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EVOLUTION ALGEBRAS THAT ARE ALMOST JORDAN

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Abstract. In this paper, we give a necessary and sufficient condition for an almost Jordan algebra, which is also called Lie triple algebra to be an evolution algebra. We study the nilpotency, power associativity of such algebras. We specify the necessary and sufficient conditions for such an algebra to be baric. Along the way we show that train evolution algebras which are almost Jordan algebras are of rank at most 5. We prove that any finite-dimensional non-nil-evolution algebra verifying the identity of almost Jordan algebras has at least one non-zero idempotent. Finally, we study the derivations of this class of algebras.

1. Introduction

Let K be a commutative field and A a commutative K-algebra which is not necessarily associative. For any x in A, and for any integer $n \geq 1$, the *principal powers* of x are defined by

$$x^1 = x \text{ and } x^{k+1} = x^k x \text{ for } k \geq 1$$

and those of A are defined by

$$A^1 = A \text{ and } A^{k+1} = A^k A \text{ for } k \geq 1.$$

We say that a K-algebra A is an almost Jordan or Lie triple algebra if for all $x, y \in A$ we have

$$x^3 y + 2((xy)x)x - 3(x^2 y)x = 0. \tag{1.1}$$

Almost Jordan algebras have been studied by Osborn [8], Petersson [11], Sidorov [12, 13], Hentzel and Peresi [7], Bayara and al. [2, 3]. In [7], the authors showed the existence of an ideal L, generated by the associators (x^2, x, x), such that $L^2 = 0$, and then established that A/L is a Jordan algebra. In [3], the authors showed that every non-nil almost Jordan algebra contains either a non-zero idempotent or a pseudo-idempotent. In [5], the authors studied derivations of almost Jordan algebras, distinguishing between those containing non-zero idempotents and those containing pseudo-idempotents. Unlike almost Jordan algebras defined by a polynomial identity, evolution algebras, introduced by Tian([14]), are defined

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using a so-called natural basis. A K-algebra of finite dimension n, is said to be an evolution algebra if it admits a basis $B = \{e_1, \cdots, e_n\}$ such that

$$e_i e_j = 0 \text{ and } e_i^2 = \sum_{k=1}^{n} a_{ik} e_k \text{ for } 1 \leq i \neq j \leq n \tag{1.2}$$

Such a basis is called the natural basis of the evolution algebra A and the matrix $M = (a_{ik})_{1 \leq i,k \leq n}$ is the matrix of structural constants of A relative to the natural basis B. Evolution algebras are commutative, they are not associative in general [14].

In Section 2, we give a necessary and sufficient condition for an evolution algebra to satisfy almost Jordan identity. We study the nilpotency, power-associativity and existence of weight function of such algebras. We show that nil-evolution algebras of nil-index at most 4 are almost Jordan. In Section 3, we show that any finite-dimensional non nil-evolution algebra satisfying the almost Jordan identity admits at least one non-zero idempotent. In Section 4, we show that the computation of derivations of evolution algebras which are almost Jordan reduces to that of derivations of nil-evolution algebras of nil index at most 4.

2. Characterisation theorem and general results

From now on, let K be an infinite commutative field with characteristics other than 2, 3 and 5.

2.1. Characterization theorem and nilpotency.

Theorem 2.1. *Let A be an evolution K-algebra with the natural basis $B = (e_1, \cdots, e_n)$ whose multiplication table is given by (1.2). Then the algebra A is almost Jordan if and only if the following assertions hold:*

- **(a):** $e_k^3 e_j - 3(e_k^2 e_j)e_k = 0$;
- **(b):** $(e_j^2 e_k)e_k - (e_k^2 e_j)e_j = 0$;
- **(c):** $(e_k^2 e_j)e_i = 0$;

where $i, j, k \in \{1, \cdots, n\}$ are pairwise distinct.

