|g\|_2 \left(\int_{\mathcal{K}} \frac{16|z|^2|w|^2 dA(w)}{|z^2-\overline{w}^2|^4}\right)^{\frac{1}{2}},\\
&\leq C_{\mathcal{K}} \|f\|_2 \|g\|_2 \left(\int_{\mathcal{K}} \frac{16|z|^2|w|^2 dA(w)}{|z^2-\overline{w}^2|^4}\right)^{\frac{1}{2}}, \qquad (4.3)
\end{aligned}
$$

where the last inequality follows from Inequality (2.7) of Lemma 2.2. Thus, by Fubini's theorem we infer that

$$
\begin{aligned}
\|T_g f\|_2^2 &= \int_{\mathcal{Q}} |T_g f(z)|^2 dA(z) \leq \int_{\mathcal{Q}} C_{\mathcal{K}}^2 \|f\|_2^2 \|g\|_2^2 \int_{\mathcal{K}} \frac{16|z|^2|w|^2 dA(w) dA(z)}{|z^2-\overline{w}^2|^4},\\
&= C_{\mathcal{K}}^2 \|f\|_2^2 \|g\|_2^2 \int_{\mathcal{K}} \int_{\mathcal{Q}} \frac{16|z|^2|w|^2 dA(z) dA(w)}{|z^2-\overline{w}^2|^4},\\
&= C_{\mathcal{K}}^2 \|f\|_2^2 \|g\|_2^2 \int_{\mathcal{K}} \frac{|w|^2 dA(w)}{4(\operatorname{Im} w \operatorname{Re} w)^2},\\
&\leq C_{\mathcal{K}} \|f\|_2^2 \|g\|_2^2. \qquad (4.4)
\end{aligned}
$$

Hence, we infer that $\|T_g\| \leq C_{\mathcal{K}} \|g\|_2$.

The following assertion characterizes boundedness and compactness of Toeplitz operators with harmonic symbols.

Theorem 4.3. *Let $g \in L^p(\mathcal{Q})$, with $1 \leq p \leq \infty$, be harmonic. Then, we have*

1. *T_g is bounded if and only if g is bounded. In such case, its norm is given by $\|T_g\| = \|g\|_\infty$.*
2. *If T_g is compact, then $\lim_{z\to\partial\mathcal{Q}} g(z) = 0$. Furthermore, if $\lim_{|z|\longrightarrow\infty} g(z) = 0$, then T_g is compact if and only if $g \equiv 0$.*

Proof. 1) The if part is obvious. For the only if part, suppose that T_g is bounded. Then, since g is harmonic, it is invariant under the action of the Berezin transform by Theorem 3.12, and thus we obtain $|g(z)| = |\mathcal{B}g(z)| = |\langle T_g k_z, k_z\rangle| \leq \|T_g\|$. We conclude that g is bounded, with $\|g\|_\infty \leq \|T_g\|$. Whence, by item (1) of Proposition 4.2, we deduce that the desired equality holds.

2) If T_g is compact, then, by Theorem 3.12, we have

$$|g(z)| = |\mathcal{B}g(z)| = |\langle T_g k_z, k_z\rangle| \leq \|T_g k_z\|_2 \longrightarrow 0 \text{ as } z \longrightarrow \partial\mathcal{Q},$$

because $k_z \longrightarrow 0$ weakly as $z \longrightarrow \partial\mathcal{Q}$, by Proposition 2.7. Furthermore, if $\lim_{|z|\longrightarrow\infty} g(z) = 0$, then, by the maximum principle for harmonic functions, we conclude that $g \equiv 0$ on $\mathcal{Q}$. □

The next assertion shows that the Berezin symbol of a bounded Toeplitz operator coincides with the Berezin transform of its defining symbol.

