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A STUDY ON COARSE DEG-CENTRIC GRAPHS

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Abstract. The coarse deg-centric graph of a simple, connected graph G, denoted by G_{cd}, is a graph constructed from G such that $V(G_{cd}) = V(G)$ and $E(G_{cd}) = \{v_iv_j : d_G(v_i, v_j) > \deg_G(v_i)\}$. This paper introduces and discusses the concepts of coarse deg-centric graphs and iterated coarse deg-centrication of a graph. It also presents the properties and structural characteristics of coarse deg-centric graphs of some graph families.

1. Introduction

For terminology in graph theory, we refer to [1, 2, 13]. A graph is assumed to be a simple, connected, and undirected graph throughout this paper. The *size* of G is denoted by $\varepsilon(G)$. The eccentricity of a vertex $v_i \in V(G)$, denoted by $e(v_i)$, is the farthest distance from v_i to some vertices of G. Vertices at a distance $e(v_i)$ from v_i are called the eccentric vertices of v_i. An *eccentric graph* of a graph G, denoted by G_e, is obtained from the same set of vertices as G with two vertices v_i and v_j being adjacent in G_e if and only if v_j is an eccentric vertex of v_i or v_i is an eccentric vertex of v_j (see[1]). The *iterated eccentric graph* of G, denoted by G_{e^k}, is defined (see[7, 8]) as the derived graph obtained by taking the eccentric graph successively k-times; that is, $G_{e^k}= ((Ge)e \dots)e$, (k-times).

Similarly, a particular type of newly derived graphs based on the vertex degrees and distances in graphs called *deg-centric graphs*, have been introduced (see[11]) as follows, The *degree centric graph* or *deg-centric graph* of G is the graph G_d with $V(G_d) = V(G)$ and $E(G_d) = \{v_iv_j : d_G(v_i, v_j) \leq \deg_G(v_i)\}$ (see[11]). Let G be a graph and G_d be the deg-centric graph of G. Then, the *successive iteration deg-centric graph* of G, denoted by G_{d^k}, is the derived graph obtained by taking the deg-centric graph successively k times; that is, $G_{d^k}= ((G_d)_d ...)_d$, (k-times). This process is known as *deg-centrication process* (see[11]). The *exact degree centric graph* or *exact deg-centric graph* of a graph G and denoted by G_{ed}, is the graph with $V(G_{ed}) = V(G)$ and $E(G_{ed}) = \{v_iv_j : d_G(v_i, v_j) = d\}$ and the studies mentioned above transformation is called *exact deg-centrication* (see[12]). Let G be a graph and G_{ed} be the exact deg-centric graph of G. Then, the iterated *exact deg-centric graph* of G, denoted by G_{ed^k}, is defined as the graph obtained by

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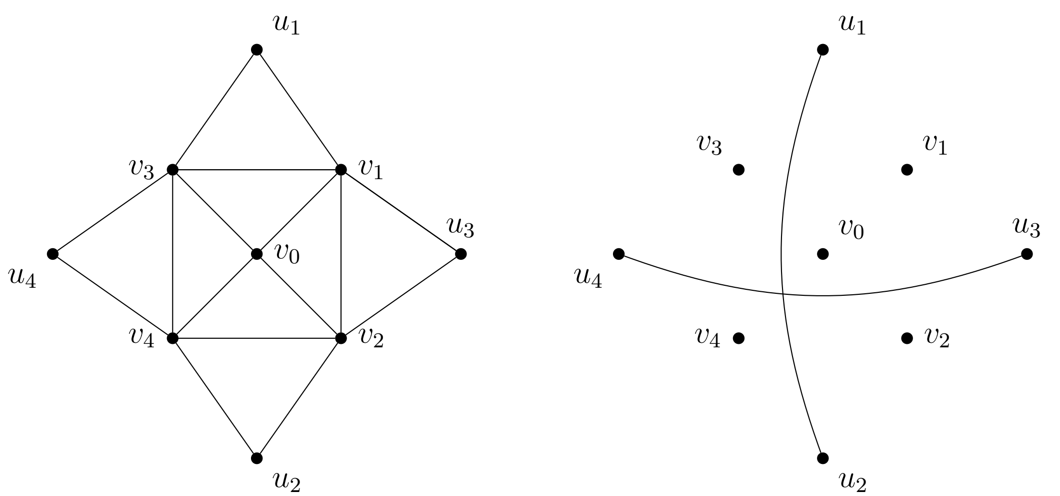
applying *exact deg-centrication* successively k-times. That is, $G_{ed^k} = ((G_{ed})_{ed} \ldots)_{ed}$, ($k$-times) (see[12]).

Motivated by the above-mentioned studies, in this paper, we introduce a new transformed graph called the coarse deg-centric graph and investigate the properties and structural characteristics of this type of transformed graph concerned.