Proof. Suppose that the algebra A is almost Jordan, a linearization of the identity (1.1) gives, for all $x, y, z \in A$,

$$(x^2 z)y + 2((xz)x)y + 2((zy)x)x + 2((xy)z)x + 2((xy)x)z - 3(x^2 y)z - 6((xz)y)x = 0 \tag{2.1}$$

and

$$\begin{aligned} &2((xw)z)y + 2((wz)x)y + 2((xz)w)y + 2((zy)w)x + 2((zy)x)w + 2((wy)z)x + \\ &2((xy)z)w + 2((wy)x)z + 2((xy)w)z - 6((xw)y)z - 6((wz)y)x - 6((xz)y)w = 0 \end{aligned} \tag{2.2}$$

By setting $x = e_k$ and $y = e_j$ for all $1 \leq j, k \leq n$ in the identity (1.1), we obtain the relation (a). Similarly, by setting $x = e_k,\ y = z = e_j$ for all $1 \leq j, k \leq n$ in the identity (2.1), we obtain the relation (b). By setting $x = e_k,\ y = e_i$ and $z = e_j$

in the identity (2.1) then $y = w = e_k,\ x = e_i$ and $z = e_j$ in the identity (2.2) for $i, j, k \in \{1, \dots, n\}$ pairwise distinct, we obtain respectively

$$(e_k^2 e_i)e_j - 3(e_k^2 e_j)e_i = 0 \tag{2.3}$$

and

$$(e_k^2 e_j)e_i + (e_k^2 e_i)e_j = 0 \tag{2.4}$$

The difference between equations (2.3) and (2.4) gives (c) because $char(K) \neq 2$. Conversely, suppose that the relations (a), (b) and (c) hold. Then for all $x = \sum_{k=1}^n x_k e_k,\ y = \sum_{k=1}^n y_k e_k$ in A, we have

$$\begin{aligned} x^3y + 2((xy)x)x - 3(x^2y)x &= \sum_{1\leq k\neq j\leq n} 2(x_k^2 x_j y_j + x_j^2 x_k y_k)((e_j^2 e_k)e_k - (e_k^2 e_j)e_j) + \\ &\quad \sum_{1\leq k\neq j\leq n} (x_k^3 y_j - x_k^2 x_j y_k)(e_k^3 e_j - 3(e_k^2 e_j)e_k) + \\ &\quad \sum_{k=1}^{n} \sum_{\substack{1\leq i\neq j\leq n \\ i\neq k \\ j\neq k}} x_k^2 x_j y_i(-3(e_k^2 e_i)e_j + (e_k^2 e_j)e_i) \\ &= 0. \end{aligned}$$

Thus, the algebra A is almost Jordan. □

Definition 2.2. Let A be a finite-dimensional K-algebra. *The annihilator* of A is defined by

$$ann(A) = \{x \in A \ \mid\ xA = 0\}.$$

In [6, Lemma 2.7], the authors show that $ann(A) = \{e_i \in B \ \mid\ e_i^2 = 0\}$ where $B = (e_1, \dots, e_n)$ is a natural basis of the evolution algebra A.

Definition 2.3. A K-algebra A is said to be:

- **(i):** *nilpotent*, if there exists an integer $r \geq 1$ such that $A^r = 0$. The smallest integer r is called *the nilpotency index* of A;
- **(ii):** *nil*, if for any a in A, there exists an integer $s \geq 1$ such that $a^s = 0$. The smallest integer s is called *the nilpotency index of a*;
- **(iii):** *nil-algebra of bounded index*, if the nilpotency index of all the elements of A are bounded by the same integer p. The smallest integer p is called *the nil-index of A*.

Theorem 2.4 ([4, Theorem 2.7])**.** *Let A be an evolution algebra with the natural basis $B = (e_1, \cdots, e_n)$ whose multiplication table is given by (1.2). Then the following assertions are equivalent:*

- **(i):** *The matrix of structural constants of A can be written in the form*

$$\begin{pmatrix} 0 & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & 0 & a_{23} & \cdots & a_{2n} \\ 0 & 0 & 0 & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix};$$

(ii): *A is a nilpotent algebra;*
(iii): *A is a nil-algebra.*

Proposition 2.5 ([10, Lemma 2.3])**.** *Let A be a nil-evolution algebra of nil-index m. Then A is nilpotent of nilpotency index m.*

Remark 2.6 ([10, Page 45]). Let A be a nil-evolution algebra with the natural basis $B = (e_1, \cdots, e_n)$. Then $A^4 = 0$ if and only if $(e_i^2 e_j)e_k = 0$ for all $1 \le i, j, k \le n$.