Proposition 4.4. *Let $1 \leq p \leq \infty$, and let T_f be bounded, where $f \in L^p(\mathcal{Q})$. Then, for the corresponding Berezin symbols, we have $\widetilde{T_f} = \widetilde{f}$.*

Proof. Direct applications of the formulas(3.1) and (3.2), we see that:

$$\widetilde{T}_f(z) = \langle T_f k_z, k_z\rangle = \int_{\mathcal{Q}} f(w)\,|k_z(w)|^2\, dA(w) = \mathcal{B}f(z), \text{ for all } z \in \mathcal{Q}. \qquad \square$$

The injectivity of the Berezin transform on the operator algebra $\mathscr{B}(L^2_a(\mathcal{Q}))$ is established in Proposition 3.5. Consequently, its injectivity on the function space $L^p(\mathcal{Q})$ follows as a corollary from Propositions 3.5 and 4.4.

Corollary 4.5. *The Berezin transform is one-to-one on $L^p(\mathcal{Q})$, for $1 \leq p \leq \infty$.*

Proof. Suppose that $\mathcal{B}(f) = 0$. Then, by Proposition 4.4, we see that $\widetilde{f} = \widetilde{T}_f = 0$. Hence, by Proposition 3.5, we infer that $T_f = 0$. Taking into account (2.1), (2.2) and (2.9), it follows that

$$\langle T_f e_n, e_m\rangle = 16\sqrt{(n+1)(m+1)} \int_{\mathcal{Q}} \zeta^n \overline{\zeta}^m (f\circ\Phi^{-1})(\zeta) dA(\zeta) = 0, \forall n, m \in \mathbb{N}.$$

By density of polynomials (in ζ and $\overline{\zeta}$) in $L^p(\mathbb{D})$, we infer that $f\circ\Phi^{-1} = 0$ *a.e.* on $\mathbb{D}$. Consequently, $f = 0$ *a.e.* on $\mathcal{Q}$. $\square$

Toeplitz operators with symbols that are compactly supported in $\mathcal{Q}$ are compact:

Proposition 4.6. *Suppose that $\mathcal{K}$ is a compact subset of $\mathcal{Q}$ and let $g \in L^2(\mathcal{Q}) \cup L^\infty(\mathcal{Q})$ with the property that $g = 0$ on $\mathcal{Q}\setminus\mathcal{K}$. Then, T_g is compact.*

Proof. Consider a norm bounded sequence $\{f_n\} \subset L^2_a(\mathcal{Q})$. Then, for any $z \in \mathcal{K}$, by (2.6) of Lemma 2.2, we have that

$$|f_n(z)| = |\langle f_n, K_z\rangle| \leq \|f_n\|_2\|K_z\|_2 \leq \frac{\|f_n\|_2}{4\delta},$$

with $\delta = \operatorname{dist}(\partial\mathcal{K}, \partial\mathcal{Q}) = \inf\{|z-w| : z \in \partial\mathcal{K} \text{ and } w \in \mathcal{Q}\}$. Whence, the sequence $\{f_n\}$ constitutes a normal family, and thus it contains a subsequence $\{f_{n_k}\}$ converging uniformly in $\mathcal{K}$ to some $f \in L^2_a(\mathcal{Q})$. The same steps (4.3)–(4.4) yield the following estimate

$$\|T_g f_{n_k} - T_g f\|_2 \leq C_{\mathcal{K}}\|f_{n_k} - f\|_2\|g\|_2.$$

Thus, $\{T_g(f_{n_k})\}$ converges to $T_g f$ in $L^2_a(\mathcal{Q})$, completing the proof. $\square$

Proposition 4.6 provides a necessary and sufficient condition for the compactness of a Toeplitz operator with continuous symbol vanishing on the boundary of $\widehat{\mathcal{Q}}$:

Proposition 4.7. *Let $g \in \mathscr{C}(\widehat{\mathcal{Q}})$ be such that $\lim_{|z|\longrightarrow\infty} g(z) = 0$. Then, T_g is compact if and only if $g \in \mathscr{C}_0(\widehat{\mathcal{Q}})$.*

Proof. Proceeding in the same manner as in the proof of Theorem 4.3, observe that if T_g is compact, then $\widetilde{g}(z) \longrightarrow 0$ as $z \longrightarrow \partial\mathcal{Q}$. Thus, Proposition 3.8 implies that $g \in \mathscr{C}_0(\widehat{\mathcal{Q}})$. Conversely, suppose that $g \in \mathscr{C}_0(\widehat{\mathcal{Q}})$ and consider the following closed square of $\mathcal{Q}$:

$$\mathcal{K}_n = \{z = x+iy \in \mathbb{C} : \ \frac{1}{n} \leq x \leq n, \quad \frac{1}{n} \leq y \leq n\}, \text{ for } n \in \mathbb{N}.$$

