Gulf Journal of Mathematics
Vol 7, Issue 1 (2019) 56-67

ON MULTIPLICATIVE GENERALIZED (α, β)-DERIVATIONS IN PRIME RINGS

ASMA ALI[1*], AMBREEN BANO[2] AND MD. HAMIDUR RAHAMAN[3]

Abstract. Let R be an associative ring with center $Z(R)$. A mapping $F : R \to R$ is called a multiplicative generalized derivation on R if there exists a derivation d on R such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$. Let α, β be endomorphisms of R. A mapping $F : R \to R$ (not necessarily additive) is said to be a multiplicative generalized (α, β)-derivation if there exists an (α, β)-derivation d on R such that $F(xy) = F(x)\alpha(y) + \beta(x)d(y)$ holds for all $x, y \in R$. In this paper we investigate some identities involving multiplicative generalized (α, β)-derivation in a prime ring R and obtain commutativity of R.

1. Introduction and preliminaries

Throughout the paper, R will denote an associative ring with center $Z(R)$. Recall that a ring R is prime if for any $a, b \in R$, $aRb = 0$ implies that either $a = 0$ or $b = 0$ and is called semiprime if for any $a \in R$, $aRa = 0$ implies that $a = 0$. We shall write for any pair of elements $x, y \in R$ the commutator $[x, y] = xy - yx$ and skew commutator $x \circ y = xy + yx$. We will frequently use the basic commutator and skew commutator identities:

$$\begin{aligned} [xy, z] &= x[y, z] + [x, z]y \text{ and } [x, yz] = y[x, z] + [x, y]z. \\ x \circ (yz) &= (x \circ y)z - y[x, z] = y(x \circ z) + [x, y]z. \\ (xy) \circ z &= x(y \circ z) - [x, z]y = (x \circ z)y + x[y, z]. \end{aligned}$$

for all $x, y, z \in R$.

By a derivation, we mean an additive mapping $d : R \to R$ such that $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. The concept of derivation was extended to generalized derivation in [7] by Bresar. An additive mapping $F : R \to R$ is said to be a generalized derivation if there exists a derivation $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$. In [2] Hvala gave the algebraic study of generalized derivation in prime rings. Obviously every derivation is a generalized derivation of R. Many papers in literature have investigated the commutativity of prime and semiprime rings satisfying certain functional identities involving

Date: Received: Feb 11, 2016; Accepted: Jan 4, 2017.
* Corresponding author.
2010 *Mathematics Subject Classification.* Primary 16W25; Secondary 16N60,16U80.
Key words and phrases. Prime rings, (α, β)-derivations, Multiplicative generalized (α, β)-derivations.

derivations or generalized derivations (see [1], [4], [6], [10], [15]). To the best of my knowledge, the concept of multiplicative derivation appeared for the first time in the work of Daif [8]. Then the complete description of those maps was given by Goldmann and Semrl in [3]. Further, Daif and Tammam-El-Sayiad [9] extended the notion of multiplicative derivation to multiplicative generalized derivation as follows. A mapping $F : R \to R$ (not necessarily additive) is called a multiplicative generalized derivation if it satisfies $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$, where d is a derivation on R. Obviously, every generalized derivation is a multiplicative generalized derivation on R. Chang [4] introduced the notion of a generalized (α,β)-derivation of a ring R and investigated some properties of such derivations. Let α,β be mappings of R into itself. An additive mapping g of R into itself is called a generalized (α,β)-derivation of R, if there exists an (α,β) derivation d of R such that $g(xy) = g(x)\alpha(y) + \beta(x)d(y)$ for all $x, y \in R$. Obviously this notion covers the notion of a generalized derivation (in case $\alpha = \beta = 1$), notion of a derivation (in case $g = d, \alpha = \beta = 1$), notion of a left centralizer (in case $d = 0, \alpha = 1$), notion of (α,β)-derivation (in case $g = d$) and the notion of left α-centralizer (in case $d = 0$). Thus it is interesting to investigate properties of this general notion. For more properties of generalized (α,β)-derivations we refer to ([11], [13], [14], [15]) and references therein. A map $F : R \to R$ (not necessarily additive) is called a multiplicative generalized (α,β)-derivation associated with a nonzero (α,β)-derivation d if it satisfies $F(xy) = F(x)\alpha(y) + \beta(x)d(y)$ for all $x, y \in R$. Obviously every generalized (α,β) derivation is a multiplicative generalized (α,β) derivation. In the present paper, our aim is to investigate some algebraic identities involving multiplicative generalized (α,β)-derivation on some suitable subsets in prime rings.

The following Lemmas are used in the proof of main results:

Lemma 1.1. [5, Lemma 3] If a prime ring R contains a commutative nonzero left ideal I, then R is commutative.

Lemma 1.2. [12, Lemma 1] Let R be a prime ring, α an epimorphism on R and I a nonzero left ideal of R such that $\alpha(I) \neq \{0\}$. If d is an (α,α)-derivation on R such that $d(I) = \{0\}$, then $d = 0$.