Proposition 2.7. *Let A be a finite dimensional nil-evolution algebra of nil-index at most 4. Then A is an almost Jordan algebra.*

Proof. Since A is a finite dimensional nil-evolution algebra of nil-index at most 4, then it is nilpotent of nilpotency index at most 4. It follows that for all $x, y \in A$, we have $((xy)x)x = (x^2y)x = x^3y = 0$. We deduce that A is an almost Jordan algebra. □

Lemma 2.8. *Let A be a finite-dimensional evolution K-algebra admitting a basis $B = (e_1, \cdots, e_n)$ such that $a_{ii} = 0$ for all $1 \le i \le n$. If the algebra A is almost Jordan, then it is a nil-algebra of nil-index at most 4.*

Proof. We suppose that A is an evolution almost Jordan algebra with the natural basis $B = \{e_1, \cdots, e_n\}$ such that $a_{ii} = 0$ for all $1 \le i \le n$. By Theorem 2.1, it tells us that for all $1 \le i, j, k \le n$ pairwise distinct, we have:

$$(e_k^2 e_j)e_k = 0 \quad (2.5)$$

$$(e_k^2 e_j)e_i = 0 \quad (2.6)$$

Thus $0 = e_i^3 = (e_k^2 e_i)e_j = (e_k^2 e_j)e_k$ for all $1 \le i, j, k \le n$ pairwise distinct. From this we deduce that A is a nil-algebra of nil-index at most 4 because $(e_k^2 e_i)e_j = 0$ for all $1 \le i, j, k \le n$. □

Remark 2.9. Let A be a non nil-evolution algebra with the natural basis $B = \{e_1, \cdots, e_n\}$ and satisfying the identity (1.1). Then there exists $i_0 \in \{1, \cdots, n\}$ such that $a_{i_0 i_0} \neq 0$. Therefore, $e_{i_0}^2 \neq 0$.

2.2. **Power associativity.**

Definition 2.10. A K-algebra A is power-associative if for any $x \in A$, $x^i x^j = x^{i+j}$ for all integers $i, j \ge 1$.

Theorem 2.11 ([1])**.** *Let K be a commutative field of $Char(K) \neq 2, 3, 5$. The algebra A is power-associative if and only if $x^2x^2 = x^4$ for any $x \in A$.*

Proposition 2.12 ([9, Corollary 1])**.** *Let A be a nil-evolution algebra with the basis $(e_1, \cdots, e_n)$. Then A is power-associative if and only if the following conditions are satisfied:*
(1): *$e_i^2 e_j^2 = 0$ for all $1 \le i \le j \le n$;*
(2): *$(e_i^2 e_j)e_k = 0$ for all $1 \le i, j, k \le n$.*

Remark 2.13. A power-associative nil-evolution algebra has nil-index at most 4. It follows that any power-associative nil-evolution algebra is almost Jordan.

Proposition 2.14 ([9, Theorem 5 (wedderburn)]). *Let A be a power-associative non-nil-evolution algebra. Then A admits s pairwise orthogonal idempotents $u_1, u_2, \cdots, u_s$ such that*

$$A = Ku_1 \oplus Ku_2 \oplus \cdots \oplus Ku_s \oplus N$$

direct sum of algebras, with $s \geq 1$ an integer and N is either zero or a power-associative nil-algebra. Moreover, $Ku_1 \oplus Ku_2 \oplus \cdots \oplus Ku_s$ is the semi-simple component of A and $N = Rad(A)$ is the nil radical of A.

Proposition 2.15. *Let A be a finite-dimensional power-associative non nil-evolution algebra. Then the algebra A is almost Jordan.*

Proof. Let $A = Ku_1 \oplus \cdots \oplus Ku_s \oplus N$ be the Wedderburn decomposition of A with N its nil-radical and $Ku_1 \oplus \cdots \oplus Ku_s$ the semi-simple component. Let $x = \sum_{k=1}^{s} x_k u_k + z_x,\ y = \sum_{k=1}^{s} y_k u_k + z_y \in A$ with $z_x, z_y \in N$. We have

$$x^3y \ = \ \sum_{k=1}^{n} x_k^3 y_k u_k + z_x^3 z_y \ = \ \sum_{k=1}^{n} x_k^3 y_k u_k \ ; \ (x^2y)x \ = \ \sum_{k=1}^{n} x_k^3 y_k u_k + (z_x^2 z_y) z_x \ = \ \sum_{k=1}^{n} x_k^3 y_k u_k$$

$$\text{and } ((xy)x)x \ = \ \sum_{k=1}^{n} x_k^3 y_k u_k + ((z_x z_y) z_x) z_x \ = \ \sum_{k=1}^{n} x_k^3 y_k u_k.$$

