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RELATIVELY PRIME RESTRAINED DETOUR DOMINATION NUMBER OF A GRAPH

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ABSTRACT. Consider a connected graph $G = (V, E)$ with a minimum of two vertices. A subset $S \subseteq V$ is termed a relatively prime restrained detour dominating set of G if it fulfills two conditions: firstly, S must be a relatively prime detour dominating set, and secondly, the induced subgraph $< V - S >$ should not contain any isolated vertices. The relatively prime restrained detour domination number, denoted as $\gamma_{rprdn}(G)$, is defined as the minimum cardinality of such a set that satisfies these conditions. Precise values for certain standard graphs, limits and some interesting results are established.

1. Introduction and preliminaries

A finite undirected graph, characterized by numerous edges and the absence of loops, is represented as $G = (V, E)$. In graph theory terminology, we adopt the notation from Chartrand and Lesniak's book [2]. Our primary focus is on connected graphs, defined as those containing two or more vertices. Detour distance in graphs was a concept that G. Chartrand et al. first developed in [3]. The detour number of a graph was a concept that Gray Chartrand et al. first developed in [1]. Fundamentals of domination in graph was a concept that studied in [4, 10, 11]. J. John et al. introduced the idea of a graph's detour domination number [8]. Restrained domination number of a graph was a concept that studied in [9, 12]. The notion of relatively prime dominating sets in graphs was first described in [7]. Jayasekaran et al. proposed the Relatively prime detour domination number of a graph [5, 6]. In this paper, we indicates that γ_{dn}-set is a minimal detour dominating set, whereas γ_{rpdn}-set is a minimal relatively prime detour dominating set and further introduce the idea of relatively prime restrained detour dominating set and also we find the number $\gamma_{rprdn}(G)$ of some standard graphs.

Theorem 1.1. [8] *Each end vertex of a graph G belongs to every detour dominating set of G.*

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Theorem 1.2. [5] *Each end vertex of a graph G belong to every relatively prime detour dominating set of G.*

2. Main results

Definition 2.1. A set $S \subseteq V$ is termed a relatively prime restrained detour dominating set of graph G if it qualifies as a relatively prime detour dominating set and additionally, the induced subgraph $< V - S >$ absence isolated vertices.

Definition 2.2. The $\gamma_{rprdn}(G)$ denotes the relatively prime restrained detour domination number signifies the lowest cardinality of such a set. A set of order $\gamma_{rprdn}(G)$ fulfilling these conditions is referred to as a γ_{rprdn}-set of G. If there is no relatively prime restrained detour dominating set, the relatively prime restrained detour domination number is zero.

Example 2.3. Considering the graph G in Figure 1. The set $S = \{v_3, v_5, v_6, v_7, v_8, v_9, v_{10}\}$ is a relatively prime detour dominating set but $< V - S >= K_2 \cup K_1$ contains an isolate vertex. Therefore, S cannot be a γ_{rprdn}-set. The set $S_1 = \{v_l : 3 \leq l \leq 10\}$ is a γ_{rpdn}-set and also $< V - S_1 >= P_2$ has no vertex of degree zeros. Therefore, S_1 is a γ_{rprdn}-set and hence $\gamma_{rprdn}(G) = 8$.

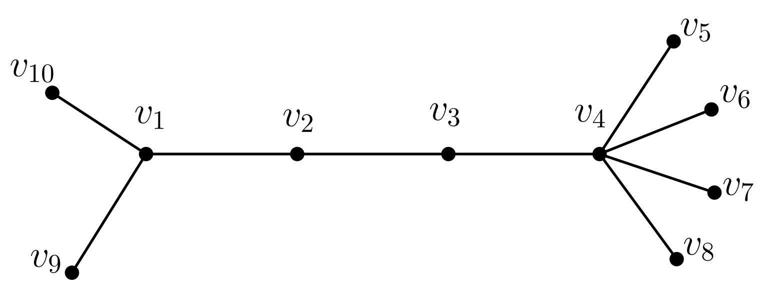

Figure 1: G

Proposition 2.4. *Every end vertex of a graph G belongs to every relatively prime restrained detour dominating set of G.*

Proof. The theorem follows from Theorems 1.1 and 1.2. □

Remark 2.5. Every relatively prime restrained detour dominating set is a relatively prime detour dominating set but the converse need not be true.