2. Coarse deg-centric graphs

Definition 2.1. The *coarse degree centric graph* or *coarse deg-centric graph* of a graph G, denoted by G_{cd}, is the graph with $V(G_{cd}) = V(G)$ and $E(G_{cd}) = \{v_iv_j : d_G(v_i, v_j) > \deg_G(v_i)\}$. This graph transformation is called *coarse deg-centrication* of the graph. Note that this process is independent of the choice of v_i or v_j in the above sets.

An example of the coarse deg-centric graph is given in Figure 1.



(A) G (B) G_{cd}

Figure 1 A graph G and its coarse deg-centric graph

Observation 2.2. The coarse deg-centric graph of a complete graph K_n of order $n \geq 3$ is an empty graph $\overline{K}_n$.

Observation 2.3. If there exists a vertex $v_i \in V(G)$ with $\deg_G(v_i) > e_G(v_i)$, then v_i cannot form an edge in G_{cd}.

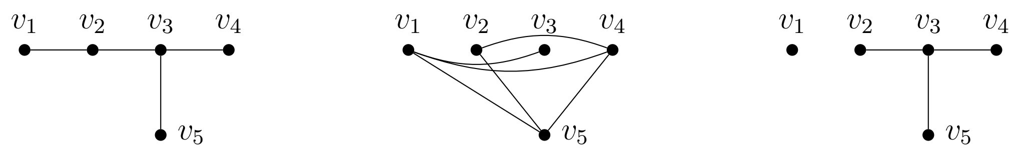
Theorem 2.4. *If for G of order n with minimum degree, $\delta(G) > diam(G)$, then $G_{cd} \cong \overline{K}_n$.*

Proof. If any vertex say, v_i incident at least one edge say v_iv_j it implies that $\deg_G(v_i) \geq d_G(v_i, v_j)$. Subsequently, the edge v_iv_k where $e(v_i) = d_G(v_i, v_k)$ must form as well. The above-stated implies that either $\deg_G(v_i) < \delta(G)$ or $e(v_i) > diam(G)$. In both cases, we have a contradiction. Hence, the result follows. $\square$

Definition 2.5. Let G be a graph and G_{cd} be the coarse deg-centric graph of G. Then, the iterated *coarse deg-centric graph* of G, denoted by G_{cd^k}, is defined as the graph obtained by applying *coarse deg-centrication* successively k-times. That is,

$$G_{cd^k} = ((G_{cd})_{cd} \cdots)_{cd},\ (k\text{-times}).$$

An example of the coarse deg-centrication process is given in Figure 2.



(A) G (B) G_{cd} (C) G_{cd^2}

FIGURE 2 Example of the coarse deg-centrication process of G.

Lemma 2.6. *Let G be a graph of the order n, which has at least one pendant vertex. Let $P(G) = \{w_i : \deg_G(w_i) = 1\}$. Then, $\deg_{G_{cd}}(w_j) = n - 2$, for all $w_j \in P(G)$.*

Proof. For a graph, G of order n has at least one vertex w with degree one. Then, by Definition 2.1, in G_{cd}, vertex w is adjacent, with all the vertices having a degree greater than one. Then, $\deg_{G_{cd}}(w) = n - 2$. □

For convenience, a path P_n is depicted on a horizontal line, and the vertices are labelled from left to right as v_1, v_2, $v_3, \ldots,$ v_n.

Proposition 2.7. *For a path P_n, $n \geq 4$, $\varepsilon((p_n)_{cd}) = 2 + \frac{r(r+1)}{2}$, where $r = n - 3$.*

Proof. By Lemma 2.6, the degree of each of v_1, v_n is $n - 2$. Then, as a direct consequence of Definition 2.1, the number of edges in a coarse deg-centric graph that is $\varepsilon((p_n)_{cd}) = 2 + \frac{r(r+1)}{2}$, where $r = n - 3$. □

Observation 2.8. For a path graph P_n, $n \geq 4$. Then, the vertices v_1, v_n of the coarse deg-centric graph have a degree of $n - 2$.

Observation 2.9. The $(p_n)_{cd^k}$ of a path graph P_n is an empty graph $\overline{K}_n$.

A *star graph*, denoted by $k_{1,n}$, $n \geq 0$, is obtained by attaching n pendant vertices (also called leaves) to a central vertex v_0.