Proof. For all $x \in I$ and all $r \in R$, $0 = d(rx) = d(r)\alpha(x) + \alpha(r)d(x)$. This implies that $d(r)\alpha(x) = 0$. Since $\alpha(I) \neq \{0\}$ is a left ideal of R, we get $d(r) = 0$ for all $r \in R$. Hence $d = 0$. □

2. Main results

Theorem 2.1. Let R be a prime ring, α an epimorphism on R and I a nonzero left ideal of R such that $\alpha(I) \neq \{0\}$. Suppose that R admits a multiplicative generalized (α,α)-derivation F associated with a nonzero (α,α)-derivation d such that $d(Z(R)) \neq \{0\}$. If $d(x)F(y) \pm \alpha(xy) \in Z(R)$ for all $x, y \in I$, then R is commutative.

Proof. It is easy to check that $d(Z(R)) \subseteq Z(R)$. Since $d(Z(R)) \neq \{0\}$, there exists $0 \neq c \in Z(R)$ such that $0 \neq d(c) \in Z(R)$. By hypothesis, we have

$$d(x)F(y) \pm \alpha(xy) \in Z(R) \text{ for all } x, y \in I. \tag{2.1}$$

Replacing y by cy in (2.1), we get

$$(d(x)F(y) \pm \alpha(xy))\alpha(c) + d(x)\alpha(y)d(c) \in Z(R) \text{ for all } x, y \in I. \tag{2.2}$$

Combining (2.1) and (2.2) and using the fact that $\alpha(c) \in Z(R)$,we get

$$[d(x)\alpha(y)d(c), r] = 0 \text{ for all } x, y \in I \text{ and } r \in R.$$

This implies that

$$[d(x)\alpha(y), r]d(c) = 0 \text{ for all } x, y \in I \text{ and } r \in R.$$

Since R is prime and $0 \neq d(c) \in Z(R)$, we have

$$[d(x)\alpha(y), r] = 0 \text{ for all } x, y \in I \text{ and } r \in R. \tag{2.3}$$

Replacing y by yx in (2.3), we get

$$d(x)\alpha(y)[\alpha(x), r] + [d(x)\alpha(y), r]\alpha(x) = 0 \text{ for all } x, y \in I \text{ and } r \in R.$$

Using (2.3), we obtain

$$d(x)\alpha(y)[\alpha(x), r] = 0 \text{ for all } x, y \in I \text{ and } r \in R. \tag{2.4}$$

Replacing r by $r\alpha(z)$ in (2.4) and using (2.4), we get

$$d(x)\alpha(y)r[\alpha(x), \alpha(z)] = 0 \text{ for all } x, y, z \in I \text{ and } r \in R.$$

$$d(x)\alpha(y)R[\alpha(x), \alpha(z)] = \{0\} \text{ for all } x, y, z \in I.$$

The primeness of R implies that for each $x \in I$, either $d(x)\alpha(y) = 0$ or $[\alpha(x), \alpha(z)] = 0$. Set $I_1 = \{x \in I | \, d(x)\alpha(y) = 0 \text{ for all } y \in I\}$ and $I_2 = \{x \in I | \, [\alpha(x), \alpha(z)] = 0\}$. Then I_1 and I_2 are both additive subgroups of I such that $I = I_1 \cup I_2$. By Brauer's trick, we have either $I = I_1$ or $I = I_2$. If $I = I_1$, then $d(x)\alpha(y) = 0$ for all $x, y \in I$. Now replacing y by ry in the above relation, we obtain

$$d(x)r\alpha(y) = 0 \text{ for all } x, y \in I \textit{ and } r \in R.$$

Using primeness we have either $d(x) = 0$ or $\alpha(y) = 0$ for all $x, y \in I$. If $d(I) = \{0\}$, then by Lemma 1.2, $d = 0$, a contradiction. On the other hand if $\alpha(y) = 0$ for all $y \in I$, then $\alpha(I) = \{0\}$ again a contradiction.
If $I = I_2$, then $[\alpha(x), \alpha(z)] = 0$. Now replacing z by rz in precceding relation, we get $\alpha(r)[\alpha(x), \alpha(z)] + [\alpha(x), \alpha(r)]\alpha(z) = 0$. This gives $[\alpha(x), \alpha(r)]\alpha(z) = 0$. Now replacing z by tz, where $t \in R$, we get $[\alpha(x), \alpha(r)]R\alpha(z) = \{0\}$. Using Primeness of R we get either $\alpha(I) = \{0\}$, a contradiction or we have $[\alpha(x), \alpha(r)] = 0$ for all $x \in I$ and $r \in R$. In the later case, we have $\alpha(I) \subseteq Z(R)$. Since $\alpha(I)$ is a nonzero left ideal of R hence by Lemma 1.1, R is commutative. □

Theorem 2.2. Let R be a prime ring, α an epimorphism on R and I a nonzero left ideal of R such that $\alpha(I) \neq \{0\}$. Suppose that R admits a multiplicative generalized (α, α)-derivation F associated with a nonzero (α, α)-derivation d such that $d(Z(R)) \neq \{0\}$. If $d(x)oF(y) \pm \alpha(x \circ y) \in Z(R)$ for all $x, y \in I$, then R is commutative

