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GENERALIZED JORDAN TRIPLE DERIVATIONS ASSOCIATED WITH HOCHSCHILD 2-COCYCLES IN SEMIPRIME RINGS

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Abstract. In this article, under some conditions, we prove that every generalized Jordan triple derivation associated with Hochschild 2-cocycle in a 2-torsion free semiprime ring is a generalized derivation associated with Hochschild 2-cocycle.

1. Introduction

Throughout, R denotes an associative ring with center Z. A ring R is prime whenever $t_1, t_2 \in R$, $t_1Rt_2 = (0)$ denotes either $t_1 = 0$ or $t_2 = 0$, and is semiprime whenever $t_1 \in R$, $t_1Rt_1 = (0)$ denotes $t_1 = 0$. A ring R is m-torsion free, where $m \neq 0$ is an integer if whenever $ml = 0$ with $l \in R$, then $l = 0$. An R-bimodule W is a left and right R-module such that $l(wq) = (lw)q$ for all $w \in W$ and $l, q \in R$. A bi-additive map γ: $R \times R \longrightarrow W$ is called a Hochschild 2-cocycle, if $l\gamma(q, t) - \gamma(lq, t) + \gamma(l, qt) - \gamma(l, q)t = 0$ for all $l, q, t \in R$, and γ is called symmetric if $\gamma(l, q) = \gamma(q, l)$ for all $l, q \in R$. Nakajima [10] has introduced the notions of generalized derivations and generalized Jordan derivations associated with Hochschild 2-cocycles. An additive map G: $R \longrightarrow W$ is called a generalized derivation associated with a Hochschild 2-cocycle γ: $R \times R \longrightarrow W$ if $G(lq) = G(l)q + lG(q) + \gamma(l, q)$ for all $l, q \in R$, and G is called a generalized Jordan derivation associated with Hochschild 2-cocycle γ if $G(l^2) = G(l)l + lG(l) + \gamma(l, l)$ for all $l \in R$. If $\gamma = 0$, then G means the usual derivation and Jordan derivation.

In [3] Ezzat and Nabiel have introduced the notion of generalized Jordan triple derivations associated with Hochschild 2-cocycles. A map G: $R \longrightarrow W$ is called a generalized Jordan triple derivation associated with a Hochschild 2-cocycle γ: $R \times R \longrightarrow W$ if G additive and $G(lql) = G(l)ql + lG(q)l + lqG(l) + \gamma(l, q)l + \gamma(lq, l)$ for all $l, q \in R$.

Motivated by the results in [[1], [2], [4], [5], [6], [7], [9]], we prove that: If a ring R is a 2-torsion free semiprime. Then every generalized Jordan triple derivation on R associated with a Hochschild 2-cocycle γ is a generalized derivation associated

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with Hochschild 2-cocycle γ if one of the following holds. (i) γ is symmetric. (ii)There is $x, y, z \in R$ with $T(x, y, z) = xyz - zyx$ is a nonzero divisor.

2. Main results

Throughout this section, R is a 2-torsion free semiprime, $G\colon\ R \longrightarrow R$ is a generalized Jordan triple derivation associated with Hochschild 2-cocycle γ where $\gamma\colon\ R\times R \longrightarrow R$ and K, T are the maps from $R\times R\times R$ into R such that $K(l, q, t) = G(lql) - G(l)ql - lG(q)l - lqG(l) - \gamma(l, q)l - \gamma(lq, l)$ and $T(l, q, t) = lqt - tql$.

The following lemmas are useful to our result.

Lemma 2.1. *$K(l, q, t)T(x, y, z) = 0$ for all $l, q, t, x, y, z \in R$.*

Proof. We have for all $l, q \in R$

$$G(lql) = G(l)ql + lG(q)l + lqG(l) + \gamma(l, q)l + \gamma(lq, l) \tag{2.1}$$

We perform two computations of $G((l + t)q(l + t))$ for all $l, q, t \in R$. Firstly, we obtain $G((l + t)q(l + t)) = G((l + t))q(l + t) + (l + t)G(q)(l + t) + (l + t)qG(l + t) + \gamma(l + t, q)(l + t) + \gamma((l + t)q, l + t)$. Secondly, $G((l + t)q(l + t)) = G(lql) + G(lqt + tql) + G(tqt)$ for all $l, q, t \in R$. When the two expressions are compared, we arrive at

$$\begin{aligned} G(lqt + tql) = {}& G(l)qt + lG(q)t + lqG(t) + G(t)ql + tG(q)l + tqG(l) + \gamma(l, q)t \\ & + \gamma(lq, t) + \gamma(t, q)l + \gamma(tq, l) \quad \textit{for all } l, q, t \in R. \end{aligned} \tag{2.2}$$

