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KRAUSE MEAN PROCESSES GENERATED BY OFF-DIAGONALLY POSITIVE DOUBLY STOCHASTIC HYPER-MATRICES

MANSOOR SABUROV[1]*, KHIKMAT SABUROV[2], AND KHAJIBAY SABUROV[3]

ABSTRACT. Back in 1974, DeGroot first introduced the concept of achieving consensus through iterative averaging processes using square stochastic matrices. A pivotal question arises regarding the feasibility of extending the classical DeGroot model from square stochastic matrices to *higher-order stochastic hyper-matrices*. In the paper, we introduce a novel mathematical model for opinion-sharing dynamics employing *the Krause mean process* that is generated by *off-diagonally positive doubly stochastic hyper-matrices*. The principal objective is to achieve consensus within the multi-agent system.

1. Introduction

Historically, the concept of achieving consensus within a *structured, time-invariant, and synchronous environment* was initially introduced by DeGroot [4] back in 1974. Subsequently, Chatterjee and Seneta [3] in 1977 generalized DeGroot's model to encompass a *structured, time-varying, and synchronous environment*. In these models, *the backward product* of square stochastic matrices represents the opinion-sharing dynamics within a multi-agent system. In contrast, *the forward product* of square stochastic matrices illustrates a non-homogeneous Markov chain. Hence, achieving *consensus* within the multi-agent system and establishing the *ergodicity* of the Markov chain can be regarded as *dual problems* to one another. Since that time, consensus, which represents the most prevalent phenomenon in multi-agent systems, has gained popularity across diverse scientific communities, including biology, physics, control engineering, and social science. A more comprehensive model for representing the opinion-sharing dynamics is the *Krause mean process* wherein opinions are expressed as vectors. Recently, several nonlinear models have been developed to describe the dynamics of opinions within social communities (see the references [5, 6, 8, 9, 10]). For a comprehensive exposition of the Krause mean process, the reader may refer to the monograph by Krause [11].

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* Corresponding author.
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Historically, a *quadratic stochastic operator* was originally introduced by Bernstein [1] back in 1942. A *quadratic stochastic process* associated with a *cubic stochastic matrix* is considered to be the simplest *nonlinear Markov chain* [7, 12, 13, 14]. Indeed, it is noteworthy to mention that the rigorous establishment of the connection between *Krause mean processes* and *nonlinear Markov chains* was accomplished in a series of papers [2, 15, 16, 17, 18, 19, 20, 21]. In this paper, we introduce a novel mathematical model for opinion-sharing dynamics using *Krause mean processes* generated by *off-diagonally positive doubly stochastic hyper-matrices*. Subsequently, we proceed to establish consensus within the multi-agent system.

2. Background of the Study

We will initially provide a general remark regarding the novel contribution presented in this paper and then make a comparative analysis with respect to prior results presented in references [2, 15, 16, 17, 18, 19, 20, 21] concerning the consensus problem.

The classical Perron–Frobenius theorem for a *positive square stochastic matrix* plays an important role in addressing the consensus problem within multi-agent systems governed by *linear* protocols. According to this theorem, a linear stochastic operator associated with the *positive square stochastic matrix* possesses a *unique* stationary distribution (fixed point) within the simplex. Furthermore, its trajectory, starting from any point within the simplex, always *converges* to this unique stationary distribution (fixed point).

A pivotal question arises regarding the feasibility of extending the classical *DeGroot* and *Chatterjee-Seneta* models from *square stochastic matrices* to *higher-order stochastic hyper-matrices*. In this regard, it becomes imperative to explore and investigate an analogue of the classical Perron-Frobenius theorem for *positive higher-order stochastic hyper-matrices*. Specifically, one may inquire whether it holds true that *if a higher-order stochastic hyper-matrix is positive then the associated polynomial stochastic operator always has a unique stationary distribution (fixed point) within the simplex*. Furthermore, in such a scenario, it is of independent interest to determine whether *the trajectory of the polynomial stochastic operator associated with positive higher-order stochastic hyper-matrix always converges to this unique stationary distribution (fixed point)*.

However, in the nonlinear context, when dealing with polynomial stochastic operators associated with *positive* higher-order stochastic hyper-matrices, the situation is notably more intricate than one might anticipate. In other words, unlike linear case, the *positivity* of higher-order stochastic hyper-matrix fails to ensure the uniqueness of fixed points of the associated polynomial stochastic operators. Moreover, even when positivity and uniqueness of fixed points are established, it does not necessarily imply the stability of associated polynomial stochastic operators. Some concrete examples illustrating these nuances can be found in the papers [22, 23, 24]. Consequently, it becomes apparent that additional conditions need to be imposed on *positive* higher-order stochastic hyper-matrices in order to achieve consensus in nonlinear settings. This is why, at the initial stages of our investigation, we focused on studying the consensus problem within multi-agent systems governed by *positive higher-order triply stochastic hyper-matrices*. Indeed, the main results of the papers [16, 17, 18, 19, 20, 21] demonstrate that within the class of *positive higher-order triply stochastic hyper-matrices*, the uniqueness of fixed points for polynomial stochastic operators can be ensured. Furthermore, the polynomial stochastic operators associated with *positive higher-order triply stochastic hyper-matrices* exhibit stability. These findings can be regarded as groundbreaking advancements of the classical Perron-Frobenius theorem, extending its domain from *positive square stochastic matrices* to *positive higher-order triply stochastic hyper-matrices*. The subsequent step involved relaxing the *triple stochasticity* condition imposed on *positive higher-order stochastic hyper-matrices*. Indeed, in the paper [2], the consensus was established within the multi-agent system governed by *positive higher-order doubly stochastic hyper-matrices*. This can be regarded as an extension of the classical Perron-Frobenius theorem, transitioning from *positive square stochastic matrices* to *positive higher-order doubly stochastic hyper-matrices*. Following the establishment of consensus within multi-agent systems governed by *positive higher-order doubly stochastic hyper-matrices*, the subsequent research focus shifted towards relaxing the *positivity* condition of *higher-order doubly stochastic hyper-matrices*. This objective was successfully achieved for *diagonally primitive higher-order doubly stochastic hyper-matrices* in the paper [15]. It is worth noting that for *diagonally primitive higher-order doubly stochastic hyper-matrices*, off-diagonal entries may indeed be zero. As long as (some power of) the *diagonal matrix* of higher-order doubly stochastic hyper-matrix is *positive* then consensus can still be achieved within the system.

