

QUANTILE-BASED GENERALIZED ENTROPY OF ORDER (α, β) FOR DOUBLY TRUNCATED RANDOM VARIABLE

SEEMA TEHLAN^{1*} AND VIKAS KUMAR²

ABSTRACT. In this paper, we introduce the concept of quantile - based generalized entropy of order (α, β) for doubly truncated random variable and study their properties. Also, we discuss the bounds of the purposed quantile-based generalized interval entropy (QGIE). There are several models for which quantile function are available in tractable form, though their distribution functions are not available in explicit form. Further some special lifetime distribution have been studied using the proposed (QGIE). Various example are provided for illustration purpose.

1. INTRODUCTION

Suppose X be a continuous absolute random variable that does not have any negative value and which reflects the lifespan of a component with a survival function $\bar{F}(t) = P(X > t) = 1 - F(t)$. Together with the cumulative distribution function (CDF), $F(t) = P(X \leq t)$, the differential entropy function in life time data modelling and analysis provides a differential entropy function, and analysis assigns the average degree of uncertainty to a continuous random variable X as follows :

$$H(X) = - \int_0^\infty f(x) \log f(x) dx, \quad (1.1)$$

a continuous counterpart to the discrete Shannon entropy, where $f(x)$ represents the probability density function of the random variable X . Although the use of the term "entropy" has grown, the specific difficulties associated with calculating it have received very little attention. Literature has a number of suggested parameter expansions for Shannon entropy. A few of them are better suited to describe a more complicated system than Shannon's entropy. An additive generalization of Shannon entropy of order α is Renyi's (1961) entropy which is defined as

$$H^\alpha(X) = \frac{1}{1-\alpha} \left(\int_0^\infty f^\alpha(x) dx \right), \quad \alpha > 0, \quad \alpha \neq 1. \quad (1.2)$$

A lot of researchers are still interested in generalised information measures because of Renyi's work. Renyi has noted that other quantities may function as

Date: Received: Jan 24, 2024; Accepted: Mar 29, 2024.

* Corresponding author.

2020 *Mathematics Subject Classification.* Primary 62N05; Secondary 60E05.

Key words and phrases. quantile function, quantile entropy, interval entropy, characterization results.

measures of information in some situations equally or even more effectively. Furthermore, we would want a generalised measure to be flexible in order to better match the data and account for intangible aspects that are impossible to include in the absence of parameters. One important generalization of Renyi's entropy is the Varma's (1966) entropy

$$H_{\alpha}^{\beta}(X) = \frac{1}{\beta - \alpha} \log \left(\int_0^{\infty} f^{\alpha-\beta+1}(x) dx \right); \quad \beta \neq \alpha, \quad \beta - 1 < \alpha < \beta, \quad \beta \geq 1. \quad (1.3)$$

It is an essential tool for characterising many chaotic systems in fields like engineering, physics, and electronics as a measure of complexity and uncertainty. Keep in mind that these generalised measures have many additional qualities, like smoothness and a broader dynamic range under specified situations, which make them suitable for particular applications. They are also considerably more adaptable owing to the parameters. The most effective and often applied measurements in the literature have unquestionably been those developed by Shannon and Renyi. But there is no reason not to investigate the other broad measures. For this reason, the literature has put out over a dozen different information measures. Researchers in reliability, survival analysis, and many other domains are interested in capturing the impacts of an individual's age. As argued by Ebrahimi (1996), Di Crescenzo and Longobardi (2002) gives residual and past entropy. Several aspects of equation (1.3) have recently been studied by Baig and Dar (2008) and Kundu (2015) for left/right truncated random variable, called generalized residual/past entropy and is defined as

$$H_{\alpha}^{\beta}(X; t) = \frac{1}{\beta - \alpha} \log \int_t^{\infty} \left(\frac{f(x)}{\bar{F}(t)} \right)^{\alpha+\beta-1} dx, \quad \beta - 1 < \alpha < \beta, \quad \beta \geq 1 \quad (1.4)$$

$$\bar{H}_{\alpha}^{\beta}(X; t) = \frac{1}{\beta - \alpha} \log \int_0^t \left(\frac{f(x)}{F(t)} \right)^{\alpha+\beta-1} dx. \quad (1.5)$$

Now recently Kundu and Singh (2021) proposed the generalized interval entropy which is defined as

$$H_{\alpha}^{\beta}(X; t_1, t_2) = \frac{1}{\beta - \alpha} \log \int_{t_1}^{t_2} \left(\frac{f(x)}{F(t_2) - F(t_1)} \right)^{\alpha+\beta-1} dx. \quad (1.6)$$

For more recent work based on generalized entropy one may refer to Kumar and Taneja (2011), Rajesh et al. (2014), Kayal (2015), Sunoj et al. (2009), Misagh and Yari (2010, 2012), Kundu and Nanda (2015). We frequently only have information between two points, and in these circumstances, statistical measures are examined under the assumption that random variables have been doubly truncated. When the observations are taken both after the engineering system begins working and before it fails, the doubly truncated measurements are appropriate. If the lifespan of a unit is represented by the random variable X , then $X_{t_1, t_2} = (X - t_1 | t_1 \leq X \leq t_2)$ it is known as the doubly truncated (interval) residual lifetime, which is more likely to be the remaining lifespan random variable X in this particular case $t_2 \rightarrow \infty$. Additionally, we can utilise the doubly truncated past lifetime random variable $X_{t_1, t_2} = (t_2 - X | t_1 \leq X \leq t_2)$, which tends to the past lifetime random

variable X_t in the specific case $t_1 = 0$. Based on a doubly truncated random variable, Shannon entropy is extended once more, to a generalized interval entropy introduced by Misagh and Yari (2012) as define

$$H(X; t_1, t_2) = - \int_{t_1}^{t_2} \frac{f(x)}{F(t_2) - F(t_1)} \log \frac{f(x)}{F(t_2) - F(t_1)} dx, \quad (1.7)$$

$H(X; t_1, t_2)$ determine the uncertainty of a system's lifespan between t_1 and t_2 , given that it has survived up to time t_1 and has been discovered to be down at time t_2 . Misagh and Yari (2011, 2012) and Sunoj et al. (2009) both examined several facets and characteristics of $H(X; t_1, t_2)$. For more results, application and generalization of this information measure we may refer to Sankaran and Sunoj (2004), Khorashadizaeh et al. (2013), Kayal and Moharana (2016), Kundu (2017) and Singh and Kundu (2020) for various results on doubly truncated random variables. The distribution function serves as the foundation for all theoretical analysis and applications involving these information measures. A probabilistic probability distribution may be described using either the quantile function or the distribution function. The idea and approach based on distribution function are conventional even though both provide the same information about the distribution with different interpretations. Quantile-based studies were conducted when standard approaches were either challenging or failed to produce the desired results. However, as Gilchrist (2000) noted, quantile functions possess a number of distinctive characteristics that distribution functions do not, making the first-mentioned more valuable in some real-world circumstances. Statistics that are based on quantile rather than distribution function moments are usually more accurate for inference. In addition, we have other straightforward quantile functions that are excellent for generating empirical models when the distribution function is not in a tractable form. See, for example Hankin and Lee (2006), Nair et al. (2012), Staden and Loots (2009). Since this is not as affected by severe observations, it provides a straightforward evaluation with fewer data, QF is often more practical. It might be challenging to employ traditional techniques of analysis that use distribution functions in such situations, a different method of studying the quantile function as stated by.