Proof. It is easy to check that $d(Z(R)) \subseteq Z(R)$. Since $d(Z(R)) \neq \{0\}$, there exists $0 \neq c \in Z(R)$ such that $0 \neq d(c) \in Z(R)$. By assumption, we have

$$d(x) \circ F(y) \pm \alpha(x \circ y) \in Z(R) \text{ for all } x, y \in I. \tag{2.5}$$

Replacing y by cy in (2.5), we get

$$d(x) \circ F(y)\alpha(c) + d(x) \circ \alpha(y)d(c) \pm \alpha(c)\alpha(x \circ y) \in Z(R),$$

Therefore, we have

$$\alpha(c)(d(x) \circ F(y)) + [d(x), \alpha(c)]F(y) + d(c)(d(x) \circ \alpha(y)) + [d(x), d(c)]\alpha(y) \pm \alpha(x \circ y)\alpha(c) \in Z(R),$$

This implies that

$$(d(x) \circ F(y) \pm \alpha(x \circ y))\alpha(c) + (d(x) \circ \alpha(y))d(c) \in Z(R). \tag{2.6}$$

Combining (2.5) and (2.6) and using the fact that $\alpha(c) \in Z(R)$, we find

$$(d(x) \circ \alpha(y))d(c) \in Z(R).$$

This implies that

$$[d(x) \circ \alpha(y), r]d(c) = 0 \text{ for all } x, y \in I \text{ and } r \in R.$$

Since R is prime and $0 \neq d(c) \in Z(R)$, we have

$$[d(x) \circ \alpha(y), r] = 0 \text{ for all } x, y \in I \text{ and } r \in R. \tag{2.7}$$

Replacing y by yx in (2.7) and using (2.7), we get

$$[(d(x) \circ \alpha(y))\alpha(x) - \alpha(y)[d(x), \alpha(x)], r] = 0,$$

This implies that

$$(d(x) \circ \alpha(y))[\alpha(x), r] + [d(x) \circ \alpha(y), r]\alpha(x) - [\alpha(y), r][d(x), \alpha(x)] - \alpha(y)[[d(x), \alpha(x)], r] = 0. \tag{2.8}$$

Replacing y by ry in (2.8), we get

$$(d(x) \circ r\alpha(y))[\alpha(x), r] - [r\alpha(y), r][d(x), \alpha(x)] - r\alpha(y)[[d(x), \alpha(x)], r] = 0. \tag{2.9}$$

Applying (2.8) in (2.9), we find

$$[d(x), r]\alpha(y)[\alpha(x), r] = 0 \text{ for all } x, y \in I \text{ and } r \in R. \tag{2.10}$$

Again replacing y by sy and since α is an epimorphism, we obtain

$$[d(x), r]s\alpha(y)[\alpha(x), r] = 0 \text{ for all } x, y \in I \text{ and } r, s \in R.$$

$$[d(x), r]R\alpha(y)[\alpha(x), r] = \{0\} \text{ for all } x, y \in I \text{ and } r \in R.$$

The primeness of R yields that either **(i)** $[d(x), r] = 0$ for all $x \in I, r \in R$ or **(ii)** $\alpha(y)[\alpha(x), r] = 0$ for all $x, y \in I, r \in R$.

Considering the case **(i)** if $[d(x), r] = 0$ for all $x \in I, r \in R$, then $d(I) \subseteq Z(R)$. That is, we have $[r, d(xy)] = 0$ for all $x, y \in I$ and $r \in R$. This implies that

$$[r, d(x)\alpha(y) + \alpha(x)d(y)] = 0 \text{ for all } x, y \in I, r \in R. \tag{2.11}$$

Therefore, we have

$$d(x)[r,\alpha(y)]+[r,d(x)]\alpha(y)+[r,d(y)]\alpha(x)+d(y)[r,\alpha(x)]=0. \tag{2.12}$$

Since $d(I)\subseteq Z(R)$, we obtain

$$d(x)[r,\alpha(y)]+d(y)[r,\alpha(x)]=0 \text{ for all } x,y\in I, r\in R. \tag{2.13}$$

Replacing r by $r\alpha(y)$, we get

$$d(x)[r,\alpha(y)]\alpha(y)+d(y)r[\alpha(y),\alpha(x)]+d(y)[r,\alpha(x)]\alpha(y)=0. \tag{2.14}$$

Using (2.13), we have

$$d(y)r[\alpha(y),\alpha(x)]=0 \text{ for all } x,y\in I, r\in R.$$

Thus, we have

$$d(y)R[\alpha(y),\alpha(x)]=\{0\}.$$

By primeness of R we have either $d(y)=0$ for all $y\in I$ or $[\alpha(y),\alpha(x)]=0$ for all $x,y\in I$. If $d(I)=\{0\}$, then by Lemma 1.2, $d=0$, a contradiction.