In this paper, our research is focused on the examination of *off-diagonally positive higher-order doubly stochastic hyper-matrices*. More specifically, the primary aim of this paper is to establish consensus within the multi-agent system, even in the cases where *the diagonal matrix of the higher-order doubly stochastic hyper-matrix may <u>not</u> necessarily be primitive*, as long as *its off-diagonal entries are positive*. In fact, the main result (Theorem 4.2) of this paper is a further extension and generalization of some related results reported in the papers [2, 15, 16, 17, 18, 19, 20, 21].

3. The Proposed Model

We now present some definitions and notations that will be used throughout this paper. Let $\Delta^{m-1} = \left\{ x \in \mathbb{R}^m : \sum\limits_{k=1}^{m} x_k = 1,\ x_k \geq 0,\ 1 \leq k \leq m \right\}$ be an $(m-1)-$ dimensional simplex.

Definition 3.1 (Stochastic Hyper-Matrix)**.** A $(k+1)-$order $m-$dimensional hyper-matrix $\mathcal{P} = (p_{i_1\cdots i_k j})_{i_1,\cdots,i_k,j=1}^{m,\cdots,m,m}$ is called *stochastic* if one has

$$\sum_{j=1}^{m} p_{i_1\cdots i_k j} = 1, \quad p_{i_1\cdots i_k j} \geq 0, \qquad 1 \leq i_1, \cdots, i_k, j \leq m.$$

Definition 3.2 (Doubly Stochastic Hyper-Matrix)**.** A $(k+1)-$order $m-$dimensional hyper-matrix $\mathcal{P} = (p_{i_1\cdots i_k j})_{i_1,\cdots,i_k,j=1}^{m,\cdots,m,m}$ is called *doubly stochastic* if one has

$$\sum_{i_k=1}^{m} p_{i_1\cdots i_k j} = \sum_{j=1}^{m} p_{i_1\cdots i_k j} = 1, \quad p_{i_1\cdots i_k j} \geq 0, \qquad 1 \leq i_1, \cdots, i_k, j \leq m.$$

Let $\mathcal{P} = (p_{i_1\cdots i_k j})_{i_1,\cdots,i_k j=1}^{m,\cdots,m,m}$ be the $(k+1)-$order $m-$dimensional doubly stochastic hyper-matrix. We define a polynomial stochastic operator $\mathfrak{P} : \Delta^{m-1} \to \Delta^{m-1}$ associated with $(k+1)-$order $m-$dimensional doubly stochastic hyper-matrix $\mathcal{P} = (p_{i_1\cdots i_k j})_{i_1,\cdots,i_k j=1}^{m,\cdots,m,m}$ as follows

$$(\mathfrak{P}(\mathbf{x}))_l = \sum_{i_1=1}^{m} \cdots \sum_{i_k=1}^{m} p_{i_1\cdots i_k l}\ x_{i_1} \cdots x_{i_k}, \qquad 1 \leq l \leq m. \tag{3.1}$$

We now define an $m \times m$ square matrix $\mathbb{P}(\mathbf{x}) = \left(p_{ij}(\mathbf{x})\right)_{i,j=1}^{m}$ for every $\mathbf{x} \in \Delta^{m-1}$ as follows

$$p_{ij}(\mathbf{x}) = \sum_{i_1=1}^{m} \cdots \sum_{i_{k-1}=1}^{m} p_{i_1\cdots i_{k-1} j i}\ x_{i_1} \cdots x_{i_{k-1}}. \tag{3.2}$$

We show that $\mathbb{P}(\mathbf{x})$ is square doubly stochastic matrix for every $\mathbf{x} \in \Delta^{m-1}$. In fact, we have that

$$\begin{aligned}
\sum_{i=1}^{m} p_{ij}(\mathbf{x}) &= \sum_{i_1=1}^{m} \cdots \sum_{i_{k-1}=1}^{m} \left(\sum_{i=1}^{m} p_{i_1\cdots i_{k-1} j i}\right) x_{i_1} \cdots x_{i_{k-1}} \\
&= \sum_{i_1=1}^{m} \cdots \sum_{i_{k-1}=1}^{m} x_{i_1} \cdots x_{i_{k-1}} = (x_1 + \cdots + x_m)^{k-1} = 1, \\
\sum_{j=1}^{m} p_{ij}(\mathbf{x}) &= \sum_{i_1=1}^{m} \cdots \sum_{i_{k-1}=1}^{m} \left(\sum_{j=1}^{m} p_{i_1\cdots i_{k-1} j i}\right) x_{i_1} \cdots x_{i_{k-1}} \\
&= \sum_{i_1=1}^{m} \cdots \sum_{i_{k-1}=1}^{m} x_{i_1} \cdots x_{i_{k-1}} = (x_1 + \cdots + x_m)^{k-1} = 1.
\end{aligned}$$

Therefore, it can be deduced from equations (3.1) and (3.2) that

$$\mathfrak{P}(\mathbf{x}) = \mathbb{P}(\mathbf{x})\mathbf{x} \tag{3.3}$$

and the equation (3.3) is called *a density dependent matrix form* of the polynomial stochastic operator (3.1) associated with the $(k+1)-$order $m-$dimensional doubly stochastic hyper-matrix.