Proposition 2.10. *The coarse deg-centric graph of a star graph $K_{1,n}$, $n \geq 0$ is the disjoint union of the empty graph $\overline{K}_1$ and the complete graph K_n.*

Proof. Let G be a star graph $K_{1,n,}$, $n \geq 0$. Clearly, the star graph is of the order $n + 1$. Let $V(K_{1,n,}) = \{v_0, \underbrace{u_1, u_2, \ldots, u_n}_{\text{pendant vertices}}\}$. Since $\deg_G(v_0) = n > e(v_0) = 2$, no edge incident at v_0. However, since all u_i are pendant vertices, each u_i forms the edge $u_i u_j$. Then, by Definition 2.1, the n pendant vertices $u_1, u_2, \ldots, u_n$ are adjacent to all other $n-1$ pendant vertices that is $\deg_{G_{cd(u_n)}} = n-1$. This implies the disjoint union of the empty graph $\overline{K}_1$ and the complete graph K_n. □

Observation 2.11. For a star graph $k_{1,n}$, $n \geq 0$, $\varepsilon((k_{1,n})_{cd}) = \frac{n(n-1)}{2}$.

A non-trivial *bistar graph*, denoted by $S_{a,b}$, is a graph obtained by joining the centers of two non-trivial star graphs $k_{1,a}$, $a \geq 1$ and $k_{1,b}$, $b \geq 1$ with the edge v_0u_0.

Proposition 2.12. *For a bistar graph $S_{a,b}$, $a, b \geq 1$, $\varepsilon((s_{a,b})_{cd}) = \frac{(a+b)^2+a+b}{2}$.*

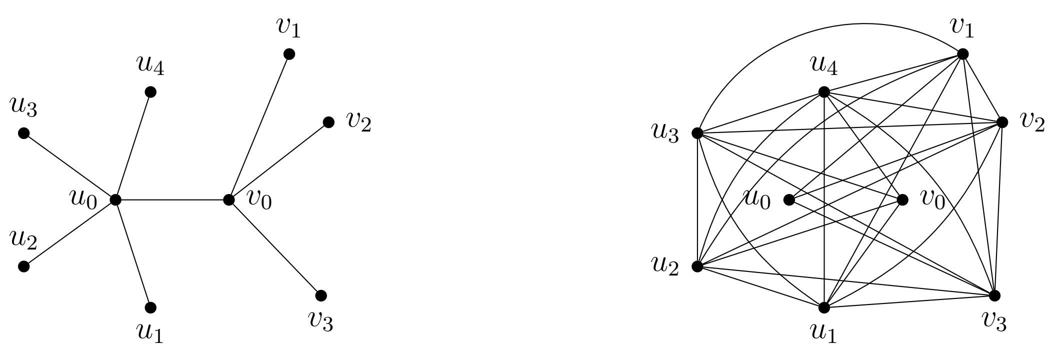
Proof. Let G be a bistar graph $S_{a,b}$, $a, b \geq 1$. Let the pendant vertices of $K_{1,a}$ be the set $X = \{v_1, v_2, \ldots, v_a\}$ and let the pendant vertices of $K_{1,b}$ be the set $Y = \{u_1, u_2, \ldots, u_b\}$. Finally let $W = \{v_0, u_0\}$. By Definition 2.1, it follows that within the set X, a total of $\frac{a(a-1)}{2}$ edges are formed in the coarse deg-centric graph. Similarly, within the set Y a total of $\frac{b(b-1)}{2}$ edges are formed in G_{cd}. Similarly, by Definition 2.1, a total of ab edges are formed between sets X and Y. Finally, between respectively, the pair of sets X, W and the pair of sets Y, W at a total of $a + b$ edges are formed in G_{cd} Clearly, all edges resulting from coarse deg-centrication have been accounted for. Therefore, after this process, it follows that:

$$\varepsilon((s_{a,b})_{cd}) = \frac{(a+b)^2 + a + b}{2}.$$

. □

Observation 2.13. For a bistar graph $S_{a,b}$, $a, b \geq 1$, $(s_{a,b})_{cd^k}$ is an empty graph.

An illustration of a proposition 2.12 is given in Figure 3.



(A) $S_{4,3}$ (B) G_{cd}

FIGURE 3 : Coarse deg-centric graphs of a bistar graph $S_{a,b}$

Proposition 2.14. *For a cycle graph C_n, $n \geq 5$, the coarse deg-centric graph, $(c_n)_{cd}$ is always a $(n-5)$-regular graph.*

Proof. Because $\deg_G(v_i) = 2$, for all $v_i \in V(G)$, any vertex v_i in G_{cd} is adjacent to all vertices $d_G(v_i, v_j) > 2$. It immediately follows that $(c_n)_{cd}$ is always a $(n-5)$-regular graph. □

Proposition 2.15. *For a cycle graph C_n, $n \geq 6$, $\varepsilon((c_n)_{cd}) = \frac{n^2-5n}{2}$.*

Proof. Consider a cycle graph C_n, $n \geq 6$, then by Proposition 2.14, the coarse deg-centric graph is always a $(n-5)$-regular graph. It immediately follows that, $\varepsilon((c_n)_{cd}) = \frac{n^2-5n}{2}$. □

A *wheel graph* denoted by $W_{1,n}$, $n \geq 3$ is obtained by taking a cycle C_n, $n \geq 3$ (the rim with rim-vertices) and adding the central vertex v_0 with *spokes* namely, edges v_0v_i, $1 \leq i \leq n$.