If $[\alpha(y),\alpha(x)]=0$ for all $x,y\in I$, replacing x by rx, we get $\alpha(r)[\alpha(y),\alpha(x)]+[\alpha(y),\alpha(r)]\alpha(x)=0$. This gives $[\alpha(y),\alpha(r)]\alpha(x)=0$. Again replacing x by tx, where $t\in R$ and α is an epimorphism, we get $[\alpha(y),\alpha(r)]R\alpha(x)=\{0\}$. Using Primeness of R, we get either $\alpha(x)=0$ for all $x\in I$, that is $\alpha(I)=\{0\}$, a contradiction or we have $[\alpha(y),\alpha(r)]=0$ for all $y\in I$ and $r\in R$. Therefore, in the later case, we have $\alpha(I)\subseteq Z(R)$. Since $\alpha(I)$ is a nonzero left ideal of R hence by Lemma 1.1, R is commutative.

Now consider the case **(ii)** if $\alpha(y)[\alpha(x),r]=0$ for all $x,y\in I, r\in R$, then replacing r by $r\alpha(z)$, we get $\alpha(y)r[\alpha(x),\alpha(z)]=0$ for all $x,y,z\in I$ and $r\in R$, hence $\alpha(y)R[\alpha(x),\alpha(z)]=\{0\}$ for all $x,y,z\in I$. Since R is prime, then either $\alpha(y)=0$ for all $y\in I$ or $[\alpha(x),\alpha(z)]=0$ for all $x,z\in I$. If $\alpha(y)=0$ for all $y\in I$ then $\alpha(I)=\{0\}$, a contradiction. If $[\alpha(x),\alpha(z)]=0$ for all $x,z\in I$, then arguing as above we get the result. □

Theorem 2.3. Let R be a prime ring, α an epimorphism on R and I a nonzero left ideal of R such that $\alpha(I)\neq\{0\}$. Suppose that R admits a multiplicative generalized (α,α)-derivation F associated with a nonzero (α,α)-derivation d such that $d(Z(R))\neq\{0\}$. If $[d(x),F(y)]\pm\alpha([x,y])\in Z(R)$ for all $x,y\in I$, then R is commutative.

Proof. It is easy to check that $d(Z(R))\subseteq Z(R)$. Since $d(Z(R))\neq\{0\}$, there exists $0\neq c\in Z(R)$ such that $0\neq d(c)\in Z(R)$. By hypothesis, we have

$$[d(x),F(y)]\pm\alpha([x,y])\in Z(R) \text{ for all } x,y\in I \tag{2.15}$$

Replacing y by cy in (2.15), we get

$$F(y)[d(x),\alpha(c)]+[d(x),F(y)]\alpha(c)+\alpha(y)[d(x),d(c)]+[d(x),\alpha(y)]d(c)\pm\alpha([x,y])\alpha(c)\in Z(R),$$

This implies that

$$([d(x),F(y)]\pm\alpha([x,y]))\alpha(c)+[d(x),\alpha(y)]d(c)\in Z(R). \tag{2.16}$$

Combining (2.15) and (2.16) and using the fact that $\alpha(c)\in Z(R)$, we find

$$[d(x),\alpha(y)]d(c)\in Z(R),$$

$$[[d(x),\alpha(y)]d(c),r]=0 \text{ for all } x,y\in I \text{ and } r\in R.$$

This implies that

$$[[d(x),\alpha(y)],r]d(c)=0 \text{ for all } x,y\in I \text{ and } r\in R.$$

Since R is prime and $0\neq d(c)\in Z(R)$, we have

$$[[d(x),\alpha(y)],r]=0 \text{ for all } x,y\in I \text{ and } r\in R. \tag{2.17}$$

Replacing y by yx in (2.17), we have

$$[\alpha(y)[d(x),\alpha(x)]+[d(x),\alpha(y)]\alpha(x),r]=0,$$

Using (2.17), we get

$$[\alpha(y),r][d(x),\alpha(x)]+\alpha(y)[[d(x),\alpha(x)],r]+[d(x),\alpha(y)][\alpha(x),r]=0. \tag{2.18}$$

Substituting ry for y in (2.18) and since α is an epimorphism on R, we get

$$\begin{aligned} r[\alpha(y),r][d(x),\alpha(x)]&+r\alpha(y)[[d(x),\alpha(x)],r] \\ &+r[d(x),\alpha(y)][\alpha(x),r]+[d(x),r]\alpha(y)[\alpha(x),r]=0. \end{aligned} \tag{2.19}$$

Using (2.18) in (2.19), we obtain

$$[d(x),r]\alpha(y)[\alpha(x),r]=0. \tag{2.20}$$

which is same as (2.10) of Theorem 2.2, now arguing in the similar manner we get the result. □

Theorem 2.4. Let R be a prime ring, α an epimorphism on R and I a nonzero left ideal of R such that $\alpha(I)\neq\{0\}$. Suppose that R admits a multiplicative generalized (α,α)-derivation F associated with a nonzero (α,α)-derivation d such that $d(Z(R))\neq\{0\}$. If $F([x,y])\pm[F(x),\alpha(y)]\in Z(R)$ for all $x,y\in I$, then R is commutative.