$$Q(u) = F^{-1}(u) = \inf\{x | F(x) \geq u\}, \quad 0 \leq u \leq 1. \quad (1.8)$$

When F is a continuous function, (1.8) yields the equation $FQ(u) = u$, where $FQ(u)$ denotes the composite function $F(Q(u))$. Define the quantile density function as $q(u) = Q'(u)$, and the density quantile function as $fQ(u) = f(Q(u))$, where prime indicates the differentiation, we have

$$q(u)fQ(u) = 1,$$

cited by Parzen (1979). In view of above discussion the quantile-based entropy function has been studied by Sunoj and Sankaran in (2012) gives quantile-based Shannon entropy and residual entropy respectively

$$H = \int_0^1 \log q(p) dp, \quad (1.9)$$

and

$$H(u) = \log(1 - u) + (1 - u)^{-1} \int_u^1 \log q(p) dp.$$

The quantile generalized entropy of order (α, β) studied by Kumar and Rani (2018) is

$$H_\alpha^\beta(X; Q(u)) = \frac{1}{\beta - \alpha} \log \left[\int_u^1 \frac{f(Q(p))^{\alpha+\beta-1} q(p)}{(1 - u)^{\alpha+\beta-1}} dp \right]. \quad (1.10)$$

We can also refer to Sunoj et al.(2013), Nanda et al. (2014), Baratpour and Khammar (2018), Sankaran and Sunoj (2017), Guoxin (2018), Kumar (2018) and Khorashadizadeh (2019). In our research, a quantile version of the generalized truncated entropy of order (α, β) was taken into consideration. We used certain theorems and models to our created entropy. We discuss the quantile generalised entropy bounds and how they work. We give an example of some distribution with their generalized quantile hazard function with doubly truncated period (u_1, u_2) . We discuss some result on model also.

2. QUANTILE-BASED GENERALIZED ENTROPY OF ORDER (α, β) FOR DOUBLY TRUNCATED RANDOM VARIABLE

Define a doubly truncated random variable $(X|u_1 \leq X \leq u_2)$ which represent the lifespan of a unit between u_1 and u_2 , where $(u_1, u_2) \in D = (u_1, u_2); Q(u_1) < Q(u_2)$. Corresponding to (1.8) we define the quantile version of generalized doubly truncated entropy of order (α, β) of the non negative random variable X is

$$H_\alpha^\beta(Q, u_1, u_2) = \frac{1}{\beta - \alpha} \log \left[\int_{u_1}^{u_2} \frac{f(Q(p))^{\alpha+\beta-1} q(p)}{(u_2 - u_1)^{\alpha+\beta-1}} dp \right]. \quad (2.1)$$

If $u_1 \rightarrow 0$ and $u_2 \rightarrow 1$ it is reduced to quantile generalized entropy of order (α, β) referring to interval Shannon entropy .

Definition 2.1. The important quantile measures useful in reliability analysis are hazard quantile function and reversed hazard quantile, defined as $A(u) = ((1 - u)q(u))^{-1}$ and $\bar{A}(u) = (uq(u))^{-1}$, respectively, corresponding to the hazard rate $a(x) = \frac{f(x)}{F(x)}$ and reverse hazard rate $\bar{a}(x) = \frac{f(x)}{F(x)}$ of X . In doubly truncation, Ruiz and Navarro (1996) define the *generalized hazard function (GHF)* given by $h_1(t_1, t_2) = \frac{f(t_1)}{(F(t_2) - F(t_1))}$ and $h_2(t_1, t_2) = \frac{f(t_2)}{(F(t_2) - F(t_1))}$, respectively. Thus, quantile generalized hazard function are defined as

$$h_1(u_1, u_2) = \frac{1}{(u_2 - u_1)q(u_1)},$$

and

$$h_2(u_1, u_2) = \frac{1}{(u_2 - u_1)q(u_2)}.$$

Example 2.2. Suppose X is a uniform distribution with density function $f(x) = \frac{1}{b-a}$ and survival function $\bar{F}(x) = \frac{b-x}{b-a}$ and quantile function $Q(u) = a + (b - a)u$.

Then, we obtain *generalized quantile hazard function*(GQHF)

$$h_i(u_1, u_2) = \frac{1}{(u_2 - u_1)(b - a)} \quad i = 1, 2$$

and the *generalized quantile hazard entropy*

$$\begin{aligned} H_\alpha^\beta(Q, u_1, u_2) &= \frac{1}{\beta - \alpha} \log \int_{u_1}^{u_2} \frac{q(p)^{\alpha+\beta-2}}{(u_2 - u_1)^{\alpha+\beta-1}} dp. \\ &= \frac{1}{\beta - \alpha} \log \frac{1}{((u_2 - u_1)(b - a))^{\alpha+\beta-2}} (u_2 - u_1). \end{aligned}$$

Hence, on using the expression of Generalized hazard quantile function we have

$$H_\alpha^\beta(Q, u_1, u_2) = \frac{1}{\beta - \alpha} \log[u_2(h_2^{\alpha+\beta-2}(u_2, u_1)) - u_1(h_1^{\alpha+\beta-2}(u_2, u_1))].$$

Example 2.3. Exponential distribution with density function $f(x) = \lambda e^{-\lambda x}$ and survival function $\bar{F}(x) = \exp(-\lambda x)$; x and $\lambda > 0$ and quantile function $Q(u) = \frac{-1}{\lambda} \log(1 - u)$. Then, we obtain the generalized quantile hazard function are

$$h_i(u_1, u_2) = \frac{\lambda(1 - u)}{(u_2 - u_1)} \quad ; \quad i = 1, 2$$

and the *generalized hazard quantile doubly truncated entropy*

$$\begin{aligned} H_\alpha^\beta(Q, u_1, u_2) &= \frac{1}{\beta - \alpha} \log \left[\frac{\lambda^{\alpha+\beta-2}}{(u_2 - u_1)^{\alpha+\beta-1}} \int_{u_1}^{u_2} (1 - p)^{\alpha+\beta-2} dp \right]. \\ &= \frac{1}{\beta - \alpha} \log \left[\frac{1}{\lambda(\alpha + \beta - 1)} (h_1^{\alpha+\beta-1}(u_1, u_2)) - (h_2^{\alpha+\beta-1}(u_1, u_2)) \right]. \end{aligned}$$