This means that

$$x^3y + 2((xy)x)x - 3(x^2y)x \ = \ 0.$$

Hence, we have the desired result. □

A finite-dimensional non-nil-evolution algebra that is almost Jordan algebra, is not necessarily power-associative.

Example 2.16. Let A be an evolution algebra with the natural basis $(e_1, \cdots, e_5)$ whose multiplication table is given by: $e_1^2 = e_1$, $e_2^2 = e_2$, $e_3^2 = e_4$, $e_4^2 = e_5$ and $e_5^2 = 0$. The algebra A is non-nil because e_1 and e_2 are non-zero idempotents of A. It is also not power-associative because $(e_3^2)^2 = e_4^2 = e_5 \neq 0 = e_3^4$. The algebra A is almost Jordan. Let $x = x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4 + x_5e_5,\ y = y_1e_1 + y_2e_2 + y_3e_3 + y_4e_4 + y_5e_5$. We have $x^3y = (x^2y)x = ((xy)x)x = x_1^3y_1e_1 + x_2^3y_2e_2$. Thus, $x^3y + 2((xy)x)x - 3(x^2y)x \ = 0$. We can also see that A is not a train evolution algebra because it has two non-zero idempotents [10, Proposition 3.13].

2.3. Baric algebras.

Definition 2.17 ([15, Definition 1.7]). A K-algebra is said to be *baric*, if there exists a non-trivial morphism of algebras $\omega : \ A \longrightarrow K$. The ω morphism is called *the weight function or weight of A*.

Proposition 2.18 ([10, Theorem 3.3]). *A K-evolution algebra with matrix $M_B = (a_{ij})$ of structural constants relative to the natural basis $B = (e_1, e_n \cdots, e_n)$ is baric if there exists $i_0 \in \{1, \ldots, n\}$ such that $a_{i_0i_0} \neq 0$ and $a_{ii_0} = 0$ for all $1 \leq i \neq i_0 \leq n$. Moreover, the corresponding weight function ω is defined by $\omega(e_{i_0}) = a_{i_0i_0}$ and $\omega(e_i) = 0$ for all $1 \leq i \neq i_0 \leq n$.*

Lemma 2.19 ([10, Lemma 3.2]). *Let (A,ω) be a baric evolution algebra. Then A admits a natural basis $\{e_1,\dots,e_n\}$ such that*

$$e_1^2 = e_1 + \sum_{k=2}^{n} a_{1k}e_k;\ e_j^2 = \sum_{k=j+1}^{n} a_{jk}e_k \quad \text{for } 2 \leqslant j \leqslant n.$$

Definition 2.20. A baric K-algebra (A,ω) is said to be train of rank $r \geq 2$ if there exist scalars $\gamma_1,\cdots,\gamma_{r-1} \in K$ such that

$$x^r + \gamma_1\omega(x)x^{r-1} + \cdots + \gamma_{r-1}\omega(x)^{r-1}x = 0 \tag{2.7}$$

for all $x \in A$ and r is the smallest such integer.

Remark 2.21. Let $A = Ke \oplus \ker\omega$ be a finite dimensional train evolution algebra of rank at most 5, with e a non-zero idempotent and $e\ker\omega = 0$. Then A is almost Jordan.