Now, we are prepared to introduce the main protocol (model) of this paper, which is the *Krause mean process* generated by *doubly stochastic hyper-matrices* (for short, we use **DSHM**).

Protocol–DSHM: Let $\mathcal{P} = \left(p_{i_1\cdots i_k j}\right)_{i_1,\cdots,i_k,j=1}^{m,\cdots,m,m}$ be a $(k+1)-$order $m-$dimensional doubly stochastic hyper-matrix and let $\mathfrak{P} : \Delta^{m-1} \to \Delta^{m-1}$ be a polynomial stochastic operator associated with $(k+1)-$order $m-$dimensional doubly stochastic hyper-matrix $\mathcal{P} = \left(p_{i_1\cdots i_k j}\right)_{i_1,\cdots,i_k,j=1}^{m,\cdots,m,m}$. Suppose that an opinion-sharing dynamics of the multi-agent system is generated by the polynomial stochastic operator as follows

$$\mathbf{x}^{(n+1)} = \mathfrak{P}\left(\mathbf{x}^{(n)}\right), \quad \mathbf{x}^{(0)} \in \Delta^{m-1}$$

where $\mathbf{x}^{(n)} = \left(x_1^{(n)}, \cdots, x_m^{(n)}\right)^T$ is the subjective distribution after n revisions.

It follows from (3.3) that the opinion-sharing dynamics of the multi-agent system given by **Protocol–DSHM** can be written as follows

$$\mathbf{x}^{(n+1)} = \mathfrak{P}(\mathbf{x}^{(n)}) = \mathbb{P}(\mathbf{x}^{(n)})\mathbf{x}^{(n)}, \quad \mathbf{x}^{(0)} \in \Delta^{m-1}$$

This means that, due to the matrix form (3.3), the opinion-sharing dynamics of the multi-agent system given by **Protocol–DSHM** generates the Krause mean process. Some special cases of **Protocol–DSHM** have been considered in the previous studies [2, 15, 16, 17, 19, 18, 20, 21]. In this paper, we unify, extend, and generalize all previous results presented in the aforementioned papers.

Definition 3.3 (Consensus)**.** We say that the opinion-sharing dynamics of the multi-agent system governed by **Protocol–DSHM** eventually achieves *consensus* if an opinion-sharing dynamics converges to the center $\mathbf{c} = (\frac{1}{m}, \cdots, \frac{1}{m})^T$ of the simplex Δ^{m-1}.

4. Results

We introduce the concept of *off-diagonally positive* stochastic hyper-matrices.

Definition 4.1. A $(k+1)-$order $m-$dimensional stochastic hyper-matrix $\mathcal{P} = \left(p_{i_1\cdots i_k j}\right)_{i_1,\cdots,i_k,j=1}^{m,\cdots,m,m}$ is called *off-diagonally positive* if one has $p_{i_1\cdots i_k j} > 0$ for any $1 \leq i_1, \cdots, i_k, j \leq m$ with at least two different indices $1 \leq i_1, \cdots, i_k \leq m$.

We are now ready to state the main result of this paper.

Let $\mathbf{e}_1 = (1, 0, 0, \cdots, 0)^T$, $\mathbf{e}_2 = (0, 1, 0, \cdots, 0)^T, \cdots, \mathbf{e}_m = (0, 0, 0, \cdots, 1)^T$ be vertices of the simplex Δ^{m-1} and $\mathbf{e}_l^{(n+1)} := \mathfrak{P}\left(\mathbf{e}_l^{(n)}\right)$, where $\mathbf{e}_l^{(0)} := \mathbf{e}_l$ for all $1 \leq l \leq m$ and $n \in \mathbb{N}$.

Theorem 4.2. *Let $\mathcal{P} = \left(p_{i_1\cdots i_k j}\right)_{i_1,\cdots,i_k,j=1}^{m,\cdots,m,m}$ be a $(k+1)-$order $m-$dimensional off-diagonally positive doubly stochastic hyper-matrix and let $\mathfrak{P} : \Delta^{m-1} \to \Delta^{m-1}$ be the associated polynomial stochastic operator given by (3.1). The opinion-sharing dynamics of the multi-agent system given by* **Protocol–DSM** *eventually reaches a consensus for any initial opinion $\mathbf{x}^{(0)} \in \mathbb{S}^{m-1}$ if and only if for each $1 \leq l \leq m$ one has $\mathbf{e}_l^{(n_l)} \in \mathbb{S}^{m-1} \setminus \{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_m\}$ for some $1 \leq n_l \leq m$.*

Proof. Let $\mathcal{P} = \left(p_{i_1\cdots i_k j}\right)_{i_1,\cdots,i_k,j=1}^{m,\cdots,m,m}$ be a $(k+1)-$order $m-$dimensional *off-diagonally positive* doubly stochastic hyper-matrix and let $\mathfrak{P} : \Delta^{m-1} \to \Delta^{m-1}$ be the associated polynomial stochastic operator given by (3.1).