Proof. Clearly $d(Z(R))\subseteq Z(R)$. Since $d(Z(R))\neq\{0\}$, there exists $0\neq c\in Z(R)$ such that $0\neq d(c)\in Z(R)$. By hypothesis, we have

$$F([x,y])\pm[F(x),\alpha(y)]\in Z(R) \text{ for all } x,y\in I. \tag{2.21}$$

Replacing y by cy in (2.21) and since $\alpha(c)\in Z(R)$, we get

$$F([x,y])\alpha(c)+\alpha([x,y])d(c)\pm\alpha(y)[F(x),\alpha(c)]\pm[F(x),\alpha(y)]\alpha(c)\in Z(R),$$

Therefore, we have

$$(F([x,y])\pm[F(x),\alpha(y)])\alpha(c)+\alpha([x,y])d(c)\in Z(R). \tag{2.22}$$

Combining (2.21) and (2.22) and using the fact that $\alpha(c)\in Z(R)$, we get that $\alpha([x,y])d(c)\in Z(R)$. Thus $[\alpha([x,y])d(c),r]=0$ for all $r\in R$ i.e. $[\alpha([x,y]),r]d(c)=0$ for all $x,y\in I$ and $r\in R$. Since R is prime and $0\neq d(c)\in Z(R)$, we have

$$[\alpha([x,y]),r]=0 \text{ for all } x,y\in I \text{ and } r\in R. \tag{2.23}$$

Substituting yx for y in (2.23) and using (2.23), we get

$$\alpha([x,y])[\alpha(x),r]=0 \text{ for all } x,y\in I \text{ and } r\in R. \tag{2.24}$$

Replacing r by $r\alpha(z)$ in (2.24) and using (2.24), we have

$$\alpha([x,y])r[\alpha(x),\alpha(z)]=0 \text{ for all } x,y,z\in I \text{ and } r\in R.$$

The primeness of R yields that, either $\alpha([x,y]) = 0$ or $[\alpha(x), \alpha(z)] = 0$. In both the cases, we get $[\alpha(x), \alpha(y)] = 0$ for all $x, y \in I$. Then arguing in the same manner as in Theorem 2.1, we get the result. □

Theorem 2.5. Let R be a prime ring, α an epimorphism on R and I a nonzero left ideal of R such that $\alpha(I) \neq \{0\}$. Suppose that R admits a multiplicative generalized (α,α)-derivation F associated with a nonzero (α,α)-derivation d such that $d(Z(R)) \neq \{0\}$. If $F(x \circ y) \pm F(x) \circ \alpha(y) \in Z(R)$ for all $x, y \in I$, then R is commutative.

Proof. By hypothesis, we have

$$F(x \circ y) \pm F(x) \circ \alpha(y) \in Z(R) \text{ for all } x, y \in I. \tag{2.25}$$

Since $d(Z(R)) \neq \{0\}$, there exists $0 \neq c \in Z(R)$ such that $0 \neq d(c) \in Z(R)$. Replacing y by cy in (2.25), we get

$$F(c(x \circ y) + [x, c]y) \pm (\alpha(c)(F(x) \circ \alpha(y)) + [F(x), \alpha(c)]\alpha(y) \in Z(R),$$

Using that $\alpha(c) \in Z(R)$, we get

$$(F(x \circ y) \pm F(x) \circ \alpha(y))\alpha(c) + \alpha(x \circ y)d(c) \in Z(R). \tag{2.26}$$

Combining (2.25) and (2.26) and using the fact that $\alpha(c) \in Z(R)$, we find

$$\alpha(x \circ y)d(c) \in Z(R).$$

This implies that

$$[\alpha(x \circ y)d(c), r] = 0 \text{ for all } x, y \in I \text{ and } r \in R.$$

$$[\alpha(x \circ y), r]d(c) = 0 \text{ for all } x, y \in I \text{ and } r \in R.$$

Since R is prime and $0 \neq d(c) \in Z(R)$, we have

$$[\alpha(x \circ y), r] = 0 \text{ for all } x, y \in I \text{ and } r \in R. \tag{2.27}$$

Replacing y by yx in (2.27) and using (2.27), we have

$$\alpha(x \circ y)[\alpha(x), r] = 0 \text{ for all } x, y \in I \text{ and } r \in R. \tag{2.28}$$

Replacing r by $r\alpha(z)$ in (2.28) and using (2.28), we have

$$\alpha(x \circ y)r[\alpha(x), \alpha(z)] = 0 \text{ for all } x, y, z \in I \text{ and } r \in R.$$

The primeness of R yields that for each $x \in I$, either $\alpha(x \circ y) = 0$ or $[\alpha(x), \alpha(z)] = 0$. In the former case, replacing x by xw and using the fact $\alpha(x \circ y) = 0$, we find that $\alpha[x,y]\alpha(w) = 0$ for all $x, y, w \in I$. Replacing w by rw and since α is an epimorphism on R, we get $[\alpha(x), \alpha(y)]R\alpha(w) = 0$, for all $x, y, w \in I$. Using Primeness of R we get either $\alpha(I) = \{0\}$, a contradiction or $[\alpha(x), \alpha(y)] = 0$ for all $x, y \in I$. If we consider the case $[\alpha(x), \alpha(y)] = 0$ for all $x, y \in I$, then arguing in the same manner as in Theorem 2.1, we get the result. □

Theorem 2.6. Let R be a prime ring, α an epimorphism on R and I a nonzero left ideal of R such that $\alpha(I) \neq 0$. Suppose that R admits a multiplicative generalized (α,α)-derivation F associated with a nonzero (α,α)-derivation d such that $d(Z(R)) \neq 0$. If $F([x,y]) \pm \alpha(x \circ y) \in Z(R)$ for all $x, y \in I$, then R is commutative.