Theorem 2.22. *Let $A = Ke_1 \oplus \ker\omega$ be a finite dimensional baric evolution algebra and ω the weight function with $e_1^2 = e_1 + z$, where $z \in ker(\omega)$ and $e_1 ker(\omega) = 0$. Then the algebra A satisfies the identity (1.1) if and only if the following conditions are satisfied:*

- **(i):** $z \in ann(\ker\omega)$;
- **(ii):** $\ker\omega$ *is a almost Jordan algebra.*

Proof. Assume that A satisfies the identity (1) and let $y \in \ker\omega$. We have: $0 = e_1^3y + 2((e_1y)e_1)e_1 - 3(e_1^2y)e_1 = e_1^3y = zy$, i.e. $zy = 0$. It follows that $z \in ann(\ker\omega)$. Since $\ker\omega$ is an evolution subalgebra of A, then it is almost Jordan. Reciprocally, we suppose that the assertions (i) and (ii) are satisfied. Since $e_1^2\ker(\omega) = z\ker(\omega) = 0$, then for all $x = ae_1 + x', y = be_1 + y' \in A$ with $a,b \in K$ and $x',y' \in ker(\omega)$, we have: $(x^2y)x = a^3be_1^2 + (x'^2y')x'$, $x^3y = a^3be_1^2 + x'^3y'$ and $((xy)x)x = a^3be_1^2 + ((x'y')x')x'$. It follow that $x^3y + 2((xy)x)x - 3(x^2y)x = x'^3y' + 2((x'y')x') - 3(x'^2y')x' = 0$. Hence, we have the desired results. □

Corollary 2.23. *Let $A = Ke_1 \oplus \ker(\omega)$ be a finite dimensional baric evolution algebra satisfying the identity (1.1) with $e_1^2 = e_1 + z$ where $z \in \ker\omega$ and $e_1\ker\omega = 0$. Then e_1^2 is a non-zero idempotent of A.*

Proof. We have $(e_1^2)^2 = (e_1 + z)^2 = e_1^2$ because $z \in ann(\ker\omega)$. □

Corollary 2.24. *Let $A = Ke_1 \oplus \ker\omega$ be a finite dimensional train evolution algebra. Then the algebra A satisfies the identity (1.1) if and only if the following conditions are satisfied:*

- **(i):** $z \in ann(ker(\omega))$;
- **(ii):** $ker(\omega)$ *is a nil-algebra of nil-index at most* 4.

Proof. As the algebra $Ke_1 \oplus \ker\omega$ is a train algebra, then it admits a unique non-zero idempotent and $\ker\omega$ is a nil-algebra. It is assumed that A satisfies identity (1.1). Then Theorem 2.22 tells us that $z \in ann(\ker\omega)$ and $\ker\omega$ satisfies the identity (1.1). Moreover, the Lemma 2.8 tells us that $ker(\omega)$ has nil-index at most 4. Reciprocally, if the assertions (i) and (ii) are satisfied, then A satisfies the identity (1.1) by the Theorem 2.22 □

The example below shows us that the assertion (i) of the Theorem 2.22 is indispensable for a finite-dimensional train evolution algebra to satisfy the identity (1.1).

Example 2.25. Let A be an evolution algebra with the natural basis $B = \{e, u, v, w\}$ whose multiplication table is given by: $e^2 = e + u,\ u^2 = v,\ v^2 = w,\ w^2 = 0$. Then the linear map $\omega : A \longrightarrow K$ defined by: $\omega(e) = 1$ and $\omega(u) = \omega(v) = \omega(w) = 0$ is a weight function of A and $N = \ker\omega = \langle u, v, w\rangle$ is a nil-algebra of nil-index 4. [10, Theorem 4.4]. It follows that (A, ω) is a train algebra of rank 5. We have $e^3u+2((eu)e)e-3(e^2u)e = u^2 = v \neq 0$. The identity (1.1) is not satisfied.

3. Idempotents and almost Jordan algebras

In this section, we assume that A is a non-nil-evolution algebra with the natural basis $B = \{e_1, \ldots, e_n\}$ and satisfying the identity (1.1). Since A satisfies the identity (1.1), then there exists $i_0 \in \{1, \ldots, n\}$ such that $a_{i_0i_0} \neq 0$. Thus, $e_{i_0}^2 \neq 0$.