Example 2.4. In this example we represent a graph of Log logistic distribution whose quantile function is $\theta \left(\frac{u}{1-u} \right)^{\frac{1}{b}}$; $b, \theta \in R$, $\alpha, \beta > 0$

$$H_\alpha^\beta(Q; u_1, u_2) = \frac{1}{\beta - \alpha} \log \int_{u_1}^{u_2} \frac{\left(\frac{\theta}{b} \right)^{2-\alpha-\beta} (p)^{\left(\frac{1}{b}-1 \right)(2-\alpha-\beta)} (1-p)^{\left(\frac{1}{b}-1 \right)(2-\alpha-\beta)}}{(u_2 - u_1)^{\alpha+\beta-1}} du.$$

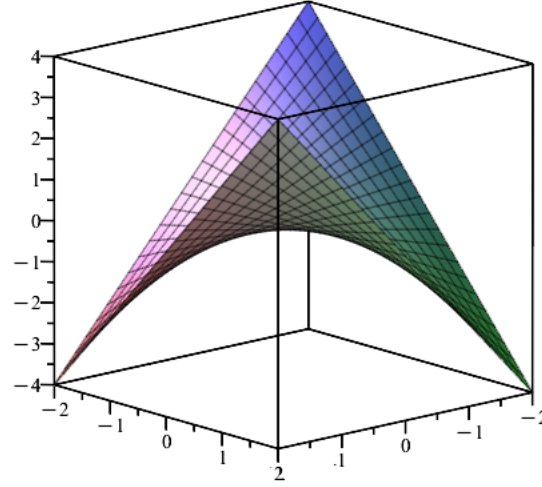


Fig 1 Plot of $H_\beta^\alpha(Q; u_1, u_2)$ for ingraind value between $[-4, 4]$ vale against $u_1, u_2 \in [-2, 2]$.

Table 1 Generalized quantile interval entropy for doubly truncated $H_\alpha^\beta(Q; u_1, u_2)$ for some lifetime distributions

Distribution	$Q(u)$	$H_\alpha^\beta(Q; u_1, u_2)$
Uniform	$a + (b-a)u$	$\frac{2-\alpha-\beta}{(\beta-\alpha)} \log((b-a)(u_2-u_1))$
Exponential	$-\lambda^{-1} \log(1-u)$	$\frac{\alpha+\beta-2}{\beta-\alpha} \log \lambda + \frac{1}{\beta-\alpha} \log \left(\frac{-(1-u_2)^{\alpha+\beta-1} + (1-u_1)^{\alpha+\beta-1}}{(u_2-u_1)^{\alpha+\beta-1} (\alpha+\beta-1)} \right)$
Pareto	$b(1-u)^{-\frac{1}{a}}$	$\frac{1}{\beta-\alpha} \log \left(\frac{a^{\alpha+\beta-1}}{(a+1)((\alpha+\beta-1)-1)b^{\alpha+\beta-2}} \left[\frac{(1-u_1)^{(\alpha+\beta-1)+(\frac{1}{a}(\alpha+\beta-2))} - (1-u_2)^{(\alpha+\beta-1)+(\frac{1}{a}(\alpha+\beta-2))}}{(u_2-u_1)^{\alpha+\beta-1}} \right] \right)$
Pareto II	$a[(1-u)^{-\frac{1}{b}} - 1]$	$\frac{1}{\beta-\alpha} \log \left(\frac{b^{\alpha+\beta-1}}{(b+1)((\alpha+\beta-1)-1)a^{\alpha+\beta-2}} \left[\frac{(1-u_1)^{(\alpha+\beta-2)+(\frac{b+1}{b})+1} - (1-u_2)^{(\alpha+\beta-2)+(\frac{b+1}{b})+1}}{(u_2-u_1)^{\alpha+\beta-1}} \right] \right)$
Generalized Pareto	$\frac{b}{a} \left[(1-u)^{-\frac{a}{a+1}} - 1 \right]$	$\frac{1}{\beta-\alpha} \log \left(\frac{(a+1)^{\alpha+\beta-1}}{(2a+1)((\alpha+\beta-1)-a)b^{\alpha+\beta-2}} \left[\frac{(1-u_1)^{(\alpha+\beta-2)(\frac{2a+1}{a+1})+1} - (1-u_2)^{(\alpha+\beta-2)+(\frac{2a+1}{a+1})+1}}{(u_2-u_1)^{\alpha+\beta-1}} \right] \right)$
Finite Range	$b \left(1 - (1-u)^{\frac{1}{a}} \right)$	$\frac{1}{\beta-\alpha} \log \left(\frac{a^{\alpha+\beta-1}}{(a-1)((\alpha+\beta-1)+1)b^{\alpha+\beta-2}} \left[\frac{(1-u_1)^{(\frac{1-a}{a})(2-\alpha-\beta)+1} - (1-u_2)^{(\frac{1-a}{a})(2-\alpha-\beta)+1}}{(u_2-u_1)^{\alpha+\beta-2}} \right] \right)$
Folder Crammer	$\frac{1}{\theta(1-u)^2}$	$\frac{1}{\beta-\alpha} \log \frac{\theta^{\alpha+\beta-2} ((1-u_1)^{2\alpha+2\beta-3} - (1-u_2)^{2\alpha+2\beta-3})}{(2\alpha+2\beta-3)(u_2-u_1)^{\alpha+\beta-1}}$
Power	$au^{\frac{1}{b}}$	$\frac{1}{(\beta-\alpha)} \log \left(\frac{(a)^{2-\alpha-\beta} (b)^{\alpha+\beta-1}}{(1-b)(2-\alpha-\beta)+b} \left(\frac{(u_2)^{(\frac{1}{b}-1)(2-\alpha-\beta)+1} - (u_1)^{(\frac{1}{b}-1)(2-\alpha-\beta)+1}}{(u_2-u_1)^{\alpha+\beta-1}} \right) \right)$
Turkey Lambda	$\frac{u^\lambda - (1-u)^\lambda}{\lambda}$	$\frac{1}{(\beta-\alpha)} \log \left(\frac{(a)^{2-\alpha-\beta} b^{\alpha+\beta-1}}{(1-b)(2-\alpha-\beta)+b} \left(\frac{(u_2)^{(\frac{1}{b}-1)(2-\alpha-\beta)+1} - (u_1)^{(\frac{1}{b}-1)(2-\alpha-\beta)+1}}{(u_2-u_1)^{\alpha+\beta-1}} \right) \right)$

Lemma 2.5. *We consider a non negative, continuous random variable X with probability density function f and distribution function F . Let $Y = \phi(X)$ where ϕ is a strictly monotonic increasing, continuous and differentiable function with derivative ϕ' . Then for all $0 < u_1 < u_2 < \infty$ we can re-write the (QGIE) for doubly truncated random variable*

$$H_\alpha^\beta(Q_Y; u_1, u_2) = H_\alpha^\beta(Q_X; u_1, u_2) + \log \int_{u_1}^{u_2} \phi'(Q(P)) dp. \quad (2.2)$$