Consider the path $P_6 : v_1v_2v_3v_4v_5v_6$. Then $S = \{v_6, v_3, v_1\}$ is a relatively prime detour dominating set of P_6 but $< V - S >= K_1 \cup K_2$. Therefore, S is not a γ_{rprdn}- set of P_6.

Proposition 2.6. *Assume that G be a connected graph with size g. The condition $dn(G) \leq \gamma_{rpdn}(G) \leq \gamma_{rprdn}(G)$ holds if $\gamma_{rprdn}(G)$ exists.*

Proof. In order for $\gamma_{rpdn}(G)$ to exist, let's assume that G is a connected graph of size g. Since every γ_{rpdn}-set is a detour set, $dn(G) \leq \gamma_{rpdn}(G)$. Since by Proposition 2.4, $\gamma_{rpdn} \leq \gamma_{rprdn}$. Thus $dn(G) \leq \gamma_{rpdn}(G) \leq \gamma_{rprdn}(G)$. □

Remark 2.7. For the path P_2, $dn(P_2) = \gamma_{rpdn}(P_2) = \gamma_{rprdn}(P_2) = 2$.This makes all the inequality conditions in Proposition 2.6 become sharp. Considering the graph G in Figure 1. The set $S = \{v_5, v_6, v_7, v_8, v_9, v_{10}\}$ is a minimum detour set of G and so $dn(G) = 6$. Now we choose the set $S_1 = \{v_3, v_{10}, v_9, v_8, v_7, v_6, v_5\}$ is a γ_{rpdn}-set of G and hence $\gamma_{rpdn}(G) = 7$. Now the set $S_2 = \{v_3, v_{10}, v_9, v_8, v_7, v_6, v_5, v_4\}$ is a γ_{rprdn}-set of G and $< V - S_2 >$ has no vertex of degree zero and hence $\gamma_{rprdn}(G) = 8$. This makes all the inequality conditions in Proposition 2.6 strict.

Theorem 2.8. *For the path graph P_g, $g \geq 2$, then*

$$\gamma_{rprdn}(P_g) = \begin{cases} 2 & if\ g = 2, 4 \\ 3 & if\ g = 3, 5, 7 \\ 0 & otherwise \end{cases}$$

.

Proof. Let $v_1 v_2 ... v_g$ be the path P_g.
Case 1. $g = 2$ or 4
Then $S = \{v_1, v_g\}$ is a γ_{rpdn}-set of P_g and $< V - S >$ has no vertex of degree zero. Therefore, S is a γ_{rprdn}-set of P_g. Thus $\gamma_{rprdn}(P_g) = 2$.
Case 2. $g = 3$ or 5 or 7
If $g = 3$ or 5, then $S = \{v_1, v_{g-1}, v_g\}$ is a γ_{rpdn}-set of P_g and $< V - S >$ has no vertex of degree zero. Therefore, S is a γ_{rprdn}-set of P_g. Thus $\gamma_{rprdn}(P_g) = 3$.
If $g = 7$, then $S = \{v_1, v_{g-3}, v_g\}$ is a γ_{rpdn}-set of P_g and $< V - S >$ has no vertex of degree zero. Therefore, S is a γ_{rprdn}-set of P_g. Thus $\gamma_{rprdn}(P_g) = 3$.
Case 3. $g \geq 8$ and $g = 6$
If $g = 6$, then $S = \{v_1, v_3, v_g\}$ is a γ_{rpdn}-set of P_g and $< V - S >= K_1 \cup K_2$. Therefore, S is not a γ_{rprdn}-set of P_g and hence $\gamma_{rprdn}(P_g) = 0$.
If $g \geq 8$, then by Proposition 2.4, any relatively prime restrained detour dominating set S must contains v_1 and v_g. Since $< V - S >$ has no vertex of degree zero, S must contain at least two internal vertices $v_l, v_k, 3 \leq l \neq k \leq g - 2$ and $(deg(v_l), deg(v_k)) = 2$ which implies that $\gamma_{rprdn}(P_g) = 0$. □