Then, Proposition 4.6 implies that $T_{g\chi_{\mathcal{K}_n}}$ is compact, with $\chi_{\mathcal{K}_n}$ being the characteristic function of the compact set $\mathcal{K}_n$. Now, for any f in $L_a^2(\mathcal{Q})$, we obtain

$$\begin{aligned}\|T_g f - T_{g\chi_{\mathcal{K}_n}} f\|_2^2 &= \|\mathcal{P}(gf) - \mathcal{P}(g\chi_{\mathcal{K}_n} f)\|_2^2 \leq \|gf - g\chi_{\mathcal{K}_n} f\|_2^2, \\ &= \int_{\mathcal{Q}} |g(z)f(z) - g\chi_{\mathcal{K}_n}(z)f(z)|^2 dA = \int_{\mathcal{Q}\setminus\mathcal{K}_n} |g(z)|^2 |f(z)|^2 dA, \\ &\leq \|f\|_2^2 \sup_{z\in\mathcal{Q}\setminus\mathcal{K}_n} |g(z)|^2.\end{aligned}$$

Thus, by assumption, we find that

$$\|T_g - T_{g\chi_{\mathcal{K}_n}}\|_2 \leq \sup_{z\in\mathcal{Q}\setminus\mathcal{K}_n} |g(z)| \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty.$$

We conclude that T_g is compact. □

For non-negative symbols, the compactness of the Toeplitz operator T_f and that of the underlying multiplication operator $\mathcal{M}_f$ are in fact very related to each other, as the next assertion shows. However, this is not the case in general, as the discussion preceding Theorem 4.3 reveals.

Theorem 4.8. *Let $f \in L^2(\mathcal{Q})$ be non-negative. Then, the Toeplitz operator T_f is compact on $L_a^2(\mathcal{Q})$ if and only if the compression of the underlying multiplication operator $\mathcal{M}_f : L_a^2(\mathcal{Q}) \longrightarrow L^2(\mathcal{Q})$ is compact.*

Proof. Since $T_f = \mathcal{P}\mathcal{M}_f$, the if part is trivial. For the only if part, suppose that T_f is compact and consider a sequence $\{g_m\} \subset L_a^2(\mathcal{Q})$, which is weakly convergent to 0 in $L_a^2(\mathcal{Q})$. Then, observe that

$$\langle T_f g_m, g_m\rangle = \langle f g_m, g_m\rangle = \int_{\mathcal{Q}} f(z)|g_m(z)|^2 dA(z) = \left\langle \sqrt{f} g_m, \sqrt{f} g_m \right\rangle = ||\mathcal{M}_{\sqrt{f}} g_m||_2^2,$$

where $\sqrt{f}$ is the positive square root of f. Now, since $T_f g_m \to 0$ strongly, we see that $\lim_{m\to\infty} ||\mathcal{M}_{\sqrt{f}} g_m||_2 = 0$. Whence $\mathcal{M}_{\sqrt{f}} : L_a^2(\mathcal{Q}) \longrightarrow L^2(\mathcal{Q})$ is compact. It follows that $\mathcal{M}_f = \mathcal{M}_{\sqrt{f}}\mathcal{M}_{\sqrt{f}}$ is compact on $L_a^2(\mathcal{Q})$. □

Acknowledgement. The author would like to extend his sincere appreciation to the Deanship of Scientific Research at King Saud University for funding this research through Research Group No. RG-1440-088.
The author would like to thank very much indeed the referee for the valuable comments and suggestions.