Let $\left\{\mathbf{x}^{(n)}\right\}_{n=0}^{\infty}$ where $\mathbf{x}^{(n+1)} = \mathfrak{P}(\mathbf{x}^{(n)})$ be a trajectory of the polynomial stochastic operator $\mathfrak{P} : \Delta^{m-1} \to \Delta^{m-1}$ starting from an initial point $\mathbf{x}^{(0)} \in \Delta^{m-1}$. Particularly, let $\left\{\mathbf{e}_l^{(n)}\right\}_{n=0}^{\infty}$ be a trajectory of the polynomial stochastic operator $\mathfrak{P} : \Delta^{m-1} \to \Delta^{m-1}$ starting from a vertex $\mathbf{e}_l$ of the simplex Δ^{m-1} for all

$1 \leq l \leq m$. According to the definition, the multi-agent system eventually reaches a consensus if and only if the sequence $\{\mathbf{x}^{(n)}\}_{n=0}^{\infty}$ converges to the center $\mathbf{c} = (\frac{1}{m}, \cdots, \frac{1}{m})^T$ of the simplex Δ^{m-1} for any initial point $\mathbf{x}^{(0)} \in \Delta^{m-1}$.

If Part: We assume that for each $1 \leq l \leq m$ one has $\mathbf{e}_l^{(n_l)} \in \Delta^{m-1} \setminus \{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_m\}$ for some $1 \leq n_l \leq m$. In that scenario, we aim to show that the sequence $\{\mathbf{x}^{(n)}\}_{n=0}^{\infty}$ converges to the center $\mathbf{c} = (\frac{1}{m}, \cdots, \frac{1}{m})^T$ of the simplex Δ^{m-1} for any initial point $\mathbf{x}^{(0)} \in \Delta^{m-1}$.

In this regard, we shall show that if $\mathbf{x} \in \Delta^{m-1} \setminus \{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_m\}$ then $\mathfrak{P}(\mathbf{x}) \in \mathrm{int}\Delta^{m-1}$. Indeed, if $\mathbf{x} \in \Delta^{m-1} \setminus \{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_m\}$ then $|\mathbf{supp}(\mathbf{x})| \geq 2$ where $\mathbf{supp}(\mathbf{x}) := \{i \in \mathbf{I}_m : x_i > 0\}$. Let us say $\alpha, \beta \in \mathbf{supp}(\mathbf{x})$ and $\alpha \neq \beta$. Since $\mathbf{p}_{i_1 \cdots i_k \bullet} > 0$ for all $i_1, \cdots, i_k \in \{\alpha, \beta\}$ at least two different indices $i_1, \cdots, i_k$ (because of off-diagonal positivity of the given doubly stochastic hyper-matrix), we obtain that

$$\mathfrak{P}(\mathbf{x}) = \sum_{i_1=1}^{m} \cdots \sum_{i_k=1}^{m} x_{i_1} \cdots x_{i_k} \mathbf{p}_{i_1 \cdots i_k \bullet} \geq \sum_{i_k \in \{\alpha,\beta\}} \cdots \sum_{i_k \in \{\alpha,\beta\}} x_{i_1} \cdots x_{i_k} \mathbf{p}_{i_1 \cdots i_k \bullet} > 0.$$

This means that $\mathfrak{P}(\mathbf{x}) \in \mathrm{int}\Delta^{m-1}$ for all $\mathbf{x} \in \Delta^{m-1} \setminus \{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_m\}$.

Particularly, if $\mathbf{x} \in \mathrm{int}\Delta^{m-1}$ then $\mathfrak{P}(\mathbf{x}) \in \mathrm{int}\Delta^{m-1}$, i.e., $\mathfrak{P}(\mathrm{int}\Delta^{m-1}) \subset \mathrm{int}\Delta^{m-1}$.

We now show that there always exists $n_0 \in \mathbb{N}$ such that for any initial point $\mathbf{x}^{(0)} \in \Delta^{m-1}$ one has $\mathbf{x}^{(n)} \in \mathrm{int}\Delta^{m-1}$ for all $n \geq n_0$. Indeed, as we already showed above that for all $\mathbf{x}^{(0)} \in \Delta^{m-1} \setminus \{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_m\}$, we have $\mathbf{x}^{(1)} \in \mathrm{int}\Delta^{m-1}$ and $\mathbf{x}^{(n)} \in \mathrm{int}\Delta^{m-1}$ for all $n \geq 1$. Hence, we can always choose $n_0 = 1$ for all $\mathbf{x}^{(0)} \in \Delta^{m-1} \setminus \{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_m\}$. Let $\mathbf{x}^{(0)} = \mathbf{e}_l$ for some $1 \leq l \leq m$. Since for each $1 \leq l \leq m$ one has $\mathbf{e}_l^{(n_l)} \in \Delta^{m-1} \setminus \{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_m\}$ for some $1 \leq n_l \leq m$, as we already showed above that for each $1 \leq l \leq m$ one has $\mathbf{e}_l^{(n_l+1)} \in \mathrm{int}\Delta^{m-1}$ and $\mathbf{e}_l^{(n)} \in \mathrm{int}\Delta^{m-1}$ for all $n \geq n_l + 1$. Let $n_0 := \max\limits_{1 \leq l \leq m} n_l + 1$. This shows that $\mathbf{e}_l^{(n)} \in \mathrm{int}\Delta^{m-1}$ for all $n \geq n_0$ and for all $1 \leq l \leq m$. Consequently, for any initial point $\mathbf{x}^{(0)} \in \Delta^{m-1}$ one has $\mathbf{x}^{(n)} \in \mathrm{int}\Delta^{m-1}$ for any $n \geq n_0 := \max\limits_{1 \leq l \leq m} n_l + 1$. It is worth mentioning that n_0 does not depend on an initial point $\mathbf{x}^{(0)} \in \Delta^{m-1}$.