Theorem 2.9. *For the star graph $K_{1,g}$ $(g \geq 2)$, $\gamma_{rprdn}(K_{1,g}) = g + 1$.*

Proof. Assume that G is a star graph with a vertex set of $V(K_{1,g}) = \{v, v_k : 1 \leq k \leq g\}$ and that $E(K_{1,g}) = \{vv_k : 1 \leq k \leq g\}$. Let S be a γ_{rpdn}-set of $K_{1,g}$. Then by Proposition 2.4, $\{v_1, v_2, ..., v_g\} \subseteq S$. Clearly, $S = \{v_1, v_2, ..., v_g\}$ itself is a γ_{rpdn}-set of G and $< V - S >$ has a vertex of degree zero. Now $S_1 = S \cup \{v\}$ is the only γ_{rprdn}-set of $K_{1,g}$ and hence $\gamma_{rprdn}(K_{1,g}) = g + 1$. □

Theorem 2.10. *For the bistar graph $B_{c,g}$, $\gamma_{rprdn}(B_{c,g}) = c + g$.*

Proof. Let G be a bistar graph with a vertex set of $V(B_{c,g}) = \{v, v_l, u, u_k : 1 \leq l \leq c, 1 \leq k \leq g\}$ and that $E(B_{c,g}) = \{uv, uu_k, vv_l : 1 \leq l \leq c, 1 \leq k \leq g\}$. Let S be a γ_{rpdn}-set of $B_{c,g}$. By Proposition 2.4, $\{v_l, u_k : 1 \leq l \leq c, 1 \leq k \leq g\} \subseteq S$. Clearly $\{v_l, u_k : 1 \leq l \leq c, 1 \leq k \leq g\}$ itself is a γ_{rpdn}-set of G and also $< V - S >= P_2$ has no vertex of degree zero and so $S = \{v_1, v_2, ..., v_c, u_1, u_2, ..., u_g\}$ itself is a γ_{rprdn}-set of G. Thus, $\gamma_{rprdn}(G) = c + g$.

□

Theorem 2.11. *For the complete bipartite graph $K_{c,g}$ which is not a star,* $\gamma_{rprdn}(K_{c,g}) = 2$ *if and only if* $(c, g) = 1$.

Proof. Let $V_1 \cup V_2$ be the bipartition of $V(K_{c,g})$ where $V_1 = \{u_l : 1 \leq l \leq c\}$ and $V_2 = \{v_k : 1 \leq k \leq g\}$. Since $K_{c,g}$ is not a star, both c and g are greater than 1. Also $deg(u_l) = g$ and $deg(v_k) = c$.

A γ_{dn}-set S of $K_{c,g}$ is $\{u_l, v_k\}$ and $(deg(u_l), deg(v_k)) = (c, g)$, where $1 \leq l \leq c, 1 \leq k \leq g$. Clearly $< V - S >= K_{c-1,g-1}$ and hence $S = \{u_l, v_k\}$ is a γ_{rprdn}-set if and only if $(c, g) = 1$. Therefore, $\gamma_{rprdn}(K_{c,g}) = 2$ if and only if $(c, g) = 1$.

□

Theorem 2.12. *For the wheel graph W_g,*

$$\gamma_{rprdn}(W_g) = \begin{cases} 2 & if\ g \not\equiv 1(mod\ 3) \\ 0 & otherwise \end{cases}.$$

Proof. Let $v_1v_2...v_{g-1}v_1$ be the cycle C_{g-1} and v be the K_1. The join $C_{g-1} + K_1$ is the wheel graph W_g. Also $deg(v) = g - 1$ and $deg(v_k) = 3$ for $1 \leq k \leq g - 1$. Clearly, $S_k = v, v_k, 1 \leq k \leq g - 1$, is a γ_{dn}-set and $< V - S_k >= P_{g-1}$ has no vertex of degree zero. This implies that S_k is a γ_{rprdn}-set of W_g if and only if $(deg(v), deg(v_k)) = (g - 1, 3) = 1$ if and only if $g - 1$ is not a multiple of 3 if and only if $g \not\equiv 1(mod3)$. Hence $\gamma_{rprdn}(W_g) = 2$ if and only if $g - 1 \not\equiv 0(mod3)$.