Proof. The probability density function of $Y = \phi(X)$ is $G(Y) = P[Y \leq y] = P[\phi(X) \leq Y] = P[X \leq \phi^{-1}(Y)] = F(\phi^{-1}(Y))$ and $G(Y) = F(\phi^{-1}(y))$. This the quantile function is $F^{-1}(u) = x$. The prob. density function on R.V. Y is $g(y) = \frac{f(\phi^{-1}(y))}{\phi'(\phi^{-1}(y))}$. Now, Generalized Quantile Interval entropy can be written as

$$H_\alpha^\beta(Q; u_1, u_2) = \frac{1}{\beta - \alpha} \log \left[\int_{u_1}^{u_2} \frac{f(Q(p))^{\alpha+\beta-1} q(p)}{(u_2 - u_1)^{\alpha+\beta-1}} dp \right]$$

$$H_\alpha^\beta(Q_Y; u_1, u_2) = H_\alpha^\beta(Q_X; u_1, u_2) + \log \int_{u_1}^{u_2} \phi'(Q(P)) dp. \quad (2.3)$$

□

Remark 2.6. For any absolutely continuous random variable X , define $Y = aX + b$, where $a, b > 0$ are constant. Then

$$H_\alpha^\beta(Y; u_1, u_2) = H_\alpha^\beta(Q_X; u_1, u_2) + \log \int_{u_1}^{u_2} a dp = H_\alpha^\beta(Q; u_1, u_2) + \log a(u_2 - u_1).$$

The quantile generalized doubly truncated entropy defined in (2.1) is invariant under location but not under scale transformation. Next, we see how the monotonicity of $H_\alpha^\beta(Q_X; u_1, u_2)$ is affected by increasing transformation.

Theorem 2.7. *Consider a non negative, continuous random variable X with quantile function $Q_X(\cdot)$ and quantile density function $q_X(\cdot)$. Define $Y = \phi(X)$, where $\phi(\cdot)$ is a nonnegative, increasing and convex (concave) function.*

(i) *For $0 < \alpha + \beta < 2$, $H_\alpha^\beta(Q_Y; u_1, u_2)$ is increasing (decreasing) in (u_1, u_2) whenever $H_\alpha^\beta(Q_X; u_1, u_2)$ is increasing (decreasing) in (u_1, u_2) .*

(ii) *For $\alpha + \beta > 2$, $H_\alpha^\beta(Q_Y; u_1, u_2)$ is decreasing (increasing) in (u_1, u_2) whenever $H_\alpha^\beta(Q_X; u_1, u_2)$ is increasing (decreasing) in (u_1, u_2) .*

Proof. (i) The probability density function of $Y = \phi(X)$ is $g(y) = \frac{f(\phi^{-1}(y))}{\phi'(\phi^{-1}(y))}$; hence density quantile function is $g(Q_Y(u)) = \frac{1}{q_Y(u)} = \frac{f(Q_X(u))}{\phi'(Q_X(u))} = \frac{1}{q_X(u)\phi'(Q_X(u))}$. Thus we have

$$H_\alpha^\beta(Q_Y; u_1, u_2) = \frac{1}{(\beta - \alpha)} \log \left(\frac{1}{(u_2 - u_1)^{\alpha+\beta-1}} \int_{u_1}^{u_2} (q_Y(p))^{2-\alpha-\beta} dp \right).$$

$$= \frac{1}{(\beta - \alpha)} \log \left(\frac{1}{(u_2 - u_1)^{\alpha+\beta-1}} \int_{u_1}^{u_2} (q_X(p)\phi'(Q_X(p)))^{2-\alpha-\beta} dp \right).$$

From the given condition, we have

$$\frac{1}{(\beta - \alpha)} \log \left(\frac{1}{(u_2 - u_1)^{\alpha + \beta - 1}} \int_{u_1}^{u_2} (q_X(p))^{2 - \alpha - \beta} dp \right) \text{ is increasing in } u_1,$$

which gives that

$$\log \left(\frac{1}{(u_2 - u_1)^{\alpha + \beta - 1}} \int_{u_1}^{u_2} (q_Y(p))^{2 - \alpha - \beta} dp \right) \text{ is increasing in } u_1 \text{ for fixed } u_2.$$

We can rewritten as

$$(\beta - \alpha) H_\alpha^\beta(Q_Y; u_1, u_2) = \log \left(\frac{1}{(u_2 - u_1)^{\alpha + \beta - 1}} \int_{u_1}^{u_2} (q_X(p) \phi'(Q_X(p)))^{2 - \alpha - \beta} dp \right). \quad (2.4)$$

Since $0 < \alpha + \beta < 2$ and ϕ is non negative, increasing convex (concave), we have $[\phi'(Q(p))]^{2 - \alpha - \beta}$ is increasing (decreasing) and is non negative. From above lemma we get desired result. When $\alpha + \beta > 2$, $[\phi'(Q(p))]^{2 - \alpha - \beta} = \frac{1}{[\phi'(Q(p))]^{\alpha + \beta - 2}}$ is decreasing in p , since ϕ is increasing and convex. Hence we have

$$H_\alpha^\beta(Q_Y; u_1, u_2) = \frac{1}{(\beta - \alpha)} \log \left(\frac{1}{(u_2 - u_1)^{\alpha + \beta - 1}} \int_{u_1}^{u_2} (q_X(p) \phi'(Q_X(p)))^{2 - \alpha - \beta} dp \right). \quad (2.5)$$

It is decreasing (increasing) with respect to u_1, u_2 . Hence proved. $\square$

The following theorem gives bounds for $h_1(u_1, u_2)$ and $h_2(u_1, u_2)$ based on the monotonicity of $H_\alpha^\beta(Q; u_1, u_2)$. Since the distribution function is characterized by the GFR uniquely, on using the theorems one may obtain a bound of the distribution also.

Theorem 2.8. *For non-negative, absolutely continuous random variable X , if $H_\alpha^\beta(Q; u_1, u_2)$ is increasing and decreasing in u_1 for fixed u_2 , then*

$$h_1(u_1, u_2) \leq (\geq) \left[(\alpha + \beta - 1)(u_2 - u_1) e^{(\beta - \alpha) H_\alpha^\beta(Q; u_1, u_2)} \right]^{\frac{1}{\alpha + \beta - 3}}.$$

for $\alpha + \beta > 3$ and

$$h_1(u_1, u_2) \geq (\leq) \left[(\alpha + \beta - 1)(u_2 - u_1) e^{(\beta - \alpha) H_\alpha^\beta(Q; u_1, u_2)} \right]^{\frac{1}{\alpha + \beta - 3}}.$$

for $\alpha + \beta < 3$.