Proposition 2.16. *The coarse deg-centric graph of a wheel graph $W_{1,n}$ is the empty graph $\overline{K}_{2n+1}$.*

Proof. Since $\delta(W_{1,n}) = 3 > 2 = diam(W_{1,n})$, then by Theorem 2.4, $(W_{1,n})_{cd} \cong \overline{K}_{2n+1}$. □

A *helm graph*, denoted by $H_{1,n,}$, $n \geq 3$ is a graph obtained from a wheel graph $W_{1,n}$ by attaching a pendant vertex u_i to the corresponding *rim vertex* v_i.

Proposition 2.17. *For a helm graph $H_{1,n,}$, $n \geq 3$, $\varepsilon((H_{1,n})_{cd}) = \frac{3n^2-n}{2}$.*

Proof. For a helm graph $H_{1,n,}$, $n \geq 3$. The helm graph is of the order $2n+1$. Let $V(H_{1,n,}) = \{v_0, v_1v_2, \ldots, v_n, \underbrace{u_1, u_2, \ldots, u_n}_{\text{pendant vertices}}\}$. Since $\deg(v_0) = n > e(v_0) = 2$ in $H_{1,n,}$, no edge incident at v_0. Also, since each $\deg(v_i) = 4 > e(v_i)$ in $H_{1,n,}$, no edge incident at v_i. However, since all u_i are pendant vertices, by Lemma 2.6, the n pendant vertices $u_1, u_2, \ldots, u_n$ are adjacent to all other $2n-1$ vertices in $(H_{1,n,})_{cd}$. That is, $\deg(u_n) = 2n-1$. Also, $\deg(v_0) = n$ and $\deg(v_n) = n-1$ in $(H_{1,n,})_{cd}$. Finally,

$$\varepsilon((H_{1,n,})_{cd}) = \frac{\sum\limits_{w_i \in V((H_{1,n,})_{cd})} \deg(w_i)}{2} = \frac{3n^2-n}{2}.$$

□

Recall that the sequence of pentagonal numbers is generated by:

$$q_n = \frac{3n^2-n}{2}; n = 0, 1, 2, \ldots.$$

In expanded form it is:

$$1, 5, 12, 22, 35, 51, 70, 92, 117, 145, 176, 210, 247, \ldots$$

The relation between the size of the coarse deg-centricated Helm graphs and the pentagon numbers follows immediately as a corollary.

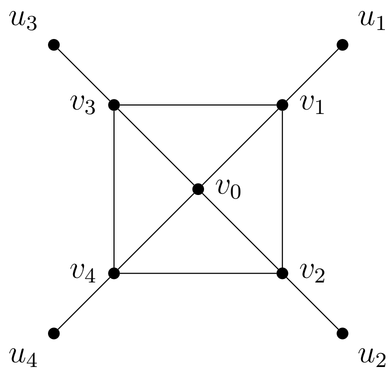
Corollary 2.18. *For a helm graph $H_{1,n,}$, $n \geq 3$,*

$$\varepsilon((H_{1,n})_{cd}) = q_n,\ n \geq 3.$$

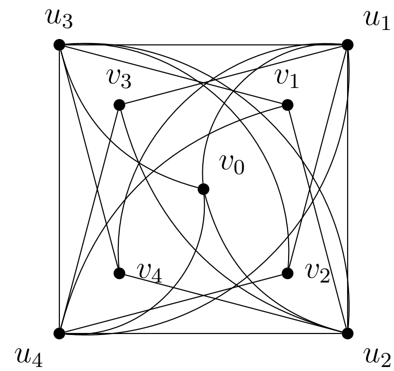
An illustration of a proposition 2.17 is given in Figure 4.

A *closed helm graph*, denoted by $CH_{1,n}$, $n \geq 3$, is the graph obtained from a helm graph $H_{1,n}$ by joining the pendant vertices, in order, forming a cycle, called the outer rim.

Proposition 2.19. *For a closed helm graph $CH_{1,n,}$, $n \geq 8$, $\varepsilon((CH_{1,n})_{cd}) = \frac{n^2-7n}{2}$.*



(A) $H_{1,4}$



(B) G_{cd}

FIGURE 4 Coarse deg-centric graph of $H_{1,4}$.