Proof. Clearly $d(Z(R)) \subseteq Z(R)$. Since $d(Z(R)) \neq \{0\}$, there exists $0 \neq c \in Z(R)$ such that $0 \neq d(c) \in Z(R)$. By hypothesis, we have

$$F([x,y]) \pm \alpha(x \circ y) \in Z(R). \tag{2.29}$$

Replacing y by cy in (2.29), we get

$$F([x,y])\alpha(c) + \alpha([x,y])d(c) \pm \alpha(c)\alpha(x \circ y) \in Z(R),$$

Therefore, we have

$$(F([x,y]) \pm \alpha(x \circ y))\alpha(c) + \alpha([x,y])d(c) \in Z(R). \tag{2.30}$$

Combining (2.29) and (2.30) and noting the fact that $\alpha(c) \in Z(R)$, we find

$$\alpha([x,y])d(c) \in Z(R).$$

This implies that

$$[\alpha([x,y]), r] = 0 \text{ for all } x,y \in I \text{ and } r \in R. \tag{2.31}$$

which is same as (2.23) of Theorem 2.4. Arguing in the similar manner we can get the required result. □

Using the same techniques with necessary variations, we can prove the following:

Theorem 2.7. Let R be a prime ring, α an epimorphism on R and I a nonzero left ideal of R such that $\alpha(I) \neq \{0\}$. Suppose that R admits a multiplicative generalized (α,α)-derivation F associated with a nonzero (α,α)-derivation d such that $d(Z(R)) \neq \{0\}$. If $F(x \circ y) \pm \alpha([x,y]) \in Z(R)$ for all $x,y \in I$, then R is commutative.

Theorem 2.8. Let R be a prime ring, α an epimorphism on R and I a nonzero left ideal of R such that $\alpha(I) \neq \{0\}$. Suppose that R admits a multiplicative generalized (α,α)-derivation F associated with a nonzero (α,α)-derivation d such that $d(Z(R)) \neq \{0\}$. If $F(x)F(y) \pm \alpha([x,y]) \in Z(R)$ for all $x,y \in I$, then R is commutative.

Proof. It is easy to check that $d(Z(R)) \subseteq Z(R)$. Since $d(Z(R)) \neq \{0\}$, there exists $0 \neq c \in Z(R)$ such that $0 \neq d(c) \in Z(R)$. By hypothesis, we have

$$F(x)F(y) \pm \alpha([x,y]) \in Z(R) \text{ for all } x,y \in I. \tag{2.32}$$

Replacing y by cy in (2.32), we get

$$(F(x)F(y) \pm \alpha([x,y]))\alpha(c) + F(x)\alpha(y)d(c) \in Z(R) \text{ for all } x,y \in I. \tag{2.33}$$

Combining (2.32) and (2.33) and using the fact that $\alpha(c) \in Z(R)$, we find that $F(x)\alpha(y)d(c) \in Z(R)$. This implies that

$$[F(x)\alpha(y)d(c), r] = 0 \text{ for all } x,y \in I \text{ and } r \in R.$$

$$[F(x)\alpha(y), r]d(c) = 0 \text{ for all } x,y \in I \text{ and } r \in R. \tag{2.34}$$

Since R is prime and $0 \neq d(c) \in Z(R)$, we have

$$[F(x)\alpha(y), r] = 0 \text{ for all } x,y \in I \text{ and } r \in R. \tag{2.35}$$

This implies that

$$F(x)[\alpha(y), r] + [F(x), r]\alpha(y) = 0. \tag{2.36}$$

Replace y by yx in (2.36), we have

$$F(x)[\alpha(y), r]\alpha(x) + F(x)\alpha(y)[\alpha(x), r] + [F(x), r]\alpha(y)\alpha(x) = 0.$$

Using (2.36), we obtain

$$F(x)\alpha(y)[\alpha(x), r] = 0 \text{ for all } x, y \in I \text{ and } r \in R. \tag{2.37}$$

Replacing y by ty in (2.37), we get

$$F(x)\alpha(t)\alpha(y)[\alpha(x), r] = 0 \text{ for all } x, y \in I \text{ and } r, t \in R, \tag{2.38}$$