□

Theorem 2.13. *For the helm graph H_g, $\gamma_{rprdn}(H_g) = g$.*

Proof. Let $v_1v_2...v_{g-1}v_1$ be the cycle C_{g-1} and v be the K_1. The join $C_{g-1} + K_1$ is the wheel graph W_g. For $1 \leq k \leq g - 1$, add u_k which is adjacent to v_k. The graph obtained is denoted as the helm H_g. Then $deg(v) = g - 1, deg(v_k) = 4$ and $deg(u_k) = 1$ for $1 \leq k \leq g - 1$.

Assume that S is a γ_{rpdn}-set of H_g. By Proposition 2.4, $\{u_k : 1 \leq k \leq g - 1\} \in S$. Clearly $S = \{v, u_k : 1 \leq k \leq g - 1\}$ is a γ_{dn}-set and $< V - S >= C_{g-1}$ has no vertex of degree zero. This implies that S is a γ_{rprdn}-set of H_g and hence $\gamma_{rprdn}(H_g) = g$.

□

Theorem 2.14. *For the fan graph $F_{1,g}$,*

$$\gamma_{rprdn}(F_{1,g}) = \begin{cases} 2 & if\ g\ is\ not\ divisible\ by\ 2 \\ 0 & if\ g\ is\ divisible\ by\ 2 \end{cases}.$$

Proof. Let $v_1v_2...v_g$ be a path P_g and let u be the K_1. The join $K_1 + P_g$ is the fan graph $F_{1,g}$. Clearly $deg(u) = g, deg(v_1) = 2 = deg(v_g)$ and $deg(v_k) = 3$ for $2 \leq k \leq g-1$. Let S be a γ_{dn}-set of $F_{1,g}$. Then S is either $\{u, v_1\}$ or $\{u, v_g\}$.
Case 1. g does not divisible by two

Then S is a γ_{rpdn}-set of $F_{1,g}$ and $< V - S >= P_{g-1}$ has no vertex of degree zero. This implies that either S is a γ_{rprdn}-set and hence $\gamma_{rprdn}(F_{1,g}) = 2$.
Case 2. g is divisible by two

Then $(deg(u), deg(v_1)) = 2$ and $(deg(u), deg(v_g)) = 2$. This implies that $F_{1,g}$ has no γ_{rprdn}-set and hence $\gamma_{rprdn}(F_{1,g}) = 0$.

□

Theorem 2.15. *For $g \geq 3$, $\gamma_{rprdn}\,(C_g \odot K_1) = g$.*

Proof. Let $v_1v_2...v_gv_1$ be the cycle C_g. Include a new vertex u_k that shares an adjacency with v_k, where $1 \leq k \leq g$. The graph obtained is denoted as the graph $C_g \odot K_1$. Then $deg(u_k) = 1$ and $deg(v_k) = 3$ for $1 \leq k \leq g$. Let S be a γ_{rpdn}-set of $C_g \odot K_1$. By Proposition 2.4, $\{u_1, u_2, ..., u_g\} \subseteq S$. Clearly $S = \{u_1, u_2, ..., u_g\}$ and $< V - S >= C_g$ has no vertex of degree zero. Therefore, S itself is a γ_{rprdn}-set of $C_g \odot K_1$. Thus, $\gamma_{rprdn}(G) = g$.

□

Theorem 2.16. *For $g \geq 2, \gamma_{rprdn}\,(P_g \odot K_1) = g$.*

Proof. Let $v_1v_2...v_g$ be the path P_g. Insert the vertex u_k such that it is adjacent to v_k for each k within the range $1 \leq k \leq g$. The graph obtained is denoted as the graph $P_g \odot K_1$. Then $deg(u_k) = 1, deg(v_1) = 2, deg(v_g) = 2$ and $deg(v_l) = 3$ for $1 \leq k \leq g$ and $2 \leq l \leq g-1$. Let S be a γ_{rpdn}-set of $P_g \odot K_1$. By Proposition 2.4, $\{u_1, u_2, ..., u_g\} \subseteq S$. Clearly, $S = \{u_1, u_2, ..., u_g\}$ and $< V - S >= P_g$ has no vertex of degree zero. Therefore, S itself is a γ_{rprdn}-set of $P_g \odot K_1$. Thus, $\gamma_{rprdn}((P_g \odot K_1) = g$.