Proof. Consider a closed helm graph $CH_{1,n}$ $n \geq 8$. The closed helm graph is clearly of the order $2n+1$. Let $V(CH_{1,n,}) = v_0, v_1, v_2, \dots, v_n, u_1, u_2, \dots, u_n$. Since $diam(CH_{1,n}) = 4$ and $\delta(CH_{1,n}) = 3$. Then, by Definition 2.1, $\deg(v_0) = n > diam(CH_{1,n}) = 4$ no edge incident at v_0 in $(CH_{1,n})_{cd}$. Also, since each $\deg(v_i) = diam(CH_{1,n}) = 4$, no edge incident at v_i in $(CH_{1,n})_{cd}$. $\deg(u_i) = 3 < diam(CH_{1,n}) = 4$ it follows that each outer-rim vertex form $(n-7)$ edges to vertices on the outer-rim. In total $n(n-7)$ such edges will form in $(CH_{1,n})_{cd}$. Therefore, a total of $\frac{n(n-7)}{2}$ edges are formed to obtain $(CH_{1,n})_{cd}$. The above said to yield the result,

$$\varepsilon((CH_{1,n})_{cd}) = \frac{n^2-7n}{2}$$

. □

Proposition 2.20. *For a closed helm graph $CH_{1,n}$, $n \geq 8$, the coarse deg-centric graph, $(CH_{1,n})_{cd}$ is always a $(n-7)$-regular graph.*

Proof. Consider a closed helm graph $CH_{1,n}$ $n \geq 8$. The closed helm graph is clearly of the order $2n+1$. Let $V(CH_{1,n,}) = v_0, v_1, v_2, \dots, v_n, u_1, u_2, \dots, u_n$. By Proposition 2.19, the number of edges in a coarse deg-centric graph is $\frac{n^2-7n}{2}$. It follows the result, $(CH_{1,n})_{cd}$ is always a $(n-7)$-regular graph. □

Observation 2.21. The coarse deg-centric graph of a closed helm graph $CH_{1,n}$ is the empty graph $\overline{K}_{2n+1}$, when $n < 8$.

A *djembe graph*, denoted by $D_{1,n}$, is obtained by joining the vertices u_i's; $1 \leq i \leq n$ of a closed helm graph $CH_{1,n}$ to its central vertex v_0. Note that, in view of Theorem 2.4, the coarse deg-centric graph of a djembe graph $D_{1,n}$, $n \geq 3$ is the empty graph $\overline{K}_{2n+1}$.

If the edge v_1v_3 joins vertices v_1 and v_3, then the *subdivision* of v_1v_3 replaces v_1v_3 by a new vertex v_2 and two new edges v_1v_2 and v_2v_3. A *gear graph*, denoted by G_n, $n \geq 3$, is a graph obtained by applying subdivision to each edge of the rim of a wheel graph $W_{1,n}$.

Proposition 2.22. *For a gear graph G_n, $n \geq 3$, $\varepsilon((G_n)_{cd}) = \frac{3n^2-7n}{2}$.*

Proof. For a gear graph G_n, $n \geq 3$. The gear graph is of the order $2n+1$. Let $V(G_n) = v_0, v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n$. Since $\deg_{G_n}(v_0) = n \geq 3 > e(v_0) = 2$, no edges incident at v_0 in $(G_n)_{cd}$. However, Since $\deg_{G_n}(u_i) = 2$, by Definition 2.1, $2n-5$ edges incident at u_i. Since $\deg_{G_n}(v_i) = 3$ then $n-2$ edges incident at v_i in coarse deg-centric graph. Finally,

$$\begin{aligned}\varepsilon((G_n)_{cd}) &= \frac{\sum\limits_{w_i \in V((G_n)_{cd})} \deg(w_i)}{2}\\ &= \frac{n(2n-5)+n(n-2)}{2}\\ &= \frac{3n^2-7n}{2}.\end{aligned}$$

□

A *web graph*, denoted by $Wb_{1,n}$, $n \geq 3$ is the graph obtained by attaching a pendant edge to each vertex of the outer cycle (or rim) of the closed helm graph $CH_{1,n}$.

Proposition 2.23. *For a web graph $Wb_{1,n}$, $n \geq 3$, $\varepsilon((Wb_{1,n})_{cd}) = \frac{5n^2-n}{2}$.*

Proof. For a web graph $Wb_{1,n}$, $n \geq 3$. The web graph is of the order $3n+1$. Let $V(Wb_{1,n,}) = \{v_0, v1, v_2, \ldots, v_{n-1}, v_n, u_1, u_2, u_3, \ldots, u_n, \underbrace{w_1, w_2, w_3, \ldots, w_n}_{\text{pendant vertices}}\}$. Since $\deg(v_0) = n > e(v_0) = 3$ in $Wb_{1,n}$, no edge incident at v_0 in $(Wb_{1,n})_{cd}$. Also since each $\deg(v_i) = 4 > e(v_i)$ in $Wb_{1,n}$, no edge incident at v_i in coarse deg-centric graph. Similarly, no edge incident from any vertex u_i. However, since all w_i are pendant vertices, each w_i forms the edges w_iu_j, $1 \leq j \leq n-1$ as well as w_iv_j, $1 \leq j \leq n$ as well as the edge w_iv_0 and finally, in this fashion the edges w_iw_j, $1 \leq j \leq n$, $i \neq j$. The summation of incident edges yields the result. That is,

$$\varepsilon((Wb_{1,n})_{cd}) = \frac{n(3n-1)+n(n-1)+n(n+1)}{2} = \frac{5n^2-n}{2}.$$