Left multiplying (2.37) by $\alpha(t)$ and subtracting from (2.38), we obtain

$$[F(x), \alpha(t)]\alpha(y)[\alpha(x), r] = 0. \tag{2.39}$$

Replacing y by sy and since α is an epimorphism on R, we get

$$[F(x), \alpha(t)]R\alpha(y)[\alpha(x), r] = \{0\}.$$

Using Primeness of R, we get either $[F(x), \alpha(t)] = 0$ or $\alpha(y)[\alpha(x), r] = 0$. If $[F(x), \alpha(t)] = 0$ for all $x \in I$ and $t \in R$ that is $F(x) \in Z(R)$ for all $x \in I$. Hence $F(x)F(y) \in Z(R)$ for all $x, y \in I$. Thus (2.32) yields that $\alpha[x, y] \in Z(R)$. This implies that $[\alpha([x, y]), r] = 0$ for all $x, y \in I$ and $r \in R$. Arguing in the similar manner as in the proof of Theorem 2.4, we get the result. In the other case if $\alpha(y)[\alpha(x), r] = 0$ replacing r by rs we get $\alpha(y)r[\alpha(x), s] + \alpha(y)[\alpha(x), r]s = 0$. Therefore, we have $\alpha(y)r[\alpha(x), s] = 0$ for all $x, y \in I$ and $r, s \in R$. Using primeness of R we get either $\alpha(I) = \{0\}$, a contradiction or $\alpha(I) \subseteq Z(R)$. In the later case since $\alpha(I) \neq \{0\}$ is a left ideal of R by Lemma 1.1, R is commutative. □

Theorem 2.9. Let R be a prime ring, α an epimorphism on R and I a nonzero left ideal of R such that $\alpha(I) \neq \{0\}$. Suppose that R admits a multiplicative generalized (α, α)-derivation F associated with a nonzero (α, α)-derivation d such that $d(Z(R)) \neq \{0\}$. If $F(x)F(y) \pm \alpha(x \circ y) \in Z(R)$ for all $x, y \in I$, then R is commutative.

Proof. By hypothesis, we have

$$F(x)F(y) \pm \alpha(x \circ y) \in Z(R) \text{ for all } x, y \in I. \tag{2.40}$$

Replacing y by cy in (2.40), we get

$$F(x)(F(y)\alpha(c) + \alpha(y)d(c)) \pm \alpha((x \circ y)c - y[x, c]) \in Z(R),$$

$$(F(x)F(y) \pm \alpha(x \circ y))\alpha(c) + F(x)\alpha(y)d(c) \in Z(R). \tag{2.41}$$

Combining (2.40) and (2.41) and noting that the fact $\alpha(c) \in Z(R)$, we find that $F(x)\alpha(y)d(c) \in Z(R)$. This implies that

$$[F(x)\alpha(y)d(c), r] = 0 \text{ for all } x, y \in I \text{ and } r \in R.$$

Arguing in the similar manner as in the proof of Theorem 2.8, we get the required result. □

Theorem 2.10. Let R be a prime ring, α an epimorphism on R and I a nonzero left ideal of R such that $\alpha(I) \neq \{0\}$. Suppose that R admits a multiplicative generalized (α, α)-derivation F associated with a nonzero (α, α)-derivation d such that $d(Z(R)) \neq \{0\}$. If $F(xy) \pm \alpha([x, y]) \in Z(R)$ for all $x, y \in I$, then R is commutative.

Proof. It is easy to check that $d(Z(R)) \subseteq Z(R)$. Since $d(Z(R)) \neq \{0\}$, there exists $0 \neq c \in Z(R)$such that $0 \neq d(c) \in Z(R)$. By hypothesis, we have

$$F(xy) \pm \alpha([x, y]) \in Z(R) \text{ for all } x, y \in I. \tag{2.42}$$

Replacing y by cy in (2.42), we get

$$F(xy)\alpha(c) + \alpha(xy)d(c) \pm \alpha(c)\alpha([x, y]) \in Z(R),$$

Therefore,we have

$$(F(xy) \pm \alpha([x, y]))\alpha(c) + \alpha(xy)d(c) \in Z(R). \tag{2.43}$$

Combining (2.42) and (2.43) and using the fact that $\alpha(c) \in Z(R)$, we get

$$\alpha(xy)d(c) \in Z(R) \text{ for all } x, y \in I.$$

This yields that

$$[\alpha(xy)d(c), r] = 0 \text{ for all } x, y \in I \text{ and } r \in R.$$

$$[\alpha(xy), r]d(c) = 0 \text{ for all } x, y \in I.$$

Since R is prime and $0 \neq d(c) \in Z(R)$, we have

$$[\alpha(xy), r] = 0 \text{ for all } x, y \in I \text{ and } r \in R. \tag{2.44}$$

Replacing y by yx in (2.44), we get

$$\alpha(xy)[\alpha(x), r] + [\alpha(xy), r]\alpha(x) = 0.$$

Using (2.44), we obtain

$$\alpha(xy)[\alpha(x), r] = 0 \text{ for all } x, y \in I \text{ and } r \in R. \tag{2.45}$$

Replacing r by $r\alpha(z)$ in (2.45), we have

$$\alpha(xy)[\alpha(x), r\alpha(z)] = 0 \text{ for all } x, y, z \in I \text{ and } r \in R$$

$$\alpha(xy)[\alpha(x), r]\alpha(z) + \alpha(xy)r[\alpha(x), \alpha(z)] = 0 \text{ for all } x, y, z \in I \text{ and } r \in R. \tag{2.46}$$

Using (2.45) in (2.46), we obtain

$$\alpha(xy)R[\alpha(x), \alpha(z)] = \{0\} \text{ for all } x, y, z \in I. \tag{2.47}$$

The primeness of R yields that for each $x \in I$, either $\alpha(xy) = 0$ or $[\alpha(x), \alpha(z)] = 0$. If $\alpha(xy) = 0$ for all $x, y \in I$, then replacing y by ry and since α is an epimorphism on R, we get $\alpha(x)R\alpha(y) = \{0\}$. Using primeness in both cases we get $\alpha(I) = \{0\}$, a contradiction. In the later case for $[\alpha(x), \alpha(z)] = 0$ arguing in the same manner as in Theorem 2.1, we get the result. ∎

3. EXAMPLES

The following example show that R to be prime is not superfluous in the above theorems.