Let $r = G(lqtwtql + tqlwlqt)$, then $0 = r - r = G((lqt)w(tql) + (tql)w(lqt)) - G(l(qtwtq)l + t(qlwlq)t)$. By (2.2), we find

$$\begin{aligned} 0 = {}& (G(lqt) - G(l)qt - lG(q)t - lqG(t))wtql + lqtw(G(tql) - G(t)ql - tG(q)l \\ & - tqG(l)) + (\gamma(lqt, w) - lq\gamma(t, w))tql + (\gamma(lqtw, tql) - \gamma(lqtwtq, l)) \\ & - (lq\gamma(tw, t)ql + l\gamma(q, twt)ql + l\gamma(qtwt, q)l + \gamma(l, qtwtq)l) + (G(tql) \\ & - G(t)ql - tG(q)l - tqG(l))wlqt + tqlw(G(lqt) - G(l)qt - lG(q)t - lqG(t)) \\ & + (\gamma(tql, w) - tq\gamma(l, w))lqt + (\gamma(tqlw, lqt) - \gamma(tqlwlq, t)) - (tq\gamma(lw, l)qt \\ & + t\gamma(q, lwl)qt + t\gamma(qlwl, q)t + \gamma(t, qlwlq)t). \end{aligned} \tag{2.3}$$

Since γ is a Hochschild 2-cocycle, we obtain the following relations
(i)$\gamma((lq)t, w) - lq\gamma(t, w) = \gamma(lq, tw) - \gamma(lq, t)w$
(ii)$\gamma(lqtw, (tq)l) - \gamma((lqtw)(tq), l) = \gamma(lqtw, tq)l - lqtw\gamma(tq, l)$
(iii)$\gamma((tq)l, w) - tq\gamma(l, w) = \gamma(tq, lw) - \gamma(tq, l)w$
(iv) $\gamma(tqlw, (lq)t) - \gamma((tqlw)(lq), t) = \gamma(tqlw, lq)t - tqlw\gamma(lq, t)$.
When we substitute in (2.3) from (i-iv), we need also
(v) $\gamma(lq, tw) = l\gamma(q, tw) + \gamma(l, q(tw)) - \gamma(l, q)tw$.
(vi) $\gamma(lqtw, tq) = -lqtw\gamma(t, q) + \gamma((lqtw)t, q) + \gamma(lqtw, t)q$.
(vii) $\gamma(tq, lw) = t\gamma(q, lw) + \gamma(t, q(lw)) - \gamma(t, q)lw$.

(viii) $\gamma(tqlw, lq) = -tqlw\gamma(l,q) + \gamma((tqlw)l,q) + \gamma(tqlw,l)q$.
By (2.3), we get

$$\begin{aligned}0 =&(G(lqt) - G(l)qt - lG(q)t - lqG(t) - \gamma(lq,t) - \gamma(l,q)t)wtql\\ &+ lqtw(G(tql) - G(t)ql - tG(q)l - tqG(l) - \gamma(tq,l) - \gamma(t,q)l)\\ &+ l(\gamma(q,tw)t - q\gamma(tw,t) - \gamma(q,twt))ql + (\gamma(lqtwt,q)\\ &- l\gamma(qtwt,q) - \gamma(l,qtwtq))l + (\gamma(l,qtw)t + \gamma(lqtw,t))ql\\ &+ ((G(tql) - G(t)ql - tG(q)l - tqG(l)) - \gamma(tq,l) - \gamma(t,q)l)wlqt\\ &+ tqlw(G(lqt) - G(l)qt - lG(q)t - lqG(t) - \gamma(lq,t) - \gamma(l,q)t)\\ &+ t(\gamma(q,lw)l - q\gamma(lw,l) - \gamma(q,lwl))qt + (\gamma(tqlwl,q)\\ &- t\gamma(qlwl,q) - \gamma(t,qlwlq))t + \gamma(t,qlw)lqt + \gamma(tqlw,l)qt.\end{aligned} \tag{2.4}$$