Proof. $H_\alpha^\beta(Q; u_1, u_2)$ is increasing (decreasing) in u_1 then from we have

$$-h_1^{\alpha + \beta - 1}(u_1, u_2)(u_2 - u_1)^{-1} e^{-(\beta - \alpha) H_\alpha^\beta(Q; u_1, u_2)} + (\alpha + \beta - 1) h_1(u_1, u_2) \geq (\leq) 0$$

$$h_1^{\alpha + \beta - 2}(u_1, u_2)(u_2 - u_1)^{-1} e^{-(\beta - \alpha) H_\alpha^\beta(Q; u_1, u_2)} (\leq) (\geq) (\alpha + \beta - 1) h_1(u_1, u_2)$$

$$h_1^{\alpha + \beta - 3}(u_1, u_2)(u_2 - u_1)^{-1} e^{-(\beta - \alpha) H_\alpha^\beta(Q; u_1, u_2)} (\leq) (\geq) (\alpha + \beta - 1)$$

For $\alpha + \beta > 3$ we have

$$h_1(u_1, u_2) \leq (\geq) \left[(\alpha + \beta - 1)(u_2 - u_1) e^{(\beta - \alpha) H_\alpha^\beta(Q; u_1, u_2)} \right]^{\frac{1}{\alpha + \beta - 3}}.$$

Similarly for $\alpha + \beta < 3$ we obtain

$$h_1(u_1, u_2) \geq (\leq) \left[(\alpha + \beta - 1)(u_2 - u_1) e^{(\beta - \alpha)H_\alpha^\beta(Q; u_1, u_2)} \right]^{\frac{1}{\alpha + \beta - 3}}.$$

Which completes the proof of the theorem. $\square$

Theorem 2.9. *For non-negative, absolutely continuous random variable X , if $H_\alpha^\beta(Q; u_1, u_2)$ is increasing and decreasing in u_2 for fixed u_1 , then*

$$h_2(u_1, u_2) \leq (\geq) \left[(\alpha + \beta - 1)(u_2 - u_1) e^{(\beta - \alpha)H_\alpha^\beta(Q; u_1, u_2)} \right]^{\frac{1}{\alpha + \beta - 3}}.$$

For $\alpha + \beta > 3$ and

$$h_2(u_1, u_2) \geq (\leq) \left[(\alpha + \beta - 1)(u_2 - u_1) e^{(\beta - \alpha)H_\alpha^\beta(Q; u_1, u_2)} \right]^{\frac{1}{\alpha + \beta - 3}}.$$

$\alpha + \beta < 3$ respectively.

Proof. The proof is similar to the proof of the theorem above. $\square$

Now we discuss different properties and bounds of Generalized quantile interval entropy for doubly truncated random variable. In the following properties we decompose $H_\alpha^\beta(Q, u_1, u_2)$ in terms of generalized residual, past and generalized interval quantile entropy of order (α, β) measure. The proof is omitted.

Proposition 2.10. *For an arbitrary lifespan X , the function $H_\alpha^\beta(Q, u_1, u_2)$ is expressed as*

$$H_\alpha^\beta(Q; u_1, u_2) = \frac{1}{\beta - \alpha} \log[u_1^{\alpha + \beta - 1} e^{(\beta - \alpha)\bar{H}_\alpha^\beta(u_1)} + (u_2 - u_1)^{\alpha + \beta - 1} e^{\beta - \alpha} H_\alpha^\beta(Q(u_1, u_2)) + (1 - u)^{\alpha + \beta - 1} e^{(\beta - \alpha)} H_\alpha^\beta(u_2)]. \quad (2.6)$$

The identity equation (2.6), which is comparable to the formula provided in Di Criscenzo and Longobardi (2002), can be understood as follows: uncertainty regarding an item's failure may be divided into four categories: (1) the uncertainty of failure in time interval $(0, u_1)$ assuming that the item has already failed before u_1 , (2) the uncertainty of failure in time interval (u_1, u_2) given that it has failed after u_1 before u_2 , (3) the uncertainty of failure items in time interval $(u_2, +\infty)$ is failed after u_2 and (4) The quantity of uncertainty around the random factor that affects whether an item failed before u_1 , between u_1 and u_2 , or after u_2 .

Remark 2.11. We can re-write equation (2.1) as

$$e^{(\beta - \alpha)H_\alpha^\beta(Q, u_1, u_2)} = \int_{u_1}^{u_2} \left(\frac{q(p)}{u_2 - u_1} \right)^{\alpha + \beta - 1} (q(p)) dp,$$

differentiating it with respect to u_1 and u_2 , respectively, we get

$$(\beta - \alpha) \frac{\partial}{\partial u_1} H_\alpha^\beta(Q, u_1, u_2) = -h_1^{\alpha + \beta - 2}(u_1, u_2) (u_2 - u_1)^{-1} e^{-(\beta - \alpha)H_\alpha^\beta(Q, u_1, u_2)} + (\alpha + \beta - 1)h_1(u_1, u_2) \quad (2.7)$$

and

$$(\beta - \alpha) \frac{\partial}{\partial u_2} H_\alpha^\beta(Q, u_1, u_2) = -h_2^{\alpha+\beta-2}(u_1, u_2)(u_2 - u_1)^{-1} e^{-(\beta-\alpha)H_\alpha^\beta(Q, u_1, u_2)} + (\alpha + \beta - 1)h_2(u_1, u_2). \quad (2.8)$$

We give an example below so that you can understand that not all distributions are monotone in terms of $H_\alpha^\beta(Q, u_1, u_2)$.

Example 2.12. Let X be a non negative random variable having survival function $\bar{F}(x) = 1 - (1 - e^{-x})(1 - e^{-2x}), x > 0$. Then, the density function $f(x)$ of X is defined as $f(x) = e^{-x} + 2e^{-2x} - 3e^{-3x}, x > 0$. Then

$$e^{(\beta-\alpha)H_\alpha^\beta(Q, u_1, u_2)} = \int_{u_1}^{u_2} \frac{(e^{\frac{-1}{2}(\log(1-u))^2} + 2e^{(\log(1-u))^2} - 3e^{\frac{-1}{2}(\log(1-u))^2})^{\alpha+\beta-2}}{(u_2 - u_1)^{\alpha+\beta-1}} du. \quad (2.9)$$