1): Let $i_1 \in \{1, \ldots, n\}\backslash\{i_0\}$ be such that $a_{i_1i_1} \neq 0$. We have $e_{i_1}^2 \neq 0$ and the relation (a) of Theorem 2.1 gives

$$0 = e_{i_0}^3 e_{i_1} - 3(e_{i_0}^2 e_{i_1})e_{i_0} = a_{i_0i_1}(a_{i_0i_0}e_{i_1}^2 - 3a_{i_1i_0}e_{i_0}^2) \tag{3.1}$$

$$0 = e_{i_1}^3 e_{i_0} - 3(e_{i_1}^2 e_{i_0})e_{i_1} = a_{i_1i_0}(a_{i_1i_1}e_{i_0}^2 - 3a_{i_0i_1}e_{i_1}^2) \tag{3.2}$$

Assuming $a_{i_0i_1}a_{i_1i_0} \neq 0$, then the identity (3.1) leads to $a_{i_0i_0}e_{i_1}^2 - 3a_{i_1i_0}e_{i_0}^2 = 0$ and multiplying this equality by e_{i_0}, we obtain $0 = 2a_{i_0i_0}a_{i_1i_0}e_{i_0}^2$: this is absurd and we deduce that $a_{i_0i_1}a_{i_1i_0} = 0$. If $a_{i_0i_1} = 0$, the identity (3.2) leads to $a_{i_1i_0} = 0$ and if $a_{i_1i_0} = 0$, the identity (3.1) gives $a_{i_0i_1} = 0$. It follows that $a_{i_1i_0} = a_{i_0i_1} = 0$..

2): Let $i_2 \in \{1, \ldots, n\}$ be such that $a_{i_2i_2} = 0$ and $e_{i_2}^2 \neq 0$. The relation (a) of Theorem 2.1 gives

$$0 = e_{i_2}^3 e_{i_0} - 3(e_{i_2}^2 e_{i_0})e_{i_2} = 3a_{i_2i_0}a_{i_0i_2}e_{i_2}^2, \text{ i.e. } a_{i_2i_0}a_{i_0i_2} = 0 \text{ because } char(K) \neq 3$$

$$0 = e_{i_0}^3 e_{i_2} - 3(e_{i_0}^2 e_{i_2})e_{i_0} = a_{i_0i_0}a_{i_0i_2}e_{i_2}^2, \text{ i.e. } a_{i_0i_2} = 0.$$

The relation (b) from Theorem 2.1 gives $0 = (e_{i_0}^2 e_{i_2})e_{i_2} - (e_{i_2}^2 e_{i_0})e_{i_0} = -a_{i_2i_0}a_{i_0i_0}e_{i_0}^2$, i.e. $a_{i_2i_0} = 0$. We deduce that $a_{i_0i_2} = a_{i_2i_0} = 0$.

From the above, we deduce that

$$e_{i_0}^2 = a_{i_0i_0}e_{i_0} + z_{i_0} \text{ with } z_{i_0} \in ann(A) \text{ and } e_j^2 = \sum_{1\leq k\neq i_0\leq n} a_{jk}e_k \text{ for } 1 \leq j \neq i_0 \leq n \tag{3.3}$$

Lemma 3.1. *Let A be a finite-dimensional non-nil-evolution algebra satisfying the identity (1.1). Then A possesses a non-zero idempotent with extension property.*

Proof. Since the algebra A is not a nil-algebra, then there exists $i_0 \in \{1, \ldots, n\}$ such that the relation (3.3) is satisfied. Without loss of generality, let $i_0 = 1$. The vector $u_1 = a_{11}^{-1}e_1 + a_{11}^{-2}z_1$ is a non-zero idempotent of A. Moreover, the subalgebra Ku_1 has the extension property. Indeed, the vector subspace $vect(e_2, \ldots, e_n)$ is an evolution algebra and the family $(u_1, e_2, \ldots, e_n)$ is a natural basis of A. Let $\alpha_1, \ldots \alpha_n$ be scalars such that

$$0 = \alpha_1 u_1 + \alpha_2 e_2 + \ldots + \alpha_n e_n \tag{3.4}$$

By multiplying the identity (3.4) by u_1, we obtain $0 = \alpha_1 u_1^2 = \alpha_1 u_1$, i.e. $\alpha_1 = 0$ and as the set of vectors $(e_2, \ldots, e_n)$ is linearly independent, it follows that $\alpha_2 = \cdots = \alpha_n = 0$. Thus, $(u_1, e_2, \ldots, e_n)$ is a basis of A. Moreover, we have $u_1 e_i = 0$ for all $2 \leq i \leq n$. We deduce that the family is a natural basis of A. Hence, we have the desired result. □