We now show that for any $\mathbf{x}^{(0)} \in \Delta^{m-1}$ the omega limit set $\omega\left(\left\{\mathbf{x}^{(n)}\right\}\right)$ of the sequence $\{\mathbf{x}^{(n)}\}_{n=0}^{\infty}$ is a subset of the interior $\mathrm{int}\Delta^{m-1}$ of the simplex Δ^{m-1} i.e., $\omega\left(\left\{\mathbf{x}^{(n)}\right\}\right) \Subset \mathrm{int}\Delta^{m-1}$. This indeed follows from the previous observation that $\mathfrak{P}^{n_0}(\Delta^{m-1}) \Subset \mathrm{int}\Delta^{m-1}$ for $n_0 := \max\limits_{1 \leq l \leq m} n_l + 1$ and $\mathfrak{P}^{n}(\Delta^{m-1}) \Subset \mathrm{int}\Delta^{m-1}$ for any $n \geq n_0 := \max\limits_{1 \leq l \leq m} n_l + 1$. Since the image of simplex under the polynomial stochastic operator is a compact set, there exists $\alpha > 0$ such that

$$\mathfrak{P}^{n_0}(\mathbf{x}) \geq \alpha \mathbf{e} := (\alpha, \alpha, \cdots, \alpha)^T \qquad \forall\, \mathbf{x} \in \Delta^{m-1}.$$

On the other hand, since the interior $\mathrm{int}\Delta^{m-1}$ of the simplex Δ^{m-1} is an invariant set and $\mathfrak{P}^{n}(\Delta^{m-1}) \Subset \mathfrak{P}^{n_0}(\Delta^{m-1}) \subset \Delta_\alpha$ for any $n > n_0 := \max\limits_{1 \leq l \leq m} n_l + 1$, we have that $\{\mathbf{x}^{(n)}\}_{n=n_0}^{\infty} \subset \Delta_\alpha$, i.e., $\mathbf{x}^{(n)} \geq \alpha \mathbf{e}$ for any $n \geq n_0$ where

$$\Delta_\alpha := \{\mathbf{x} \in \Delta^{m-1} : \mathbf{x} \geq \alpha \mathbf{e}\}.$$

Consequently, the omega limit set $\omega\left(\{\mathbf{x}^{(n)}\}\right)$ of the sequence $\{\mathbf{x}^{(n)}\}_{n=0}^{\infty}$ is a subset of the set Δ_α, i.e., $\omega\left(\{\mathbf{x}^{(n)}\}\right) \subset \Delta_\alpha \subset \mathrm{int}\Delta^{m-1}$ for any $\mathbf{x}^{(0)} \in \Delta^{m-1}$.

As we showed in the previous step $\mathfrak{P}^n(\Delta^{m-1}) \subset \Delta_\alpha$ for any $n > n_0 := \max\limits_{1\le l\le m} n_l + 1$, it is therefore enough to study the dynamics of the polynomial stochastic operator over the set Δ_α which is an invariant set. Let $\mathbf{x}^{(0)} \in \Delta_\alpha$. Then $\mathbf{x}^{(n)} \in \Delta_\alpha$, i.e., $\mathbf{x}^{(n)} \ge \alpha\mathbf{e}$ for any $n > n_0 := \max\limits_{1\le l\le m} n_l + 1$. It follows from the matrix form (3.3) of the polynomial stochastic operator that

$$\mathbf{x}^{(n+1)} = \mathfrak{P}(\mathbf{x}^{(n)}) = \mathbb{P}\left(\mathbf{x}^{(n)}\right)\mathbf{x}^{(n)} = \mathbb{P}\left(\mathbf{x}^{(n)}\right)\cdots\mathbb{P}\left(\mathbf{x}^{(1)}\right)\mathbb{P}\left(\mathbf{x}^{(0)}\right)\mathbf{x}^{(0)}$$

where $\mathbb{P}\left(\mathbf{x}^{(n)}\right)$ is the square doubly stochastic matrix. Let us set for any two integer numbers $n > r$

$$\mathbb{P}^{\left[\mathbf{x}^{(n)},\mathbf{x}^{(r)}\right]} := \mathbb{P}\left(\mathbf{x}^{(n)}\right)\mathbb{P}\left(\mathbf{x}^{(n-1)}\right)\cdots\mathbb{P}\left(\mathbf{x}^{(r+1)}\right)\mathbb{P}\left(\mathbf{x}^{(r)}\right).$$

Then for any $n \ge r \ge 0$, we obtain

$$\mathbf{x}^{(n+1)} = \mathbb{P}^{\left[\mathbf{x}^{(n)},\mathbf{x}^{(0)}\right]}\mathbf{x}^{(0)} = \mathbb{P}^{\left[\mathbf{x}^{(n)},\mathbf{x}^{(r)}\right]}\mathbf{x}^{(r)}.$$

We set $\rho := \min\limits_{\substack{1\le i_1,\cdots,i_k,j\le m \\ \text{at least two } i_{l_1}\neq i_{l_2}}} p_{i_1\cdots i_k j} > 0$.