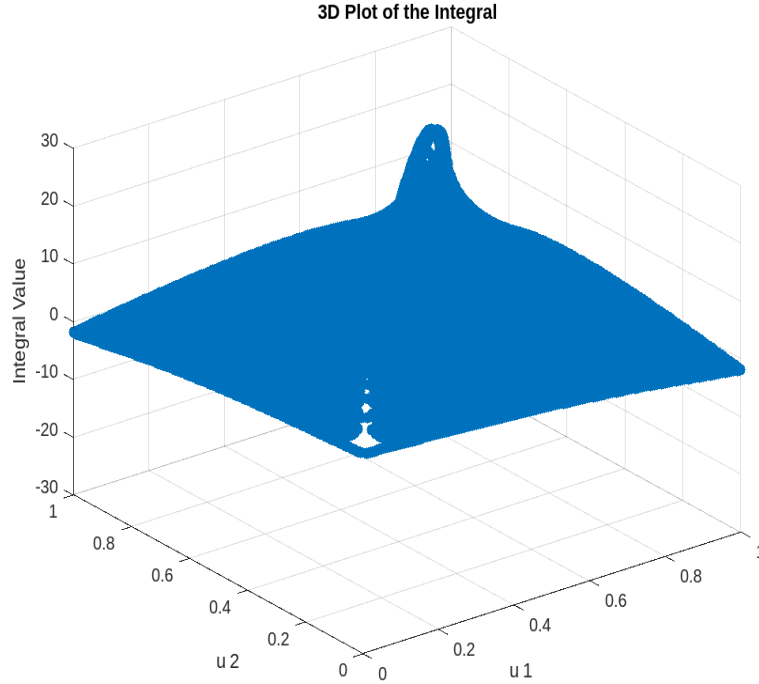


Fig 2

which is not monotone in given figure. Hence $H_\alpha^\beta(Q; u_1, u_2)$ is also not monotone. Plot of $H_\alpha^\beta(Q; u_1, u_2)$ for $\alpha = 1.7$ and $\beta = 2$ against $u_1, u_2 \in [0, 1]$.

Certain probability models may not have a closed form distribution function when applied to real-world data, but they may have quantile functions. In this case, we

take the following model into consideration and derive the generalised entropy for doubly truncated for which $q(\cdot)$ exists using a quantile basis. Accordingly some distribution have been presented here with quantile-based generalized entropy for doubly truncated random variable.

$$H(Q; u_1, u_2) = \frac{1}{\beta - \alpha} \log \int_{u_1}^{u_2} \frac{(q(p))^{2-\alpha-\beta}}{(u_2 - u_1)^{\alpha+\beta-1}} du.$$

3. ENTROPY OF A GENERALIZED FAMILY OF DISTRIBUTION

Consider a *Quantile density function (qdf)*

$$q(u) = (-1)^G A(-\log(1-u))^B (1-u)^C u^D (1+u)^E (\log u)^{F+G} \quad (3.1)$$

where A, B, C, D, E, F and G are the parameters of this model. The generalized model of quantile-based generalized entropy is

$$\begin{aligned} H_\alpha^\beta(Q; u_1, u_2) \\ = \frac{1}{\beta - \alpha} \log \int_{u_1}^{u_2} \frac{((-1)^G A(-\log(1-u))^B (1-u)^C u^D (1+u)^E (\log u)^{F+G})^{2-\alpha-\beta}}{(u_2 - u_1)^{\alpha+\beta-1}} du, \end{aligned} \quad (3.2)$$

The expression in (3.2) have been evaluated for different valves of the parameter in table (2) for various type of lifespan distribution function. In some situations, using a quantile function-based strategy yields better results than using the cumulative distribution function. Since quantile functions are less influenced by extreme observation. Cite Hankel and Lee (2006), Van Staden and Loots (2009), and Nair for examples of models that include simple quantile functions or quantile density functions but no closed form equation for the *CDF* or *PDF*. Accordingly, some distribution have been presented here, where we obtain in table (2).

Example 3.1. The Davis Distribution, which Hankel and Lee (2006) developed, is a lambda family of distribution with a quantile density function that is relevant to reliability is

$$q(u) = C u^{\lambda_1-1} (1-u)^{-\lambda_2-1} \{\lambda_1(1-u) + \lambda_2 u\} : C, \lambda_1, \lambda_2 \geq 0. \quad (3.3)$$

The exponential, gamma, lognormal, and weibull distributions are all well approximated by this family of right-skewed non-negative data. The Generalized quantile interval entropy for Davis distribution is given as,

$$H_\alpha^\beta(Q; u_1, u_2) = \frac{1}{\beta - \alpha} \log \int_{u_1}^{u_2} \frac{(C u^{\lambda_1-1} (1-u)^{-\lambda_2-1} \{\lambda_1(1-u) + \lambda_2 u\})^{2-\alpha-\beta}}{(u_2 - u_1)^{\alpha+\beta-1}} du, \quad (3.4)$$

which can be easily computed numerically. As $\lambda_2 \rightarrow 0$, (3.4) is reduced to table (1) corresponding to the Power distribution. Also as $\lambda_1 \rightarrow 0$, (3.4) is reduced to table (1) the Pareto 1 distribution. When $\lambda_1 = \lambda_2 > 0$, (3.4) then is reduced to $\frac{1}{\beta-\alpha} (b)^{2-\alpha-\beta} \log \int_{u_1}^{u_2} \frac{(1-u)^{-2(2-\alpha-\beta)}}{(u_2-u_1)^{\alpha+\beta-1}} du$ corresponding to log logistic distribution.

Table 2 Quantile-based generalized truncated entropy in generalized model form with different life time distribution