□

A *double wheel* DW_n is obtained by taking two copies of a wheel W_n $n \geq 3$ and *merging* the two central vertices. Note that, in view of Theorem 2.4, the coarse deg-centric graph of a double wheel DW_n, $n \geq 3$ is the empty graph $\overline{K}_{2n+1}$.

A *flower graph*, $F_{1,n}$, $n \geq 3$ is a graph obtained from a helm graph $H_{1,n,}$ by joining each of its pendant vertices u_i's to its central vertex v_0. Note that, in view of Theorem 2.4, the coarse deg-centric graph of a flower graph $F_{1,n}$, $n \geq 3$ is the empty graph $\overline{K}_{2n+1}$.

The *sunflower graph*, denoted by $SF_{1,n}$, $n \geq 3$ is obtained from the wheel $W_{1,n}$ by attaching n vertices u_i, $1 \leq i \leq n$ such that each u_i is adjacent to v_i and v_{i+1} and count the suffix is taken modulo n.

Proposition 2.24. *For a sunflower graph $SF_{1,n,}$, $n \geq 3$, $\varepsilon((SF_{1,n})_{cd}) = \frac{3n^2-11n}{2}$.*

Proof. For a sunflower graph $SF_{1,n,}$, $n \geq 3$. The sunflower graph is of the order $2n+1$. Let $V(SF_{1,n,}) = v_0, v_1 v_2, \ldots, v_n, u_1, u_2, \ldots, u_n$. Since $\deg(v_0) = n > e(v_0) = 2$ in $SF_{1,n,}$, no edge incident at v_0 in $(SF_{1,n,})_{cd}$. Also since each $\deg(v_i) = 5 > e(v_i)$ in $SF_{1,n}$, no edge incident at v_i in $(SF_{1,n})_{cd}$. However, since $\deg(u_i) = 2$, each u_i forms the edge with at least a distance of two vertices from u_i. Then, by Definition 2.1, the n vertices $u_1, u_2, \ldots, u_n$ are adjacent with all other $2n-7$ vertices that is $\deg(u_n) = 2n-7$ in $(SF_{1,n})_{cd}$. Also vertices $v_1, v_2, v_3 \ldots v_n$ are adjacent with $n-4$ vertices hence, $\deg(v_n) = n-4$ in $(SF_{1,n})_{cd}$. Finally,

$$\varepsilon((SF_{1,n})_{cd}) = \frac{\sum\limits_{w_i \in V(SF_{1,n})_{cd}} \deg(w_i)}{2} = \frac{n(2n-7)+n(n-4)}{2} = \frac{3n^2-11n}{2}$$

. □

A *closed sunflower graph* $CSF_{1,n}$ is obtained by adding the edge $u_i u_{i+1}$ of the sunflower graph. Note that, in view of Theorem 2.4, the coarse deg-centric graph of a closed sunflower graph $CSF_{1,n}$, $n \geq 3$ is the empty graph $\overline{K}_{2n+1}$.

A *blossom graph*, denoted by $Bl_{1,n}$, is obtained by making each u_i adjacent to the central vertex of the closed sunflower graph. Note that, in view of Theorem 2.4, the coarse deg-centric graph of a blossom graph Bl_n, $n \geq 3$ is the empty graph $\overline{K}_{2n+1}$.

A *sunlet graph* , denoted by Sl_n, $n \geq 3$, is a graph obtained by attaching a pendant vertex to every vertex of a cycle graph c_n, $n \geq 3$. In other words, a sunlet graph on $2n$ vertices is obtained by taking the corona product $C_n \circ K_1$.