Theorem 3.2 ([3, Lemme 2.2])**.** *Let A be a non-nil-almost Jordan algebra containing a non-zero idempotent e. Then A admits the following Peirce decomposition: $A = A_1 \oplus A_{\frac{1}{2}} \oplus A_0$ where $A_\lambda = \{a \in A; ea = \lambda a\}$ with $\lambda \in \{0,\ 1,\ \frac{1}{2}\}$. In addition*

- **(i):** $A_{\frac{1}{2}} A_{\frac{1}{2}} \subseteq A_1 + A_0$;
- **(ii):** $A_\lambda A_\lambda \subseteq A_\lambda,\ A_\lambda A_{\frac{1}{2}} \subseteq A_{\frac{1}{2}},\ A_\lambda A_{1-\lambda} = \{0\},\ (\lambda = 0,\ 1)$.

Theorem 3.3. *Let A be a finite-dimensional non nil-evolution algebra and satisfying identity (1.1). Then A admits $s \geq 1$ idempotents $(u_1, \ldots, u_s)$ which are pairwise orthogonal such that*

$$A = Ku_1 \oplus \cdots \oplus Ku_s \oplus N \tag{3.5}$$

direct sum of algebras and N is either zero or a finite dimensional nil-evolution algebra of nil-index at most 4.

Proof. By Lemma 3.1, A has a non-zero idempotent u_1, with the extension property. Let the family $(u_1, e_2, \cdots, e_n)$ be a natural basis of A and let $A = A_1(u_1) \oplus A_0(u_1) \oplus A_{\frac{1}{2}}(u_1)$ be the Peirce decomposition of A relative to the idempotent u_1. Let $x = x_1 u_1 + x_2 e_2 + \cdots + x_n e_n \in A_1(u_1)$ and $x = u_1 x = x_1 u_1^2 = x_1 u_1$ leads to $x_2 = \cdots = x_n = 0$. Now let $x = x_1 u_1 + x_2 e_2 + \cdots + x_n e_n \in A_0(u_1)$ and we have $0 = u_1 x = x_1 u_1$ resulting in $x_1 = 0$ and $A_0(u_1) = vect(e_2, \cdots, e_n)$. We deduce that $A_{\frac{1}{2}}(u_1) = \{0\}$ and $A = Ku_1 \oplus A_0(u_1)$ with $A_0(u_1)$ which is an evolution algebra satisfying the identity (1.1). If $A_0(u_1)$ is a nil-algebra, it is finite and $s = 1$. Otherwise we repeat the process on $A_0(u_1)$. As the dimension of A is finite, the process will stop after a certain number of steps. Hence, we have the desired results. □

Remark 3.4. In the identity (3.5), the algebra N is the nil-radical of A.

Proposition 3.5. *Let A be a finite-dimensional non-nil-evolution algebra satisfying the identity (3.5). Then A is power-associative if and only if its nil-radical is power-associative.*

Proof. Let $x = z + \sum_{k=1}^{s} x_k u_k$ where z is an element of the nil-radical of A. We have $x^4 = z^4 + \sum_{k=1}^{s} x_k^4 u_k$ and $x^2 x^2 = z^2 z^2 + \sum_{k=1}^{s} x_k^4 u_k$. Therefore $x^4 = x^2 x^2$ if and only if $z^2 z^2 = z^4$. Hence, we have the desired results. □

Proposition 3.6. *Let A be a finite-dimensional non nil-almost Jordan evolution algebra. If A has one and only one non-zero idempotent, then A is train of rank at most 5.*

Proof. It is assumed that A has one and only one idempotent. Then there exists an nil-evolution algebra of nil-index at most 4 such that $A = Ku_1 \oplus N$ with $u_1^2 = u_1$ and $u_1N = 0$. The application $\omega : A \longrightarrow K$ defined by $\omega(u_1) = 1$ and $\omega(x) = 0$, $\forall x \in N$ is an algebra morphism and $N = ker(\omega)$. It follows that (A, ω) is train of rank at most 5 because N is nil, of nil-index at most 4. □

Remark 3.7. The nil-evolution algebras of dimension $\leqslant 2$ is power-associative.