For a stochastic matrix $\mathbb{P}(\mathbf{x}^{(n)}) = \left(p_{ij}(\mathbf{x}^{(n)})\right)_{i,j=1}^{m}$, it follows from (3.2) that

$$\begin{aligned} p_{ij}(\mathbf{x}^{(n)}) &= \sum_{i_1=1}^{m}\cdots\sum_{i_{k-1}=1}^{m} p_{i_1\cdots i_{k-1}ji}\; x_{i_1}^{(n)}\cdots x_{i_{k-1}}^{(n)} \\ &\ge \sum_{\substack{1\le i_1,\cdots,i_{k-1}\le m \\ \text{at least two } i_{l_1}\neq i_{l_2}}} p_{i_1\cdots i_{k-1}ji}\; x_{i_1}^{(n)}\cdots x_{i_{k-1}}^{(n)} \ge \rho\alpha^k > 0 \end{aligned}$$

for any $1 \le i,j \le m$ and $n \ge n_0 := \max\limits_{1\le l\le m} n_l + 1$. Consequently, $\mathbb{P}(\mathbf{x}^{(n)}) = \left(p_{ij}(\mathbf{x}^{(n)})\right)_{i,j=1}^{m}$ is a *uniformly positive* square stochastic matrix for any $\mathbf{x}^{(0)} \in \Delta_\alpha$ and for any $n > n_0 := \max\limits_{1\le l\le m} n_l + 1$.

Let $\delta(\mathbb{P}) = \frac{1}{2}\max\limits_{i_1,i_2}\sum\limits_{j=1}^{m}|p_{i_1j} - p_{i_2j}|$ be Dobrushin's ergodicity coefficient (see [25]) of a square stochastic matrix $\mathbb{P} = (p_{ij})_{i,j=1}^{m}$. Since $\mathbb{P}(\mathbf{x}^{(n)}) = \left(p_{ij}(\mathbf{x}^{(n)})\right)_{i,j=1}^{m}$ is positive and its entries are uniformly bounded away from zero for any $\mathbf{x}^{(0)} \in \Delta_\alpha$ and for any $n \ge n_0 := \max\limits_{1\le l\le m} n_l + 1$, we obtain

$$\delta\left(\mathbb{P}(\mathbf{x}^{(n)})\right) \le \lambda < 1, \quad \forall\, n \ge n_0 := \max_{1\le l\le m} n_l + 1. \tag{4.1}$$

Hence, we have

$$\delta\left(\mathbb{P}^{\left[\mathbf{x}^{(n)},\mathbf{x}^{(0)}\right]}\right) \le \lambda^{n-n_0}, \quad \forall\, n \ge n_0 := \max_{1\le l\le m} n_l + 1, \qquad \lim_{n\to\infty}\delta\left(\mathbb{P}^{\left[\mathbf{x}^{(n)},\mathbf{x}^{(0)}\right]}\right) = 0.$$

Therefore, the backward product of doubly stochastic matrices $\{\mathbb{P}_{\mathbf{x}^{(n)}}\}_{n=1}^{\infty}$ is weakly ergodic (see [25]) which is also strongly ergodic (see [25]), i.e., we have

$$\lim_{n\to\infty} \mathbb{P}^{[\mathbf{x}^{(n)},\mathbf{x}^{(0)}]} = m\mathbf{c}^T\mathbf{c}, \quad \lim_{n\to\infty} \mathbf{x}^{(n+1)} = \lim_{n\to\infty} \mathbb{P}^{[\mathbf{x}^{(n)},\mathbf{x}^{(0)}]}\mathbf{x}^{(0)} = \mathbf{c}, \quad \forall\ \mathbf{x}^{(0)} \in \Delta_\alpha,$$

where $\mathbf{c} = (\frac{1}{m}, \cdots, \frac{1}{m})^T$. This means that the opinion-sharing dynamics of the multi-agent system given by **Protocol–DSM** eventually reaches a consensus for any initial opinion $\mathbf{x}^{(0)} \in \Delta^{m-1}$.

Only If Part: We now assume that the opinion-sharing dynamics of the multi-agent system given by **Protocol–DSM** eventually reaches a consensus for any initial opinion $\mathbf{x}^{(0)} \in \Delta^{m-1}$. This means that the sequence $\{\mathbf{x}^{(n)}\}_{n=0}^{\infty}$ converges to the center $\mathbf{c} = (\frac{1}{m}, \cdots, \frac{1}{m})^T$ of the simplex Δ^{m-1} for any initial point $\mathbf{x}^{(0)} \in \Delta^{m-1}$. Particularly, if we choose the vertex $\mathbf{e}_l$ (for all $1 \le l \le m$) of the simplex Δ^{m-1} as an initial point, i.e., $\mathbf{x}^{(0)} := \mathbf{e}_l$, then its orbit $\{\mathbf{e}_l^{(n)}\}_{n=0}^{\infty}$ must converge to the center $\mathbf{c} = (\frac{1}{m}, \frac{1}{m}, \cdots, \frac{1}{m})$ of the simplex Δ^{m-1}. Since $\mathbf{c} = (\frac{1}{m}, \frac{1}{m}, \cdots, \frac{1}{m}) \in \mathrm{int}\Delta^{m-1}$, we can deduce that for all $1 \le l \le m$ one has $\mathbf{e}_l^{(n_l)} \in \mathrm{int}\Delta^{m-1}$ for some $n_k \in \mathbb{N}$. Since $\mathcal{P} = \left(p_{i_1\cdots i_k j}\right)_{i_1,\cdots,i_k,j=1}^{m,\cdots,m,m}$ is an *off-diagonally positive* doubly stochastic hypermatrix, as we already showed that in order to get $\mathbf{e}_l^{(n_l)} \in \mathrm{int}\Delta^{m-1}$ for some $n_l \in \mathbb{N}$, it is sufficient to have $\mathbf{e}_l^{(n_l)} \in \Delta^{m-1} \setminus \{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_m\}$ for some $1 \le n_l \le m$. This completes the proof. $\square$

5. Discussions

Now, we proceed to discuss the main result of this paper and compare it with previous findings [2, 15, 16, 17, 18, 19, 20, 21] on the consensus problem. For the sake of simplicity, we consider the case $k = 2$ and $m = 3$.