Also, we have
(ix) $\gamma(q,tw)t - q\gamma(tw,t) - \gamma(q,(tw)t) = -\gamma(q(tw),t)$.
(x) $\gamma(l(qtwt),q) - l\gamma(qtwt,q) - \gamma(l,(qtwt)q) = -\gamma(l,qtwt)q$.
(xi) $\gamma(q,lw)l - q\gamma(lw,l) - \gamma(q,(lw)l) = -\gamma(q(lw),l)$.
(xii) $\gamma(t(qlwl),q) - t\gamma(qlwl,q) - \gamma(t,(qlwl)q) = -\gamma(t,qlwl)q$.
By (ix-xii) and (2.4), we get

$$\begin{aligned}0 =&K(l,q,t)wtql + lqtwK(t,q,l) - l\gamma(qtw,t)ql - \gamma(l,qtwt)ql\\ &+ \gamma(l,qtw)tql + \gamma(lqtw,t)ql + K(t,q,l)wlqt + tqlwK(l,q,t)\\ &- t\gamma(qlw,l)qt - \gamma(t,qlwl)qt + \gamma(t,qlw)lqt + \gamma(tqlw,l)qt.\end{aligned} \tag{2.5}$$

By our hypothis, we find
$-l\gamma(qtw,t) - \gamma(l,(qtw)t) + \gamma(l,qtw)t + \gamma(l(qtw),t) = 0$ and $-t\gamma(qlw,l) - \gamma(t,(qlw)l) + \gamma(t,qlw)l + \gamma(t(qlw),l) = 0$.
From (2.5), we arrive at $K(l,q,t)wtql + lqtwK(t,q,l) + K(t,q,l)wlqt + tqlwK(l,q,t) = 0$. Equation (2.2) gives $K(l,q,t) = -K(t,q,l)$. Then $K(l,q,t)wtql - lqtwK(l,q,t) - K(l,q,t)wlqt + tqlwK(l,q,t) = 0$. Therefore,

$$K(l,q,t)wT(l,q,t) + T(l,q,t)wK(l,q,t) = 0 \ \textit{ for all } l,q,t,w \in R. \tag{2.6}$$

By equation (2.6) and [2, Lemma 1.1], we get $K(l,q,t)wT(l,q,t) = 0$ for all $l,q,t,w \in R$. It is clear that K,T are tri-additive maps. Then [2, Lemma 1.2] gives $K(l,q,t)wT(x,y,z) = 0$ for all $l,q,t,w,x,y,z \in R$. Again [2, Lemma 1.1] implies $K(l,q,t)T(x,y,z) = T(x,y,z)K(l,q,t) = 0$ for all $l,q,t,x,y,z \in R$. □

Lemma 2.2. *$K(l,q,t) \in Z$ for all $l,q,t \in R$.*

Proof. Lemma 2.1 gives for all $l,q,t,x,y,z \in R$,

$$K(l,q,t)(xyz - zyx) = 0. \tag{2.7}$$

Putting zr in place of z in (2.7) yields for all $l,q,t,x,y,z,r \in R$.

$$K(l,q,t)z[yx,r] = 0. \tag{2.8}$$

Therefore,

$$K(l,q,t)[yx,r]=0. \tag{2.9}$$

Substituting xd for x in (2.9), we find for all $l,q,t,r,d\in R$

$$K(l,q,t)yx[d,r]=0. \tag{2.10}$$

Using [2, Lemma 1.1] twice gives

$$K(a,b,c)[d,r]=[d,r]K(l,q,t)=0 \tag{2.11}$$

From [11, Lemma 1.3], we obtain $K(l,q,t)\in Z$ for all $l,q,t\in R$. □

For a ring R the set $S=\{s\in Z : sR\subseteq Z\}$ is an ideal of R and $sqt=tqs$ for all $s\in S$, $q,t\in R$, i.e., $T(s,q,t)=0$ for all $s\in S$ and $q,t\in R$.