Parameters	Distribution	$Q(u)$	$H_{\alpha}^{\beta}g(Q; u_1, u_2)$
$A = \beta - \alpha; \beta > \alpha,$ $B = C = D = E = F = G = 0$	Uniform	$\alpha + (\beta - \alpha)u$	$\frac{2-\alpha-\beta}{\beta-\alpha} \log(b-a)(u_2 - u_1)$
$A = \frac{1}{\theta}; \theta > 0, C = -1,$ $B = D = E = F = G = 0$	Exponential	$-\theta \log(1-u)$	$\frac{2-\alpha-\beta}{\beta-\alpha} \log \frac{1}{\theta} + \frac{1}{\beta-\alpha} \log \frac{(1-u_1)^{\alpha+\beta-1} - (1-u_2)^{\alpha+\beta-1}}{(\alpha+\beta-1)(u_2-u_1)^{\alpha+\beta-1}}$
$A = \frac{1}{\theta}; \theta > 0, C = -(A+1),$ $B = D = E = F = G = 0$	Classical Pareto	$(1-u)^{-\frac{1}{\theta}}$	$\frac{1}{\beta-\alpha} \log \left[\frac{(1-u_1)^{-\left(\frac{1}{\lambda}+1\right)(2-\alpha-\beta)+1} - (1-u_2)^{-\left(\frac{1}{\lambda}+1\right)(2-\alpha-\beta)+1}}{(\lambda^{1-\alpha-\beta}(-1+\lambda)(2-\alpha-\beta)+\lambda)(u_2-u_1)^{\alpha+\beta-1}} \right]$
$A = \frac{b}{c}; C = -\left(\frac{1}{c}+1\right); b, c > 0,$ $B = D = E = F = G = 0$	Pareto-I	$b(1-u)^{-\frac{1}{c}}$	$\frac{1}{\beta-\alpha} \log \left[\frac{(b)^{1-\alpha-\beta}(1-u_1)^{\frac{(2-\alpha-\beta)(\frac{1}{c}-1)+1}{c}} - (1-u_2)^{\frac{(2-\alpha-\beta)(\frac{1}{c}-1)+1}{c}}}{c^{2-\alpha-\beta}((2-\alpha-\beta)(-c-b)+b)(u_2-u_1)^{\alpha+\beta-1}} \right]$
$A = \frac{b}{c}; C = -\left(\frac{1}{c}+1\right); b, c > 0,$ $B = D = E = F = G = 0$	Pareto-II	$b((1-u)^{-\frac{1}{c}} - 1)$	$\frac{1}{\beta-\alpha} \log \left[\frac{b^{1-\alpha-\beta}}{c^{2-\alpha-\beta}(2-\alpha-\beta)(-c-b)+b} \left\{ \frac{(1-u_1)^{\frac{(2-\alpha-\beta)(\frac{1}{c}-1)+1}{c}} - (1-u_2)^{\frac{(2-\alpha-\beta)(\frac{1}{c}-1)+1}{c}}}{(u_2-u_1)^{\alpha+\beta-1}} \right\} \right]$
$A = \frac{b}{a+1}, C = -\left(\frac{2a+1}{a+1}\right); a > -1,$ $b > 0, B = D = E = F = G = 0$	Generalized Pareto	$\frac{b}{a} \left[(1-u)^{-\frac{a}{a+1}} - 1 \right]$	$\frac{1}{\beta-\alpha} \log \left[-\frac{b^{2-\alpha-\beta}(a+1)^{\alpha+\beta-1}}{(2a+1)(2-\alpha-\beta)+(a+1)} \left\{ \frac{(1-u_1)^{-\frac{2a+1}{a+1}(2-\alpha-\beta)+1} - (1-u_2)^{-\frac{2a+1}{a+1}(2-\alpha-\beta)+1}}{(u_2-u_1)^{\alpha+\beta-1}} \right\} \right]$
$A = \frac{1}{\sigma \lambda^{\frac{1}{\sigma}}}, B = \frac{1}{\sigma} - 1; \lambda, \sigma > 0,$ $C = -1, D = E = F = G = 0$	Weibull	$\left(\frac{-\log(1-u)}{\lambda} \right)^{\frac{1}{\sigma}}$	$\frac{1}{\beta-\alpha} \log \left[\left(\frac{1}{\sigma \lambda^{\frac{1}{\sigma}}} \right)^{2-\alpha-\beta} \int_{u_1}^{u_2} \frac{(-\log(1-u))^{\frac{1}{\sigma}} (1-u)^{-1} (1-u)^{2-\alpha-\beta}}{(u_2-u_1)^{\alpha+\beta-1}} du \right]$
$A = \frac{R}{c}, C = \frac{1}{c} - 1; c, R > 0,$ $B = D = E = F = G = 0$	Rescaled Beta	$R[1 - (1-u)^{\frac{1}{c}}]$	$\frac{1}{\beta-\alpha} \log \int_{u_1}^{u_2} \frac{\left\{ \frac{R}{c} (1-u)^{\frac{1}{c}-1} \right\}^{2-\alpha-\beta}}{(u_2-u_1)^{\alpha+\beta-1}} du$
$A = \frac{1}{\theta}; \theta > 0, C = -2,$ $B = D = E = F = G = 0$	Folder Crammer	$\frac{u}{\theta(1-u)}$	$\frac{1}{\beta-\alpha} \log \int_{u_1}^{u_2} \frac{\left(\frac{1}{\theta(1-u)^2} \right)^{2-\alpha-\beta}}{(u_2-u_1)^{\alpha+\beta-1}} du$
$A = \sigma b(b+1), C = 1, \sigma, b > 0$ $D = b-1, B = E = F = G = 0$	Govindarajulu's	$\theta + \sigma \{ (b+1)u^b - bu^{b+1} \}$	$\frac{1}{\beta-\alpha} \log \int_{u_1}^{u_2} \frac{(\sigma b(b+1)(1-u)u^{b-1})^{2-\alpha-\beta}}{(u_2-u_1)^{\alpha+\beta-1}} du$
$A = \frac{1}{b\theta^{\frac{1}{b}}}, C = -\left(\frac{1}{b}+1\right); \theta, b > 0,$ $D = \frac{1}{b} - 1, B = E = F = G = 0$	Log Logistic	$\left[\frac{u}{\theta(1-u)} \right]^{\frac{1}{b}}$	$\frac{1}{\beta-\alpha} \log \int_{u_1}^{u_2} \frac{\left(\left((1-u)^{-\frac{1}{b}+1} u^{\frac{1}{b}-1} \frac{1}{b\theta^{\frac{1}{b}}} \right)^{2-\alpha-\beta} \right)}{(u_2-u_1)^{\alpha+\beta-1}} du$
$A = \frac{\sigma}{c}, D = \frac{1}{c} - 1; \sigma, c > 0,$ $B = C = E = F = G = 0$	Power	$\sigma u^{\frac{1}{c}}$	$\frac{1}{\beta-\alpha} \log \frac{\sigma^{2-\alpha-\beta} c^{\alpha+\beta-1}}{((1-c)(2-\alpha-\beta)+c)} \left\{ \frac{\left(\frac{1}{c} - 1 \right) (2-\alpha-\beta) + 1}{(u_2-u_1)^{\alpha+\beta-1}} \right\} du$
$A = 2\sigma; \sigma > 0, C = E = -1,$ $B = D = F = G = 0$	Half Logistic	$\sigma \log \left(\frac{1+u}{1-u} \right)$	$\frac{1}{\beta-\alpha} \log \int_{u_1}^{u_2} \frac{((2\sigma)(1+u)^{-1}(1-u)^{-1})^{2-\alpha-\beta}}{(u_2-u_1)^{\alpha+\beta-1}} du$
$A = \lambda; \lambda > 0, D = -1, F = -2,$ $B = C = E = G = 0$	Inverted Exponential	$\frac{-\lambda}{\log u}$	$\log \frac{1}{\beta-\alpha} \int_{u_1}^{u_2} \frac{\left(\left(\frac{\lambda}{u} \right) \left(\frac{1}{\log u} \right)^2 \right)^{2-\alpha-\beta}}{(u_2-u_1)^{\alpha+\beta-1}} du$
$A = \frac{\sigma}{\lambda}, D = -1, G = -\left(\frac{1}{\lambda}+1\right);$ $\sigma, \lambda > 0, B = C = E = F = 0$	Inverse Weibull	$\sigma(-\log u)^{-\frac{1}{\lambda}}$	$\frac{1}{\beta-\alpha} \log \int_{u_1}^{u_2} \frac{\left(\frac{\sigma}{\lambda u} \right) ((-1) \log u)^{-\frac{1}{\lambda}+1} (1-u)^{2-\alpha-\beta}}{(u_2-u_1)^{\alpha+\beta-1}} du$
$A = \frac{1}{\lambda}; \lambda > 0, C = D = -1,$ $B = E = F = G = 0$	Logistic	$\frac{1}{\lambda} \log \left(\frac{u}{1-u} \right)$	$\frac{1}{\beta-\alpha} \log \int_{u_1}^{u_2} \frac{\left(\frac{1}{\lambda(1-u)(u)} \right)^{2-\alpha-\beta}}{(u_2-u_1)^{\alpha+\beta-1}} du$