Let $\mathcal{P} = (\mathbb{P}_{1\bullet\bullet}|\mathbb{P}_{2\bullet\bullet}|\mathbb{P}_{3\bullet\bullet})$ be a cubic doubly stochastic matrix defined by square doubly stochastic matrices $\mathbb{P}_{1\bullet\bullet}$, $\mathbb{P}_{2\bullet\bullet}$, and $\mathbb{P}_{3\bullet\bullet}$ as follows

$$\mathbb{P}_{1\bullet\bullet} = \begin{pmatrix} p_{111} & p_{112} & p_{113} \\ p_{121} & p_{122} & p_{123} \\ p_{131} & p_{132} & p_{133} \end{pmatrix}, \ \mathbb{P}_{2\bullet\bullet} = \begin{pmatrix} p_{211} & p_{212} & p_{213} \\ p_{221} & p_{222} & p_{223} \\ p_{231} & p_{232} & p_{233} \end{pmatrix}, \ \mathbb{P}_{3\bullet\bullet} = \begin{pmatrix} p_{311} & p_{312} & p_{313} \\ p_{321} & p_{322} & p_{323} \\ p_{331} & p_{332} & p_{333} \end{pmatrix}.$$

Protocol–DSM defined by the quadratic stochastic operator $\mathcal{Q} : \mathbb{S}^2 \to \mathbb{S}^2$ takes the following form

$$\mathcal{Q}(\mathbf{x}) = x_1\left(\mathbb{P}_{1\bullet\bullet}^T\mathbf{x}\right) + x_2\left(\mathbb{P}_{2\bullet\bullet}^T\mathbf{x}\right) + x_3\left(\mathbb{P}_{3\bullet\bullet}^T\mathbf{x}\right) = \mathbb{P}\left(\mathbf{x}\right)\mathbf{x}$$

where $\mathbb{P}\left(\mathbf{x}\right) = x_1\mathbb{P}_{1\bullet\bullet}^T + x_2\mathbb{P}_{2\bullet\bullet}^T + x_3\mathbb{P}_{3\bullet\bullet}^T$ is a square doubly stochastic matrix.

It was initially established in the series of papers [16, 17, 18, 19, 20, 21] that the consensus is achieved in the system described by **Protocol–DSM** when the square doubly stochastic matrices $\mathbb{P}_{1\bullet\bullet} > 0$, $\mathbb{P}_{2\bullet\bullet} > 0$, $\mathbb{P}_{3\bullet\bullet} > 0$ are positive and

$$\mathbb{P}_{1\bullet\bullet} + \mathbb{P}_{2\bullet\bullet} + \mathbb{P}_{3\bullet\bullet} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}. \tag{5.1}$$

Subsequently, this result was further extended and generalized in the paper [2]. Specifically, without the constraint (5.1), the consensus was still achieved for positive square doubly stochastic matrices $\mathbb{P}_{1\bullet\bullet} > 0$, $\mathbb{P}_{2\bullet\bullet} > 0$, $\mathbb{P}_{3\bullet\bullet} > 0$.

The next research focus shifted towards relaxing the *positivity* condition of *cubic doubly stochastic matrices*. This objective was successfully achieved for *diagonally primitive cubic doubly stochastic matrices* in the paper [15]. Indeed, the consensus is established within the system governed by **Protocol–DSM** even when the square doubly stochastic matrices $\mathbb{P}_{1\bullet\bullet}$, $\mathbb{P}_{2\bullet\bullet}$, and $\mathbb{P}_{3\bullet\bullet}$ are not necessarily positive and without enforcing the constraint (5.1). Namely, the consensus still holds as long as the cubic doubly stochastic matrix $\mathcal{P}$ is solely *diagonally primitive*, i.e., for some $s \in \mathbb{N}$ we have

$$[\operatorname{diag}(\mathcal{P})]^s = \begin{pmatrix} p_{111} & p_{112} & p_{113} \\ p_{221} & p_{222} & p_{223} \\ p_{331} & p_{332} & p_{333} \end{pmatrix}^s > 0.$$

It is worth noting that for *diagonally primitive cubic doubly stochastic matrices*, some off-diagonal entries of cubic doubly stochastic matrix $\mathcal{P}$ may indeed be zero.

It is important to emphasize that there is still room for further enhancement of these findings. In this paper, our research is focused on the examination of *off-diagonally positive cubic doubly stochastic matrices*. More specifically, the primary aim of this paper is to establish consensus within the multi-agent system, even in the cases where *the diagonal matrix of the cubic doubly stochastic matrix may <u>not</u> necessarily be primitive*, as long as *its off-diagonal entries are positive*.

In fact, Theorem 4.2 represents a further extension and generalization of some related results reported in the papers [2, 15, 16, 17, 18, 19, 20, 21]. To illustrate it, let us now provide some examples.