Lemma 2.3. *If γ symmetric, then $K(l,q,t)=0$ for all $l,q,t\in R$.*

Proof. Equation (2.2) implies that $K(l,q,t)=-K(t,q,l)$ for all $l,q,t\in R$. Thus, for all $s\in S$ and $q,t\in R$, we have
$2K(s,q,t)^2=(2K(s,q,t))K(s,q,t))=(K(s,q,t)-K(t,q,s))K(s,q,t)$
$=(G(sqt)-G(s)qt-sG(q)t-sqG(t)-\gamma(s,q)t-\gamma(sq,t)-G(tqs)+G(t)qs+tG(q)s+tqG(s)+\gamma(t,q)s+\gamma(tq,s))K(s,q,t)$
$=(G(sqt)-G(tqs)+T(t,q,G(s))+T(t,G(q),s)+T(G(t),q,s)-\gamma(s,q)t-\gamma(sq,t)+\gamma(t,q)s+\gamma(tq,s))K(s,q,t)$
By Lemma 2.1, we have $T(x,y,z)K(l,q,t)=0$ for all $l,q,t,x,y,z\in R$. So, we arrive at
$2K(s,q,t)^2=(-\gamma(s,q)t-\gamma(sq,t)+\gamma(t,q)s+\gamma(tq,s))K(s,q,t)$
Since $-\gamma(s,q)t-\gamma(sq,t)=-\gamma(s,qt)-s\gamma(q,t)$, then
$2K(s,q,t)^2=(-\gamma(s,qt)-s\gamma(q,t)+\gamma(t,q)s+\gamma(tq,s))K(s,q,t)$.
Since γ is symmetric and $s\in S$, then $2K(s,q,t)^2=(-\gamma(s,qt)+\gamma(tq,s))K(s,q,t)$.
But, we have $\gamma(s,qt)=\gamma(sq,t)+\gamma(s,q)t-s\gamma(q,t)$ and $\gamma(tq,s)=\gamma(t,qs)+t\gamma(q,s)-\gamma(t,q)s$. Thus,
$2K(s,q,t)^2=(-\gamma(sq,t)-\gamma(s,q)t+s\gamma(q,t)+\gamma(t,qs)+t\gamma(q,s)-\gamma(t,q)s)K(s,q,t)$.
Hence
$2K(s,q,t)^2=(-\gamma(s,q)t+t\gamma(q,s))K(s,q,t)=[t,\gamma(q,s)]K(s,q,t)$. By equation (2.11), we arrive at $2K(s,q,t)^2=0$. Since R is a 2-torsion free, then $K(s,q,t)^2=0$. From Lemma 2.2 and the simprimeness of R, we obtain $K(s,q,t)=0$. That is $G(sqt)=G(s)qt+sG(q)t+sqG(t)+\gamma(s,q)t+\gamma(sq,t)$ for all $s\in S$ and $q,t\in R$. Since S is an ideal, then we replace s by sxl for $x,l\in R$. So, we have $G((sxl)qt)=G(sxl)qt+sxlG(q)t+sxlqG(t)+\gamma(sxl,q)t+\gamma(sxlq,t)=(G(s)xl+sG(x)l+sxG(l)+\gamma(s,x)l+\gamma(sx,l))qt+sxlG(q)t+sxlqG(t)+\gamma(sxl,q)t+\gamma(sxlq,t)$.
Also, $G(sx(lqt))=G(s)xlqt+sG(x)lqt+sxG(lqt)+\gamma(s,x)lqt+\gamma(sx,lqt)$. From the two expressions, we find $0=sx(G(lqt)-G(l)qt-lG(q)t-lqG(t))+\gamma(sx,lqt)-\gamma(sx,l)qt-\gamma(sxl,q)t-\gamma(sxlq,t)$. Since $\gamma(sx,lqt)-\gamma(sxlq,t)=\gamma(sx,lq)t-sx\gamma(lq,t)$, then $0=sx(G(lqt)-G(l)qt-lG(q)t-lqG(t))+\gamma(sx,lq)t-sx\gamma(lq,t)-\gamma(sx,l)qt-\gamma(sxl,q)t$. Again, we have $\gamma(sx,lq)-\gamma(sxl,q)=\gamma(sx,l)q-sx\gamma(l,q)$.
Hence, $0=sx(G(lqt)-G(l)qt-lG(q)t-lqG(t))-sx\gamma(lq,t)-\gamma(sx,l)qt+\gamma(sx,l)qt-sx\gamma(l,q)t$. Therefore $0=sx(G(lqt)-G(l)qt-lG(q)t-lqG(t)-\gamma(lq,t)-\gamma(l,q)t)$. That is

$$0=sxK(l,q,t)\ \text{ for all } s\in S \text{ and } l,q,t,x\in R. \tag{2.12}$$

Using equation(2.11) and $K(l,q,t) \in Z$ imply that $K(l,q,t)xy = K(l,q,t)yx = yK(l,q,t)x$, i.e., $K(l,q,t) \in S$. Replacing s by $K(l,q,t)$ in (2.12), we get $K(l,q,t) = 0$ for all $l,q,t \in R$. □