Example 3.2. Consider Van Staden-Loots distribution with quantile function

$$Q(u) = \lambda_1 + \lambda_2 \left[\left(\frac{1-\lambda_3}{\lambda_4} (u^{\lambda_4} - 1) - \left(\frac{\lambda_3}{\lambda_4} \right) \{ (1-u)^{\lambda_4} - 1 \} \right], \quad \text{where } \lambda_i > 0 \text{ for } i = 1, 2, 3, 4. \quad (3.5)$$

and quantile density function

$$q(u) = \lambda_2[(1 - \lambda_3)u^{\lambda_4-1} + \lambda_3(1 - u)^{\lambda_4-1}]. \quad (3.6)$$

Thus the generalized quantile interval entropy for this distribution is given as

$$H_\alpha^\beta(Q; u_1, u_2) = \frac{1}{\beta - \alpha} \log \int_{u_1}^{u_2} \frac{(\lambda_2((1 - \lambda_3)u^{\lambda_4-1} + \lambda_3(1 - u)^{\lambda_4-1}))^{2-\alpha-\beta}}{(u_2 - u_1)^{\alpha+\beta-1}} du, \quad (3.7)$$

which can be easily computed numerically. As $\lambda_4 \rightarrow 1$, (3.7) is reduced to Uniform distribution which is in table (1). Also as $\lambda_1 \rightarrow 0$ (3.7) is reduced to Exponential distribution as in table (1) and as $\lambda_2 = 2, \lambda_3 = \frac{1}{2}, \lambda_4 = 0$, (3.7) is reduced to the Logistic distribution with parameter equal to 1.

Example 3.3. With a quantile density function, consider the generalised Lambda distribution.

$$q(u) = \frac{1}{\lambda_2}[\lambda_3 u^{\lambda_3-1} + \lambda_4(1 - u)^{\lambda_4-1}]. \quad (3.8)$$

Thus the generalized quantile interval entropy for this distribution

$$H_\alpha^\beta(Q; u_1, u_2) = \frac{1}{\beta - \alpha} \log \int_{u_1}^{u_2} \frac{(\frac{1}{\lambda_2}[\lambda_3 u^{\lambda_3-1} + \lambda_4(1 - u)^{\lambda_4-1}])^{2-\alpha-\beta}}{(u_2 - u_1)^{\alpha+\beta-1}} du, \quad (3.9)$$

for this distribution is can be easily computed numerically. As $\lambda_2 = \lambda_3 = \lambda_4 = \lambda$, (3.9) reduce to

$$\frac{1}{\beta - \alpha} \log \int_{u_1}^{u_2} \frac{[u^{\lambda-1} + (1 - u)^{\lambda-1}]^{2-\alpha-\beta}}{(u_2 - u_1)^{\alpha+\beta-1}} du \quad (3.10)$$

the lambda distribution.(3.10) further is reduced to the as $\lambda \rightarrow 0$, example (2.3) corresponding to the Logistic distribution with parameter equals to 1, (3.10) is reduced as $\lambda \rightarrow 1$ corresponding to Uniform distribution as in table (1) in [-1,1].

Example 3.4. A five-parameter Gilchrist (2000) presented the Lambda family of distributions, which has the density quantile function as

$$q(u) = \lambda_2 \left[\frac{1 - \lambda_3}{2} u^{\lambda_4-1} + \frac{1 - \lambda_3}{2} (1 - u)^{\lambda_5-1} \right]. \quad (3.11)$$

Utilising five parameter Lambda family, Tarsitano (2005) provided some close approximations to a variety of symmetric and asymmetric distributions and advised utilising this model when a specific distributional form could not be inferred from the physical circumstances under discussion. Thus the (2.1) and (3.11) is reduced the generalized quantile interval entropy as

$$H_\alpha^\beta(Q; u_1, u_2) = \frac{1}{\beta - \alpha} \log \int_{u_1}^{u_2} \frac{(\frac{1}{\lambda_2}[\frac{1-\lambda_3}{2} u^{\lambda_4-1} + \frac{1-\lambda_3}{2} (1 - u)^{\lambda_5-1}])^{2-\alpha-\beta}}{(u_2 - u_1)^{\alpha+\beta-1}} du, \quad (3.12)$$

this can be solved numerically. As $\lambda_3 \rightarrow 0$, (3.12) is reduced

$\frac{1}{\beta - \alpha} \log \int_{u_1}^{u_2} \frac{(\frac{\lambda_2}{2}[u^{\lambda_4-1} + \frac{1}{2}(1 - u)^{\lambda_5-1}])^{2-\alpha-\beta}}{(u_2 - u_1)^{\alpha+\beta-1}} du$ to the Generalized Tuckey Lambda family of distributions. This family (3.12) also includes the exponential distribution when $\lambda_4 \rightarrow \infty, \lambda_5 \rightarrow 0$ which is as in table (1), the generalized Pareto distribution

when $\lambda_4 \rightarrow \infty$ and $|\lambda_5| < \infty$, and power distribution when $\lambda_5 \rightarrow \infty$ and $|\lambda_4| < \infty$ as in table (1).

4. CONCLUSION

Quantile entropy is one of the information measures based on quantile functions that has recently attracted a lot of attention. In the context of reliability theory and survival analysis, the interval entropy is crucial when a system has a life span between two time points (t_1, t_2) . The current study developed a quantile-based alternative method for measuring interval entropy. Researchers studying the many aspects of a system that fails between two time instants may find the proposed measurements useful. The results that are provided here generalise previous findings in the setting order (α, β) and in quantile entropy.

5. ACKNOWLEDGEMENT

The authors would like to express their gratitude to the reviewers and the editor in chief for their valuable comments, which have considerably improved the earlier version of the article. The corresponding author wishes to acknowledge the Council of Scientific and Industrial Research (CSIR), Govt of India for the financial assistance (Ref. No. 09/0328(13919)/2022-EMR-1) for carrying out this research work.

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¹ DEPARTMENT OF APPLIED SCIENCES, UNIVERSITY INSTITUTE OF ENGINEERING AND TECHNOLOGY, M. D. UNIVERSITY, ROHTAK-124001, INDIA.
Email address: msseema.rs.uiet@mdurohtak.ac.in

² DEPARTMENT OF APPLIED SCIENCES, UNIVERSITY INSTITUTE OF ENGINEERING AND TECHNOLOGY, M. D. UNIVERSITY, ROHTAK-124001, INDIA.
Email address: vkumar.uiet@mdurohtak.ac.in