Proposition 2.25. *For a sunlet graph Sl_n, $n \geq 3$,*

$$\varepsilon((Sl_n)_{cd}) = \begin{cases} \frac{3n^2-3n}{2} & \text{if } 3 \leq n \leq 7. \\ \frac{4n^2-10n}{2} & \text{if } n \geq 8. \end{cases}$$

Proof. (a) If $3 \leq n \leq 7$. For a sunlet graph Sl_n, $n \geq 3$. The sunlet graph is of the order $2n$. Let $V(Sl_n) = \{v_1, v_2, \ldots, v_n, \underbrace{u_1, u_2, \ldots, u_n}_{\text{pendant vertices}}\}$. Since all u_i's are pendant vertices, then by Lemma 2.6, each u_i's are adjacent to all other $2n-2$ vertices in $(Sl_n)_{cd}$ that is, $\deg(u_n) = 2n-2$. All other n vertices are adjacent with $n-1$ vertices. Hence, $\deg(v_n) = n-1$ in $(Sl_n)_{cd}$. Finally,

$$\varepsilon((Sl_n)_{cd}) = \frac{\sum\limits_{w_i \in V(Sl_n)_{cd}} \deg(w_i)}{2} = \frac{3n^2-3n}{2}.$$

(b) If $n \geq 8$, then by Lemma 2.6, the n pendant vertices $u_1, u_2, \ldots, u_n$ are adjacent to $2n-2$ vertices, that is $\deg(u_n) = 2n-2$ in $(Sl_n)_{cd}$. Since $\deg(v_n) = 3$ in Sl_n, then these n vertices are adjacent with a distance greater than three vertices in the cycle and also adjacent with each u_n vertices in $(Sl_n)_{cd}$. That is, $\deg(v_n) = 2n - 8$. Finally,

$$\varepsilon((Sl_n)_{cd}) = \frac{\sum\limits_{w_i \in V(Sl_n)_{cd}} \deg(w_i)}{2} = \frac{4n^2 - 10n}{2}$$

.

□

Observation 2.26. Let G be a sunlet graph Sl_n, $n \geq 3$. Then, $(Sl_n)_{cd^k}$ is the empty graph $\overline{K}_{2n}$.

An illustration of a proposition 2.25 is given in Figure 5.

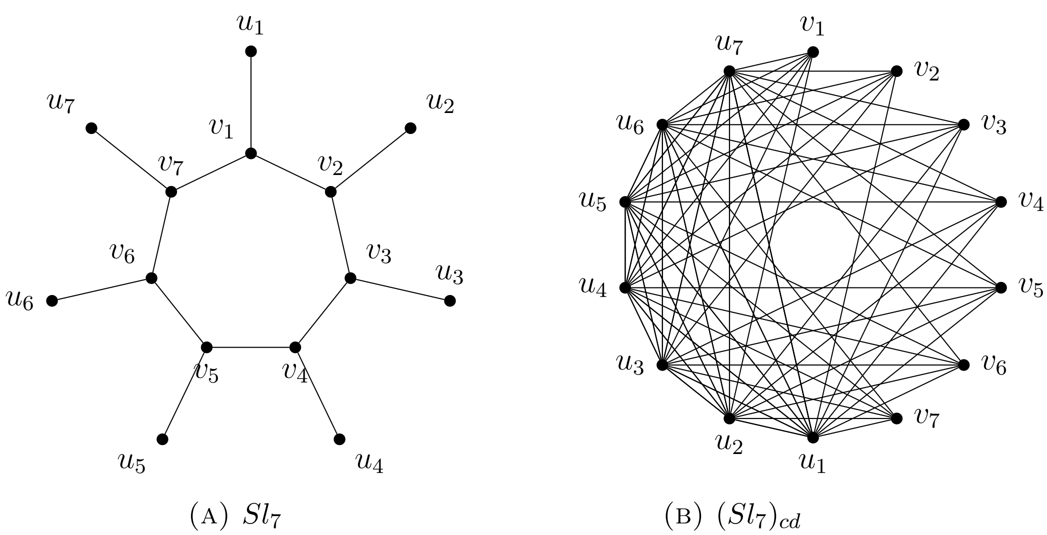


FIGURE 5 Coarse deg-centric graph of Sl_7.

An *antiprism graph*, denoted by A_n, $n \geq 3$ is a graph obtained two cycles C_n and C'_n of order n with vertex sets $V = v_1, v_2, v_3, \ldots, v_n$ and $U = u_1, u_2, u_3, \ldots, u_n$ respectively. We join the vertices $u_i v_i$ and $u_i v_{i+1}$ to form the additional edges.

Proposition 2.27. *For an antiprism graph A_n. Then,*

$$\varepsilon((A_n)_{cd}) = \begin{cases} 0 & \text{if } 3 \leq n \leq 8 \\ 2n^2 - 17n & \text{if } n \geq 9. \end{cases}$$

Proof. Let G be an antiprism graph A_n, $n \geq 3$. The antiprism graph is of the order $2n$. Let $V(A_n) = v_1, v_, \ldots, v_n, u_1, u_2, \ldots, u_n$.

(a) If $3 \leq n \leq 8$, $\deg_G(v_i) = \deg_G(u_i) = 4 > e(v_i) = e(u_i)$, then by Theorem 2.4, $(A_n)_{cd} \cong \overline{K}_{2n}$.