Lemma 2.4. *If $K(l,q,t) = 0$ for all $l,q,t \in R$, then G is a generalized derivation associated with γ.*

Proof. By our hypothis, we have for all $l,q,t \in R$

$$G(lqt) = G(l)qt + lG(q)t + lqG(t) + \gamma(l,q)t + \gamma(lq,t). \tag{2.13}$$

Now, we find $0 = G((lq)t(lq)) - G(l(qtl)q)$. By (2.13), we get
$0 = G(lq)tlq + lqG(t)lq + \gamma(lq,t)lq + lqtG(lq) + \gamma(lqt,lq) - G(l)qtlq - K(G(q)tl + qG(t)l + qtG(l) + \gamma(q,t)l + \gamma(qt,l))q - \gamma(l,qtl)q - lqtlG(q) - \gamma(lqtl,q)$. So,

$$\begin{aligned} 0 =&(G(lq) - G(l)q - lG(q))tlq + lqt(G(lq) - G(l)q - lG(q)) + (\gamma(lq,t) - l\gamma(q,t))lq \\ &+ (\gamma(lqt,lq) - \gamma(lqtl,q)) - l\gamma(qt,l)q - \gamma(l,qtl)q = 0. \end{aligned} \tag{2.14}$$

We own $\gamma(lq,t) - l\gamma(q,t) = \gamma(l,qt) - \gamma(l,q)t$ and $\gamma(lqt,lq) - \gamma((lqt)l,q) = \gamma(lqt,l)q - lqt\gamma(l,q)$.
Thus, (2.14) becomes

$$\begin{aligned} 0 =&(G(lq) - G(l)q - lG(q) - \gamma(l,q))tlq + lqt(G(lq) - G(l)q - lG(q) - \gamma(l,q)) \\ &+ \gamma(l,qt)lq + \gamma(lqt,l)q - l\gamma(qt,l)q - \gamma(l,qtl)q. \end{aligned}$$

Also, we have $\gamma(l,qt)l + \gamma(lqt,l) - l\gamma(qt,l) - \gamma(l,qtl) = 0$. Therefore

$$0 =(G(lq) - G(l)q - lG(q) - \gamma(l,q))t(lq) + (lq)t(G(lq) - G(l)q - lG(q) - \gamma(l,q)).$$

By [2, Lemma 1.1], we find $(G(lq)-G(l)q-lG(q)-\gamma(l,q))tlq = 0$ for all $l,q,t \in R$. [2, Lemma 1.2] gives $(G(lq)-G(l)q-lG(q)-\gamma(l,q))t(xd) = 0$ for all $l,q,t,x,d \in R$. By using [2, Lemma 1.1], we obtain $(G(lq) - G(l)q - lG(q) - \gamma(l,q))td = 0$ for all $l,q,t,d \in R$. Replacing d by $(G(lq) - G(l)q - lG(q) - \gamma(l,q))$, we get $(G(lq) - G(l)q - lG(q) - \gamma(l,q))t(G(lq) - G(l)q - lG(q) - \gamma(l,q)) = 0$ for all $l,q,t \in R$. The semiprimeness of R implies $G(lq) = G(l)q + lG(q) + \gamma(l,q)$ for all $l,q \in R$. □

By using the previous lemmas, we prove our main result.

Theorem 2.5. *G is a generalized derivation associated with a Hochschild 2-cocycle γ in each of the following cases:*
(i) γ is symmetric.
(ii)There is $x,y,z \in R$ with $T(x,y,z) = xyz - zyx$ is a nonzero divisor.

Proof. (i) Since γ is symmetric, then Lemma 2.3 gives $K(l,q,t) = 0$ for all $l,q,t \in R$. The required result comes from Lemma 2.4.
(ii) By Lemma 2.1, we have $K(l,q,t)T(x,y,z) = 0$ for all $l,q,t,x,y,z \in R$. From our assumption, we get $K(l,q,t) = 0$ for all $l,q,t \in R$. Lemma 2.4 lieds to G is a generalized derivation associated with γ. □

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