References

1. Ahern, P., & Čučković, Ž. (2001). A theorem of Brown-Halmos type for Bergman space Toeplitz operators. Journal of Functional Analysis. 187, 200-210. https://doi.org/10.1006/jfan.2001.3811
2. Ahern, P., Flores, M., & Rudin, W. (1994). An invariant volume mean-value property. Journal of Functional Analysis. 111, 380-397. https://doi.org/10.1006/jfan.1993.1018
3. Alshormani, F. S., & Guediri, H. (2023). Products of Toeplitz operators with angular symbols. Georgian Mathematical Journal, 30(1), 19-32. https://doi.org/10.1515/gmj-2022-2198

4. Axler, S., & Zheng, D. (1998). Compact operators via the Berezin transform. Indiana University Mathematical Journal. 47, 387-400. https://doi.org/10.1512/iumj.1998.47.1407
5. Chalendar, I., Fricain, E., Gürdal, M., & Karaev, M. (2012). Compactness and Berezin symbols. Acta Scientiarum Mathematicarum (Szeged), 78(1-2), 315-329. https://doi.org/10.1007/BF03651352
6. Duren, P. L., & Schuster, A. (2004). Bergman spaces. Mathematical Surveys and Monographs, Vol. 100, American Mathematical Society, Providence, RI. https://doi.org/10.1090/surv/100
7. Engliš, M. (1996). Berezin transform and the Laplace-Beltrami operator. Saint Petersburg Mathematical Journal, 7(4), 633-647. https://www.mathnet.ru/eng/aa/v7/i4/p176
8. Engliš, M. (1994). Functions invariant under the Berezin transform. Journal of Functional Analysis, 121(1), 233-254. https://doi.org/10.1006/jfan.1994.1048
9. Garnett, J. B. (1981). Bounded analytic functions. Academic Press. https://doi.org/10.1007/0-387-49763-3
10. Hedenmalm, H., Korenblum, B., & Zhu, K. H. (2000). Theory of Bergman spaces. Graduate Texts in Mathematics, Vol. 199, Springer-Verlag, New York. https://doi.org/10.1007/978-1-4612-0497-8
11. Kang, S. H. (2007). Berezin transform and Toeplitz operator on the weighted Bergman spaces of the half-plane. Bulletin of the Korean Mathematical Society, 44(2), 281-290. https://doi.org/10.4134/BKMS.2007.44.2.281
12. Kang, S. H. (2005). Some duality of weighted Bergman spaces of the half-plane. Bulletin of the Korean Mathematical Society, 42(2), 387-396. https://doi.org/10.4134/BKMS.2005.42.2.387
13. Kang, S. H., & Kim, J. Y. (2000). Toeplitz operators on the weighted Bergman spaces of the half-plane. Bulletin of the Korean Mathematical Society, 37(3), 437-450. https://koreascience.kr/article/JAKO200011919499825.pdf
14. Kang, S. H., & Kim, J. Y. (1999). Toeplitz operators on Bergman spaces defined on upper planes. Communications of the Korean Mathematical Society, 14(1), 171-177. https://api.semanticscholar.org/CorpusID:118488711
15. Koosis, P. (1998). Introduction to H^p spaces. Cambridge University Press, Second Edition. https://doi.org/10.1017/CBO9780511470950
16. Krantz, S. (2008). A tale of three kernels. Complex Variables and Elliptic Equations, 53(11), 1059-1082. https://doi.org/10.1080/17476930802429164
17. Lysenko, Z. M. (2016). Commutative algebras of Toeplitz operators on the strip. Matematychni Studii, 45, 98-103. https://doi.org/doi:10.15330/ms.45.1.98-103
18. Stroethoff, K. (1997). The Berezin transform and operators on spaces of analytic functions. Banach Center Publications, 38(1), 361-380. http://matwbn.icm.edu.pl/ksiazki/bcp/bcp38/bcp38122.pdf
19. Vasilevski, N. L. (2008). Commutative algebras of Toeplitz operators on the Bergman space. Birkhäuser, Basel-Boston-Berlin. https://doi.org/10.1063/1.3637748
20. Zhu, K. H. (2021). The Berezin transform and its applications. Acta Mathematica Scientia, Series B (English Edition), 41(6), 1839-1858. https://doi.org/10.1007/s10473-021-0603-5
21. Zhu, K. H. (2007). Operator theory in function spaces. Second Edition, Mathematical Surveys and Monographs, Vol. 138, American Mathematical Society, Providence, R.I. https://doi.org/10.1090/surv/138

Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia
Email address: hguediri@ ksu.edu.sa