(b) If $n \geq 9$, $\deg_G(v_i) = \deg_G(u_i) = 4$, then by Definition 2.1, $\deg_{G_{cd(v_i)}} = \deg_{G_{cd(u_i)}} = 2n - 17$. That implies $\varepsilon((A_n)_{cd}) = 2n^2 - 17n$.

□

Recall that a complete bipartite graph $K_{n,m}$, $n, m \geq 1$ is a graph whose vertex set can be partitioned into two independent sets X, $|X| = n$ and Y, $|Y| = m$ and each vertex in X is adjacent to all vertices in Y. Note that, in view of Theorem 2.4, the coarse deg-centric graph of a complete bipartite graph $K_{n,m}$, $n, m \geq 3$ is the empty graph $\overline{K}_{n+m}$.

A tree denoted by T_n, $n \geq 1$ is a connected acyclic graph. It is known that a tree T_n has $n-1$ edges.

Proposition 2.28. *For a tree T_n, $n \geq 3$. Then, in the coarse deg-centric graph, at least two vertices v_i have degree $n-2$.*

Proof. Consider a tree T_n, $n \geq 3$, then by Lemma 2.6, $P(T_n) = \{w_i : \deg_{T_n}(w_i) = 1\}$. Then, $\deg_{(T_n)_{cd}}(w_j) = n-2$, for all $w_j \in P(T_n)$. □

The *ladder graph*, L_n, $n \geq 1$ is obtained by taking two copies of a path P_n with respective vertices say, $v_1, v_2, v_3, \ldots, v_n$ and $u_1, u_2, u_3, \ldots, u_n$ and adding the edges v_iu_i, $1 \leq i \leq n$. Note that $L_n \cong P_n \Box K_2$ where $\Box$ denotes the Cartesian product.

Proposition 2.29. *For a ladder $G = L_n$, $n \geq 1$ it follows that:*

$$\begin{gathered}\varepsilon(L_{1_{cd}}) = 0,\\ \varepsilon(L_{2_{cd}}) = 0,\\ \varepsilon(L_{3_{cd}}) = 2,\\ \varepsilon(L_{4_{cd}}) = 8,\\ \varepsilon(L_{5_{cd}}) = 16,\\ \varepsilon(G_{cd}) = \varepsilon(H_{cd}) + 4n - 14 \text{ where } H = L_{n-1} \text{ and } n \geq 6.\end{gathered}$$

Proof. By applying Definition 2.1 it easily follows that $\varepsilon(L_{1_{cd}}) = 0$, $\varepsilon(L_{2_{cd}}) = 0$, $\varepsilon(L_{3_{cd}}) = 2$, $\varepsilon(L_{4_{cd}}) = 8$ and $\varepsilon(L_{5_{cd}}) = 16$. Now, besides the claimed result, it is valid that for any $n \geq 6$ and $H = L_{n-1}$ the size of H_{cd} that is, $\varepsilon(H_{cd})$ can be determined by applying Definition 2.1. Consider $H = L_{n-}$ and assume that both H_{cd} and $\varepsilon(H_{cd})$ has been determined. Now consider the extension from H to $G = L_n$. Some subgraph of H_{cd} is a subgraph of G_{cd}. Note that in G, the degrees of respectively v_{n-1}, and u_{n-1} have increased to 3. Therefore, in G_{cd} the two edges $v_{n-1}u_{n-3}$ and $u_{n-1}v_{n-3}$ as well as the two edges $v_{n-1}v_{n-4}$ and $u_{n-1}u_{n-4}$ found in H_{cd} are not adjacent in G_{cd}. All other edges incident from only amongst the vertices $V(H) \subset V(G)$ replicate exactly in G_{cd}. With regards to say v_n the edges which forms are v_nu_{n-2} together with v_nu_i, $1 \leq i \leq n-2$. A similar thing can be applied to vertex u_n. Hence,

$$\varepsilon(G_{cd}) = \varepsilon(H_{cd}) + [2 \times 2(n-3) + 2 - 4] = \varepsilon(H_{cd}) + 4n - 14.$$

Finally, since an initial value, that is $\varepsilon(L_{5_{cd}}) = 16$ is known, the result for $n \geq 6$ follows through mathematical induction. □

Observation 2.30. For a ladder $G = L_n$, $n \geq 5$ it follows that, $\varepsilon(G_{cd}) = 2(n-3)^2 + 8$.

Observation 2.31. Let G be an ladder graph L_n, $n \geq 1$. Then, G_{cd^k} is the empty graph $\overline{K}_{2n}$.

3. Conclusion

The graph transformation called Coarse deg-centrication has been introduced. Various exploratory results have been presented to establish some foundation for further research. As a scope of the study, the researchers can extend the study on graph theoretical parameters to coarse deg-centric graphs of various classes of graphs and obtain fruitful results.

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