Example 3.1. Consider the ring $R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \mid a, b, c \in \mathbb{Z} \right\}$, where $\mathbb{Z}$ is the set of all integers. $I = \left\{ \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid a \in \mathbb{Z} \right\}$ a nonzero left ideal of R. Note that R is not a prime ring. Define maps F, d, $\alpha : R \to R$ by $F\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $d\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & -c \\ 0 & 0 & 0 \end{pmatrix}$, $\alpha\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -a & -b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}$. It is easy to see that F is a multiplicative generalized (α, α)-derivation associated with a nonzero (α, α)-derivation d of R such that $\alpha(I) \neq 0$. $Z(R) = \left\{ \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid b \in \mathbb{Z} \right\}$ and $d(Z(R)) \neq \{0\}$. For all $x, y \in I$, (i) $d(x)F(y) \pm \alpha(xy) \in Z(R)$, (ii) $d(x)oF(y) \pm \alpha(x \circ y) \in Z(R)$, (iii)$[d(x), F(y)] \pm \alpha([x, y]) \in Z(R)$,(iv) $F([x, y]) \pm [F(x), \alpha(y)] \in Z(R)$, (v) $F(x \circ y) \pm F(x) \circ \alpha(y) \in Z(R)$, (vi) $F([x, y]) \pm \alpha(x \circ y) \in Z(R)$, (vii) $F(x \circ y) \pm \alpha([x, y]) \in Z(R)$, (viii) $F(x)F(y) \pm \alpha([x, y]) \in Z(R)$, (ix) $F(x)F(y) \pm \alpha(x \circ y) \in Z(R)$, (x) $F(xy) \pm \alpha([x, y]) \in Z(R)$, however R is not commutative.

References

1. A. Ali, V. De Filippis and F. Shujat, *On one sided ideals of a semiprime ring with generalized derivations*, Aequationes Math., 85(3)(2013), 529-537.
2. B. Hvala, *Generalized derivations in rings* Comm. Algebra, 26 (1998), 1147-1166.
3. H. Goldman and P. Semrl, *Multiplicative derivations on $C(X)$*, Monatsh. Math. 121(3)(1996), 189-197.
4. J. C. Chang, *On the identity $h(x) = af(x) + g(x)b$*, Taiwanese J. Math. 7(1)(2003), 103-113 .
5. J. H. Mayne, *Centralizing mappings of prime rings*, Canad. Math. Bull. 27(1984), 122-126.
6. M. Ashraf, A. Ali and S. Ali, *Some commutativity theorems for rings with generalized derivation*, Southeast Asian Bull. Math. 31(2007), 415-421.
7. M. Bresar, *On the distance of the composition of two derivations to the generalized derivations*, Glasgow Math. J. 33 (1991), 89-93.
8. M. N. Daif, *When is a multiplicative derivation is additive*, Int. J. Math. and Math. Sci. 14(3)(1991), 615-618.
9. M. N. Daif and M.S. Tammam El-Sayiad, *Multiplicative generalized derivations which are additive*, East-west Journal of Math, 9(1)(1997), 31-37.
10. M. N. Daif and H. E. Bell, *Remarks on derivation on semiprime rings*, Int. J. Math and Math. Sci. 15(1)(1992), 205-206.
11. N. Argac and E. Albas, *On generalized (σ, τ)-derivations*, Siberian. Math. J. 43(6)(2002), 977-984.

12. N. Aydin and K. Kaya, *Some generalizations in prime rings with* (σ, τ) *-derivation*, Turk. J. Math. 16(1992), 169-176.
13. O. Golbasi, *Notes on generalized* (α, β)*-derivations in prime rings*, Miskolc Mathematical Notes. 8(1)(2007), 31-41.
14. O. Golbasi and N. Aydin, *Orthogonal generalized* (σ, τ)*-derivations of semiprime rings*, Siberian Math. J. 48 (6), (2007), 979-983.
15. Y. S. Jung and K. H. Park, *On generalized* (α, β)*-derivations and commutativity in prime rings*, Bull. Korean Math. Soc. 43(1)(2006), 101-106.

[1]Department of Mathematics, Aligarh Muslim University, Aligarh-202002, (INDIA).
Email address: asma_ali2@rediffmail.com

[2]Department of Mathematics, Aligarh Muslim University, Aligarh-202002, (INDIA).
Email address: ambreenbn9@gmail.com

[3]Department of Mathematics, Aligarh Muslim University, Aligarh-202002, (INDIA).
Email address: rahamanhamidmath@gmail.com