□

Theorem 2.17. *For $g \geq 1, \gamma_{rprdn}\,(K_{1,g} \odot K_1) = g + 1$.*

Proof. Consider the vertices of $K_{1,g}$ as $v, v_1, v_2, ..., v_k, ..., v_g$. Introduce vertices u_k is adjacent to v_k, where $1 \leq k \leq g$, and additionally, include a vertex u that is adjacent to v. The graph obtained is denoted as the graph $K_{1,g} \odot K_1$. Then $deg(u_k) = deg(u) = 1, deg(v_k) = 2$ and $deg(v) = g - 1$ for $1 \leq k \leq g$. Let S be a γ_{rpdn}-set of $K_{1,g} \odot K_1$. By Proposition 2.4, $\{u, u_1, u_2, ..., u_g\} \subseteq S$. Clearly $S = \{u_1, u_2, ..., u_g\}$ and $< V - S >= K_{1,g}$ has no vertex of degree zero. Therefore, S itself is a γ_{rprdn}-set of $K_{1,g} \odot K_1$. Thus $\gamma_{rprdn}(K_{1,g} \odot K_1) = g + 1$.

□

Proposition 2.18. *If G is a regular graph of degree $r \geq 2$, then $\gamma_{rprdn}(G) = 0$.*

Proof. Suppose that $\gamma_{rprdn}(G) > 0$. Any minimum relatively prime restrained detour dominating set that contains two or more vertices u, w such that $(deg(u), deg(w)) = r \neq 1$ has a contradiction if $\gamma_{rprdn}(G) > 0$. Therefore, $\gamma_{rprdn}(G) = 0$.

□

Corollary 2.19. *If G is a cycle C_g or a complete graph K_g, then $\gamma_{rprdn}(G) = 0$.*

Proof. The Theorem follows from Proposition 2.18, since cycle graph C_g and complete graph K_g are regular graphs.

□

Proposition 2.20. *Let T be an order g tree with several internal vertices and S be the set of all end vertices. If each internal vertex must be adjacent at least one end vertex in order for S to be a γ_{rprdn}-set of G.*

Proof. Suppose that T is connected acyclic graph of size g. Assuming that $S = \{v_1, v_2, ..., v_c\}$ is the set of all the end vertices of a tree T , then $deg(v_k) = 1$, $1 \leq k \leq c$. If at least one of the end vertices is adjacent to each internal vertex, then S is a γ_{dn}-set. Since T is a tree with more than one internal vertices, $< V - S >$, is also a tree has no vertex of degree zero. Therefore, S itself is a γ_{rprdn}-set of T and hence $\gamma_{rprdn}(T) = c$.

□

Proposition 2.21. *If S is a γ_{rpdn}-set of G with every vertex of degree must be one and $\gamma_{rprdn}(G)$ exists, then $\gamma_{rprdn}(G) = \gamma_{rpdn}(G)$. This holds true for any connected graph G with more than one internal vertex.*

Proof. Assuming that G is a connected graph with more than one internal vertex. Let S be a γ_{rpdn}-set of G with every vertex of degree must be one such that $\gamma_{rprdn}(G)$ exists. Given that G contains several internal vertices, $< V - S >$ does not contain a vertex of degree zero. Therefore, S is also a γ_{rprdn}-set of G. Thus $\gamma_{rprdn}(G) = \gamma_{rpdn}(G)$. □

3. Conclusion

For a relatively prime restrained detour dominating set, we have found a necessary condition in this paper. Certain well-known graphs, such as the path, star, bistar, complete bipartite, wheel graph, and helm graph, also possess relatively prime restrained detour domination numbers.

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