Example 5.1. Given the square doubly stochastic matrices

$$\mathbb{P}_{1\bullet\bullet} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & 0 \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \end{pmatrix}, \quad \mathbb{P}_{2\bullet\bullet} = \begin{pmatrix} \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \\ \frac{2}{3} & \frac{1}{3} & 0 \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \end{pmatrix}, \quad \mathbb{P}_{3\bullet\bullet} = \begin{pmatrix} \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \\ \frac{2}{3} & \frac{1}{3} & 0 \end{pmatrix}.$$

Since for any $s \in \mathbb{N}$

$$[\operatorname{diag}(\mathcal{P})]^s = \operatorname{diag}(\mathcal{P}) = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & 0 \\ \frac{2}{3} & \frac{1}{3} & 0 \\ \frac{2}{3} & \frac{1}{3} & 0 \end{pmatrix},$$

the cubic doubly stochastic matrix $\mathcal{P} = (\mathbb{P}_{1\bullet\bullet}|\mathbb{P}_{2\bullet\bullet}|\mathbb{P}_{3\bullet\bullet})$ <u>is not</u> *diagonally primitive*. Therefore, we cannot apply the main results of the papers [2, 15, 16, 17, 18, 19, 20, 21].

In contrast, since $\mathbf{p}_{ij\bullet} = (\frac{1}{6}, \frac{1}{3}, \frac{1}{2}) > 0$ for all $1 \le i, j \le 3$ with $i \neq j$, i.e., $\mathcal{P} = (\mathbb{P}_{1\bullet\bullet}|\mathbb{P}_{2\bullet\bullet}|\mathbb{P}_{3\bullet\bullet})$ is *off-diagonally positive*. Moreover, we have that

$$\mathcal{Q}(\mathbf{e}_1) = \mathcal{Q}(\mathbf{e}_2) = \mathcal{Q}(\mathbf{e}_3) = \left(\frac{2}{3}, \frac{1}{3}, 0\right) = \frac{2}{3}\mathbf{e}_1 + \frac{1}{3}\mathbf{e}_2 \in \Delta^2 \setminus \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\},$$

$$\mathcal{Q}^2(\mathbf{e}_1) = \mathcal{Q}^2(\mathbf{e}_2) = \mathcal{Q}^2(\mathbf{e}_3) = \left(\frac{4}{9}, \frac{1}{3}, \frac{2}{9}\right) = \frac{1}{9}\left(4\mathbf{e}_1 + 3\mathbf{e}_2 + 2\mathbf{e}_3\right) \in \operatorname{int}\Delta^2.$$

Therefore, according to Theorem 4.2, the opinion-sharing dynamics of the multi-agent system described by **Protocol–DSM** ultimately achieves consensus for any initial opinion $\mathbf{x} \in \Delta^2$.

Hence, the example 5.1 shows that Theorem 4.2 extends, generalizes, and unifies some related results of the papers [2, 15, 16, 17, 18, 19, 20, 21]. Most importantly, the main results of the aforementioned papers do not inherently imply the main result (Theorem 4.2) of this paper.

6. Conclusions

Historically, the concept of achieving consensus within a structured, time-invariant, and synchronous environment was initially introduced by DeGroot [4] in 1974. Subsequently, Chatterjee and Seneta [3] in 1977 extended DeGroot's model to encompass structured, time-varying, and synchronous environments.

A pivotal question arose regarding the feasibility of extending the classical *DeGroot* and *Chatterjee-Seneta* models from *square stochastic matrices* to *higher-order stochastic hyper-matrices*. In this regard, it became imperative to explore and investigate an analogue of the classical Perron-Frobenius theorem for *positive higher-order stochastic hyper-matrices*. However, in the nonlinear context, when dealing with polynomial stochastic operators associated with *positive* higher-order stochastic hyper-matrices, the situation was notably more intricate than one might anticipate. Consequently, it became apparent that additional conditions need to be imposed on *positive* higher-order stochastic hyper-matrices in order to achieve consensus in nonlinear settings. This is why, at the early stages of our investigation [16, 17, 18, 19, 20, 21], we focused on studying the consensus problem within multi-agent systems governed by *positive higher-order triply stochastic hyper-matrices*. The subsequent step involved relaxing the *triple stochasticity* condition imposed on *positive higher-order stochastic hyper-matrices*. Following the establishment of consensus within multi-agent systems governed by *positive higher-order doubly stochastic hyper-matrices* [2], the next research focus shifted towards relaxing the *positivity* condition of *higher-order doubly stochastic hyper-matrices*. This objective was successfully achieved for *diagonally primitive higher-order doubly stochastic hyper-matrices* in the paper [15]. In this paper, our research was focused on the examination of *off-diagonally positive higher-order doubly stochastic hyper-matrices*. More specifically, the primary aim of this paper was to establish consensus within the multi-agent system, even in the cases where *the diagonal matrix of the higher-order doubly stochastic hyper-matrix may not necessarily be primitive*, as long as *its off-diagonal entries are positive*. In fact, the main result (Theorem 4.2) of this paper can be considered a further extension and generalization of some related results reported in the papers [2, 15, 16, 17, 18, 19, 20, 21].

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[1] College of Engineering and Technology, American University of the Middle East, KUWAIT.
Email address: mansur.saburov@aum.edu.kw

[2] Kimyo International University in Tashkent, UZBEKISTAN; AJOU University in Tashkent, UZBEKISTAN; National University of Uzbekistan, Tashkent, UZBEKISTAN.
Email address: h.saburov@nuu.uz

[3] University of Tashkent for Applied Sciences, UZBEKISTAN.
Email address: munisamunib@gmail.com