Proposition 3.8. *Let A be a non-nil almost Jordan evolution algebra of dimension $\leqslant 4$. Then A is train or power-associative.*

Proof. If A admits a single idempotent, then A is train. And if it has four idempotents, then it is power-associative. We then assume that A admits exactly two or three idempotents. Then its nil-radical N is of dimension 1 *or* 2. The Remark 3.7 tells us that N is power-associative. Hence, we have the desired result. □

Proposition 3.9. *In dimension* 5, 6 *and* 7, *there exist to within isomorphism respectively* 1, 5 *and* 20, *non-nil almost Jordan evolution algebras which are neither power-associative, nor train.*

Proof. This result follows from Proposition 3.5; Proposition 3.6; Remark 3.7 and the classification of nil-evolution algebras which are not power-associative. [10, pages 45 and 46] □

4. Derivations of evolution algebras which are almost Jordan

Let A be an algebra and d a linear operator of A. We say that d is a derivation of A if for all u , $v \in A$ we have $d(uv) = ud(v) + d(u)v$. The space $D(A)$ with Lie bracket, is a Lie algebra, called a Lie algebra of derivations, where the bracket of two derivations d_1 and d_2 is defined by $[d_1, d_2] = d_1 \circ d_2 - d_2 \circ d_1$. Let A be an evolution algebra with natural basis $\{e_1, \cdots, e_n\}$ and let $d(e_i) = \sum_{k=1}^{n} d_{ik}e_k$ for $1 \leqslant i \leqslant n$.

Proposition 4.1. *Let A be a finite-dimensional non nil-evolution algebra satisfying the identity* (1.1) *and N its nil-radical. Then $D(A) = D(N)$.*

Proof. According to Theorem 3.3 A can be decomposed into a direct sum of algebras given by: $A = Ku_1 \oplus \cdots \oplus Ku_s \oplus N$, where N is either zero, or a finite dimensional nil-algebra of nil-index at most 4. Let $v = \sum_{k=1}^{n} \beta_k u_k$ and d be a derivation of A. We have $d(u_k) = d(u_k^2) = 2u_k d(u_k) = 2d_{kk}u_k^2 = 2d_{kk}u_k$ implies that $2d_{kk} = d_{kk}$, so $d_{kk} = 0$ and $d(u_k) = 0$. So $d(v) = \sum_{k=1}^{n} \beta_k d(u_k) = 0$, hence, we have the desired result. □

From this we deduce that the computation of derivations in evolution algebras satisfying the identity (1.1), reduces to that of derivations of nil-evolution algebras of nil index at most 4. In the example below, we give the Lie algebra of the derivations of $A = Ku_1 \oplus Ku_2 \oplus N_{3,4}$, where $N_{3,4}$: $e_1^2 = e_2$, $e_2^2 = e_3$ and $e_3^2 = 0$, the only non-nil evolution algebra of dimension 5 which is neither power-associative nor train.

Example 4.2. We have $D(A) = D(N_{3,4})$.

If d is a derivation of A, then $d(e_2) = d(e_1^2) = 2e_1 d(e_1) = 2d_{11}e_1^2 = 2d_{11}e_2$. Similarly, $d(e_3) = d(e_2^2) = 2e_2 d(e_2) = 2d_{22}e_2^2 = 2d_{22}e_3 = 4d_{11}e_3$. Furthermore, $0 = d_{1j}e_j^2 + d_{j1}e_1^2 = d_{1j}e_j^2 + d_{j1}e_2$ for $j = 2, 3$. We deduce that $d_{12} = d_{21} = 0$ and $d(e_1) = d_{11}e_1 + d_{13}e_3$. Thus, the matrix of d in the basis (e_1, e_2, e_3) is

$$\begin{pmatrix} d_{11} & 0 & 0 \\ 0 & 2d_{11} & 0 \\ d_{13} & 0 & 4d_{11} \end{pmatrix} = d_{11} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix} + d_{13} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

From this we deduce that $D(A) = \langle d_1, d_2 \rangle$, where d_1 and d_2 are the derivations of the respective matrices $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$.

We have $[d_1, d_2] = -[d_2, d_1] = d_1 \circ d_2 - d_2 \circ d_1 = 3d_2$. It follows that $D(A)$ is the non-abelian Lie algebra of dimension 2.

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